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Let D be the unit ball in 3-space and let A_k be a set of k proper disjoint arcs in D lying in the x-z plane. The group, $\mathscr{M}_{2k}$, of orientation preserving homeomorphisms of boundary D leaving the set $A_k \cap$ boundary D invariant, modulo those isotopic to the identity via an isotopy fixing the set $A_k \cap$ boundary D, is a natural homomorphic image of the $2k$ string braid group of the sphere via a homomorphism with kernel Z_2.

In this paper, finite sets of generators are explicitly determined for the subgroups of $\mathscr{M}_{2k}$ generated by

(1) homeomorphisms of D leaving the set A_k invariant, and

(2) homeomorphisms of D leaving the set A_k fixed pointwise.

1. Introduction. Let D be the unit ball in 3-space and let P be a set of n points on the boundary of D (denoted ∂D). The group of orientation perserving homeomorphisms of ∂D leaving P invariant, modulo the subgroup of those isotopic to the identity via an isotopy fixing P, will be denoted $\mathscr{M}_n$. The group $\mathscr{M}_n$ is a subgroup of index two of the full mapping class group of the sphere with n points removed. (The full group contains equivalence classes of orientation reversing homeomorphisms.) There is a natural map of the n-string braid group of the sphere onto $\mathscr{M}_n$ whose kernel is Z_2. For a definition of the braid group, the mapping class group of a surface, their relationship and for a description of a presentation of the braid group the reader is referred to Joan Birman's work ([1] and [2]).

Let A be a set of k disjoint proper arcs in D lying in the x-z plane. Suppose that P is the set of $2k$ points $A \cap \partial D$. Let $\mathscr{A}_k$ be the subgroup of $\mathscr{M}_{2k}$ generated by equivalence classes of homeomorphisms of D leaving the set A invariant, restricted to ∂D. Let $\mathscr{F}_k$ be the subgroup of $\mathscr{A}_k$ generated by homeomorphisms leaving the set A fixed pointwise. Naturally the groups $\mathscr{A}_k$ and $\mathscr{F}_k$ will depend on the choice of the set A. However, given any two such sets A and A', there is an orientation preserving homeomorphism of D taking A onto A', so the groups $\mathscr{A}_k$ and $\mathscr{A}_k'$; $\mathscr{F}_k$ and $\mathscr{F}_k'$ will be isomorphic. The purpose of this paper is to determine, explicitly, a set of generators for the group $\mathscr{A}_k$ and $\mathscr{F}_k$ for one particular choice of the set A.

The groups $\mathscr{A}_k$ and $\mathscr{F}_k$ arise in knot theory. For example, suppose we have two knots (S^3, k) and (S^3, k') represented as closed braids. Let us consider the question, "When is there an orientation preserving homeomorphism of S^3 taking k onto k'?" (i.e. When are the knots the same?) Let A_1 equal D intersected with a set of k concentric circles lying in the x-z plane centered at (1, 0, 0) and having radii between 1/2 and 1 1/2. Let (D', A_1') be another copy of (D, A) and let i be the identification map of D with D', restricted to ∂D. Since (S^3, k) and (S^3, k') are represented as closed braids, we can explicitly construct homeomorphisms φ and ψ of ∂D fixed on $\partial D \cap \{z \geqq 0\}$ such that

$$(S^3, k) = (D, A_1) \bigcup_{i\varphi} (D', A_1') \text{ and } (S^3, k') = (D, A_1) \bigcup_{i\psi} (D', A_1') .$$

The sewing homeomorphisms φ and ψ can be considered both as elements of the k-string braid group $\mathscr{C}_k$ of the disc $\partial D \cap \{z < 0\}$ and of $\mathscr{M}_{2k}$. (Unlike the case of the sphere, the group $\mathscr{C}_k$, defined in a manner analogous to $\mathscr{M}_k$, is isomorphic to the k-string braid group, so we identify the two.) The conjugacy problem has been solved in the group $\mathscr{C}_k$ (see [4]) and it is not hard to see that if $[\varphi]$ is conjugate to $[\psi]$, then $(S^3, k) \equiv (S^3, k')$. However, there are examples of the type $(D, A_1) \bigcup_{i\alpha} (D', A_1') \equiv (D, A_1) \bigcup_{i\beta} (D', A_1')$ with α and β not conjugate, (this can be done with three string, two crossing representations of the trivial knot) so it is reasonable to investigate another equivalence relation on braids with larger equivalence classes. Suppose $[\alpha] \equiv [\beta]$ in $\mathscr{M}_{2k}$ if they belong to the same double coset $\mathscr{A}_k \alpha \mathscr{A}_k$ of $\mathscr{M}_{2k}$. If $[\varphi] \equiv [\psi]$, one can explicitly construct a homeomorphism of $(S^3, k) = (D, A_1') \bigcup_{i\varphi} (D', A_1')$ onto $(S^3, k') = (D, A_1) \bigcup_{i\varphi} (D', A_1')$. The equivalence relation "$\equiv$" is somewhat analogous to the definition of equivalence of Heegaard splittings of a three manifold.

A solution of the double coset problem (Given $[\alpha]$ and $[\beta] \in \mathscr{M}_{2k}$ when does $[\alpha] \in \mathscr{A}_k[\beta]\mathscr{A}_k$?) might shed some light on the knot problem.

The organization of this paper is as follows: In § 2 we develop notation, define a set of elements of $\mathscr{A}_k$, and prove some technical lemmas. In § 3 we state and prove the main theorem of the paper. In § 4 we give generators for the groups $\mathscr{A}_k$ and $\mathscr{F}_k$. In § 5 we make concluding remarks relating the groups $\mathscr{A}_k$ and $\mathscr{F}_k$ to other problems in mathematics.

Throughout the paper we shall work in the P. L. category and all curves, homeomorphisms, isotopies, etc., will be assumed piecewise linear without it being explicitly stated. We shall assume the reader knows much of what is in [3] and we shall refer him there for definitions of such terms as isotopy, ambient isotopy, regular neighborhood, and locally unknotted.

2. Definition of the generators. In this section a set of elements of $\mathscr{A}_n$ is defined which will ultimately be shown to generate $\mathscr{A}_n$. We begin by defining certain subsets of the ball D where D is represented in 3-space as $\{(x, y, z) \mid x \leqq 0\} \cup \{\infty\}$. In the sequel we shall define sets in the manner $S = \{x, y, z$ satisfying certain conditions$\}$ rather than $\{(x, y, z) \mid x, y, z$ satisfies certain conditions$\}$. For example $D = \{x \leqq 0\} \cup \{\infty\}$. Whenever a set of objects is indexed by i, we shall assume $1 \leqq i \leqq n$ unless explicitly stated otherwise. At this point the reader should refer to Figure 1.

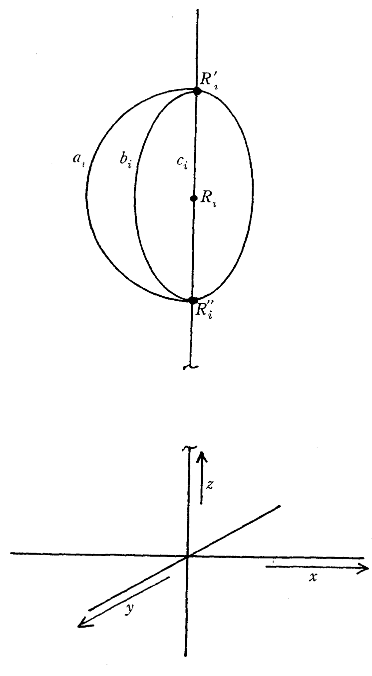

Figure 1

Let P_{zy}, P_{zx}, and Q (for quarter space) be the set $\{x = 0\}$, $\{y = 0, x \leqq 0\}$, and $\{x \leqq 0, y \geqq 0\}$ respectively. Let R_i, R_i', and R_i'' be the points $(0, 0, 8i)$, $(0, 0, 8i + 2)$, and $(0, 0, 8i - 2)$ respectively. Let c_i be the line segment $[R_i', R_i'']$ and let a_i and b_i be the semicircles with diameter c_i contained in the half planes P_{zx} and $P_{zy} \cap Q$ resrectively. Let $A = \bigcup_i a_i$, $B = \bigcup_i b_i$ and $C = \bigcup_i c_i$. Let H_i and K_i be the hemisphere, and half ball $\{\| P - R_i \| = 2\}$ and $\{\| P - R_i \| \leqq 2\}$ respectively, where $\| \quad \|$ is the usual 3-space distance. Note that a_i and b_i are contained in H_i. Finally, let $F_i = H_i \cap Q$ and $E_i = K_i \cap P_{zx}$, note that $\partial F_i = a_i \cup b_i$ and $\partial E_i = a_i \cup c_i$, and define $F = \bigcup_i F_i$ and $E = \bigcup_i E_i$.

Next we shall define a set of isotopies. In what follows a point moving on a circle in the plane P_{zy} will be said to be moving positively if it appears to be moving counterclockwise to an observer in the right half space $\{x > 0\}$. An isotopy θ_t is isometric if $\|\theta_t(P) - \theta_t(Q)\| = \|P - Q\|$ for all P and Q in the domain of definition of θ_t. An isotopy θ_t will be called a parallel isotopy if the vector $\overrightarrow{\theta_t(P)\theta_t(Q)}$ is parallel to the vector $\overrightarrow{PQ}$ for all P and Q in the domain of definition of θ_t.

1. τ_i. Let $\tau_i(t)$ (the flip of the i^{th} arc) be the isometric isotopy of the inclusion map $K_i \to D$ that rotates K_i through an angle of $180°$ about an axis perpendicular to P_{yz} through the point R_i. We insist that the point $\tau_i(t)(R'_i)$ move positively and with constant speed.

2. $\sigma_i, 1 \leqq i \leqq n - 1$. Let $\sigma_i(t)$ (the exchange of the i^{th} and $i + 1^{\text{st}}$ arcs) be the isometric isotopy of the inclusion map $K_i \cup K_{i+1} \to D$ that rotates K_i and K_{i+1} through an angle of $180°$ about an axis perpendicular to P_{zy} and through the midpoint of $[R_i, R_{i+1}]$. We insist that the point $\sigma_i(t)(R'_i)$ move positively and with constant speed.

3. μ_i. (The shrinking of the i^{th} arc.) Let $\mu_i(t)$ be the map of K_i into D such that $\|\mu_i(t)(P) - \mu_i(t)(P')\| = (1 - t)\|P - P'\|$ for $P, P' \in K_i$, $\mu_i(t)(R_i) = R_i$ for all $t \leqq 1$, and for any $\rho < 1$, $\mu_i(\rho t)$, $0 \leqq t \leqq 1$ defines a parallel isotopy of the inclusion map $K_i \to D$. The map $\mu_i(t)$ will be useful mainly in defining other isotopies.

4. $\rho_{ij}, 1 \leqq i, j \leqq n, i \neq j$. (Pull the i^{th} arc through the j^{th} arc.) Let $\rho_{ij}(t)$ be the parallel isotopy of the inclusion map $K_i \to D$ such that $\rho_{ij}(t) = \mu_i(3/2\, t)$, $0 \leqq t \leqq 1/3$; $\|\rho_{ij}(t)(P) - \rho_{ij}(t)(Q)\| = 1/2\|P - Q\|$ for any $P, Q \in K_i$, $1/3 \leqq t \leqq 2/3$; $\rho_{ij}(t)(R_i)$ moves positively with constant speed once around a circle in P_{zy} whose diameter is $[R_i, R_j]$, $1/3 \leqq t \leqq 2/3$; and $\rho_{ij}(t) = \mu_i(3/2 - 3/2\, t)$, $2/3 \leqq t \leqq 1$.

5. $\omega_{ij}, 1 \leqq i \leqq n, 0 \leqq j \leqq n, j \neq i - 1, i$. (Pull the i^{th} arc between the j^{th} and $j + 1^{\text{st}}$ arcs or around the end arcs.) Let $\omega_{ij}(t)$ be the parallel isotopy of the inclusion map $K_i \to D$ such that $\omega_{ij}(t) = \mu_i(3/2\, t)$, $0 \leqq t \leqq 1/3$; $\|\omega_{ij}(t)(P) - \omega_{ij}(t)(Q)\| = 1/2\|P - Q\|$ for any $P, Q \in K_i$, $1/3 \leqq t \leqq 2/3$; $\omega_{ij}(t)(R_i)$ moves positively with constant speed once around a circle in P_{zy} whose diameter is $[R_i, (0, 0, 8j + 4)]$, $1/3 \leqq t \leqq 2/3$; and $\omega_{ij}(t) = \mu(3/2 - 3/2\, t)$, $2/3 \leqq t \leqq 1$.

We would like to associate with each of the previous isotopies an ambient isotopy of the pair (D, A). To justify this we state the following theorem which is proved in [3], p. 154.

THEOREM 1. (The n-isotopy extension theorem.) *Let $F\colon M \times I^n \to Q \times I^n$, M and Q P. L. manifolds, M compact, be an n-isotopy which is proper and locally unknotted. Then there exists an ambient n-isotopy H of Q with $H(F_0 \times I^n) = F$.*

Although it is not stated in [7], we may also assume that the ambient isotopy H has compact support.

If φ_i is one of the isotopies τ_i, σ_i, ρ_{ij} or ω_{ij}, then we may restrict φ_i to be an isotopy of an inclusion map $H_i \to D$ or $H_i \cup H_{i+1} \to D$. We consider φ_i as an isotopy of the set H_i (resp. $H_i \cup H_{i+1}$) in the manifold $(D - A) \cup a_i$ (resp. $(D - A) \cup a_i \cup a_{i+1}$). We now use the preceding theorem to obtain an ambient isotopic extension, which we shall also call φ_i, of the manifold $(D - A) \cup a_i$ (resp. $(D - A) \cup a_i \cup a_{i+1}$). We can, by redefining if necessary, arrange that the ambient extension agree with the original isotopy on the set K_i (or $K_i \cup K_{i+1}$). Since the ambient extension has compact support, we may extend it further to all of D. We shall also denote the extension of φ_i to D by φ_i, and the homeomorphism of (D, A) it induces by the same Greek letter. (i.e. The isotopy $\rho_{ij}(t)$ extends to the ambient isotopy $\rho_{ij}(t)$ of D which induces the homeomorphism ρ_{ij} of (D, A) equal to $\rho_{ij}(1)$.) We can easily arrange, and shall assume, that the homeomorphisms ρ_{ij} and ω_{ij} are fixed in a neighborhood of A, that τ_i is fixed in a neighborhood of $A - a_i$ and is isometric in a neighborhood of a_i, that σ_i is fixed in a neighborhood of $A - (a_i \cup a_{i+1})$ and is isometric in a neighborhood of $a_i \cup a_{i+1}$, and that all the homeomorphisms ρ_{ij}, ω_{ij}, τ_i, and σ_i have compact support. Moreover, by restricting to the set $(\partial D, A \cap \partial D)$, the homeomorphisms τ_i, σ_i, ρ_{ij}, ahd ω_{ij} define homeomorphisms of $(\partial D, A \cap \partial D)$ which we continue to denote by the same letter. The next lemma shows they are well defined as elements of $\mathscr{A}_k$.

LEMMA 2. *Let M be a compact manifold and let $\varphi\colon M \times I^2 \to Q$ be a proper locally unknotted 2-isotopy such that $\varphi(m, s, 1) = \varphi(m, 1, t)$ for $0 \leqq s, t \leqq 1$ and all $m \in M$. Let α_s and β_t be ambient isotopic extensions of $\varphi(\cdot, s, 0)$ and $\varphi(\cdot, 0, t)$ respectively. Then α_1 is isotopic to β_1 via an isotopy of Q that leaves the points of the set $\varphi(M \times (1, 1))$ fixed.*

Proof. It suffices to show there is an ambient 2-isotopy of Q, $\psi\colon Q \times I^2 \to Q$ such that ψ extends φ, and

$$\psi \mid Q \times (s, 0) = \alpha_s$$
$$\psi \mid Q \times (0, t) = \beta_t\,.$$

Then $\psi \mid Q \times (1, t)$ followed by $\psi \mid Q \times (1 - s, 1)$ provides the required isotopy between α and β.

Let $H(s, t)$ be any ambient 2-isotopic extension of Q. Let $F(s, t)$ be defined as follows:

$$F(s, t) = \begin{cases} H^{-1}(s - t, 0) \circ \alpha_{s-t} & \text{if} \quad s \geqq t \\ H^{-1}(0, t - s) \circ \beta_{t-s} & \text{if} \quad t \geqq s . \end{cases}$$

Then $F(s, t)$ extends the constant isotopy of M in Q. Also $F \mid Q \times (s, 0) = H^{-1} \circ \alpha_s$ and $F \mid Q \times (0, t) = H^{-1} \circ \beta_t$. Now $\psi = H \circ F$ is the desired extension.

It follows easily from Lemma 2 that the homeomorphisms $\{\sigma_i, \tau_i, \rho_{ij}, \omega_{ij}\}$ are well defined as elements of $\mathscr{A}_n$. To see that σ_i is well defined, for example, let $\varphi(m, s, t) = \sigma_i(m, \max(s, t))$, $\alpha_s = \sigma_i'(s)$, and $\beta_t = \sigma_i''(t)$ for σ_i' and σ_i'' two different ambient isotopic extensions of σ_i and apply Lemma 2.

3. The main theorem.

Let $\mathscr{G}$ be the group of homeomorphisms consisting of all finite products of homeomorphisms on the following list and their inverses:

1. τ_i^2, $1 \leqq i \leqq n$.
2. ρ_{ij}, $1 \leqq i, j \leqq n$.
3. ω_{ij}, $1 \leqq i \leqq n$.
4. ξ_{ij}, $1 \leqq i \leqq j \leqq n$, where $\xi_{ij} = (\sigma_i \sigma_{i+1} \cdots \sigma_{j-1}) \sigma_j^2 (\sigma_i \cdots \sigma_{j-1})^{-1}$.
5. $\alpha(1)$ where $\alpha(t)$ is an A-isotopy. Any isotopy $\alpha(t)$ of D that is fixed on A and has support disjoint from $\{\infty\}$ will be called an A-isotopy. The homeomorphism $\alpha(1)$ will also be called an A-isotopy.

THEOREM 3. *If φ is a homeomorphism of D fixed on a neighborhood of A with support contained in a ball centered at* (0, 0, 0), *then* $\varphi \in \mathscr{G}$.

Proof. Suppose φ fixes the neighborhood N_1 of A. We may assume (by multiplying φ by an A-isotopy, if need be) that $\varphi(F - N_1)$ is in general position with respect to E. We wish to arrange that $\varphi(B - N_1) \cap C = \emptyset$. We shall do this by decreasing the number of points in this intersection step by step. If the set $(F - N_1) \cap \varphi^{-1}(E)$ is not empty, then it is a compact 1-manifold. Suppose $\varphi(B - N_1) \cap C \neq \emptyset$. Then for some i, $(F_i - N_1) \cap \varphi_1^{-1}(E)$ has a component that is an arc, a, (not a closed curve). The endpoints of this arc, $\{M_1, M_2\}$, define a closed subarc $[M_1, M_2]$ of b_i. We may assume that $\varphi([M_1, M_2]) \cap E = \{\varphi(M_1), \varphi(M_2)\}$. (If not, we could find another arc with this property in the disc bounded by a and $[M_1, M_2]$.) At this point we refer the reader to Figure 2. The points $\varphi(M_1)$ and $\varphi(M_2)$

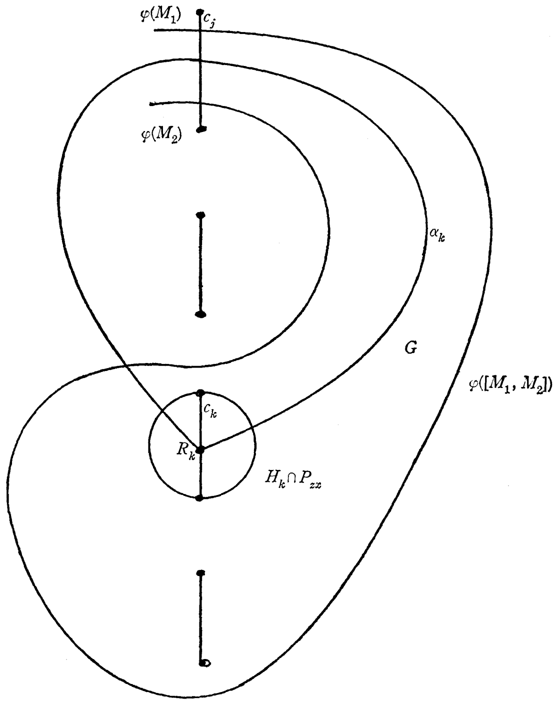

Figure 2

define a closed subarc $[\varphi(M_1), \varphi(M_2)]$ of c_j for some j. We may assume (by composing with an A-isotopy fixed on N_1 not changing the number of points in $\varphi(B - N_1) \cap C$, if necessary) that R_j belongs to the interior of this arc. If the disc G whose boundary is $[\varphi(M_1), \varphi(M_2)] \cup \varphi([M_1, M_2])$ contains none of the arcs c_k, $k = 1, \cdots, n$, $k \neq j$, it is easy to construct an A-isotopy ψ supported on a neighborhood of G such that $\varphi\psi(B - N_1) \cap C$ has two fewer points than $\varphi(B - N_1) \cup C$.

Suppose now $c_k \subset G$. We may assume (by composing with an A-isotopy fixed on C and supported on a small neighborhood of $K_k \cap P_{zy}$, if necessary) that $K_k \cap P_{zy}$ belongs to the interior of G. Let $\alpha_k(t)$, $1/3 \leqq t \leqq 2/3$ be a parameterized simple closed curve beginning and ending at R_k such that $\alpha_k(t) \cap C = \varnothing$ for all $t \neq 1/3$. $1/2$, $2/3$, such that $\alpha_k(1/2)$ lies in the interior of $[\varphi(M_1), \varphi(M_2)]$ and such that $\alpha_k(t)$ intersects ∂G only twice, the second time when $t = 7/12$. Let $\varphi(t, \varepsilon)$ be an isotopy of K_k defined by the following conditions: $\psi(t, \varepsilon)$ is a parallel isotopy, $\psi(t, \varepsilon) = \mu_k((3 - \varepsilon)t)$, $0 \leqq t \leqq 1/3$, $\psi(t, \varepsilon)(K_k)$ is a half ball of radius $2/3\varepsilon$ centered at $\alpha_k(t)$, $1/3 < t \leqq 2/3$, $\psi(t, \varepsilon) = \psi(1 - t, \varepsilon)$, $2/3 \leqq t \leqq 1$. If $0 < \varepsilon < 1$, we may, by Theorem

1, extend $\psi(t, \varepsilon)$ to an ambient isotopy of D, called $\bar{\psi}(t, \varepsilon)$ supported on a neighborhood of $K_k \cup \alpha_k[1/3, 2/3]$. By choosing ε small enough and the neighborhood of $K_k \cup \alpha_k[1/3, 2/3]$ small enough, we may assume that $\bar{\psi}\varphi(1, \varepsilon)(B - N_1) \cap (C - [\varphi(M_1), \varphi(M_2)]) = \varphi(B - N_1) \cap C - [\varphi(M_1), \varphi(M_2)]$, although new intersections of $\bar{\psi}\varphi(1, \varepsilon)(B - N_1)$ with $[\varphi(M_1), \varphi(M_2)]$ may be introduced. We can also arrange that $\bar{\psi}(1, \varepsilon)$ leaves a neighborhood, $N_2 \subset N_1$, of A fixed. Let G_1 be the disc bounded by $[\bar{\psi}\varphi(1, \varepsilon)(M_1)], \bar{\psi}\varphi(1, \varepsilon)(M_2)]$ (which is the same as $[\varphi(M_1), \varphi(M_2)]$) and $\bar{\psi}\varphi(1, \varepsilon)([M_1, M_2])$. We see that c_k is not contained in G_1, but that c_p is contained in G_1 if and only if it is contained in G. Next we must show $\bar{\psi}(1, \varepsilon)$ is in $\mathscr{G}$.

The curve $\bar{\psi}(t, \varepsilon)(R_k)$ defines an element of the grous $\pi_1((P_{zy} - C) \cup c_k \cup (c_j - \{R'_j, R''_j\}))$. This group is a free group generated by the closed curves $\omega_{kj}(t)(R_k)$, $0 \leqq j \leqq n$, $j \neq k - 1$, k and $\rho_{kj}(t)(R_k)$, and so $\bar{\psi}(t, \varepsilon)(R_k)$ is homotopic to a product of these curves, $\beta(t)$. Thus there is a map $f\colon I^2 \to (P_{zy} - C) \cup c_k \cup (c_j - \{R'_j, R''_j\})$ such that $f(0, t) = \bar{\psi}(t, \varepsilon)(R_k)$, $f(s, 0) = \beta(s)$, and $f(1, t) = f(s, 1) = R_k$ for $0 \leqq s, t \leqq 1$. There is a continuous function $\eta(s, t)$ such that $\eta(0, t) =$ radius $\bar{\psi}(t, \varepsilon)(K_k)$, $\eta(s, 0) =$ radius $\bar{\beta}(t)(K_k)$, where $\bar{\beta}(t)$ is the product of the isotopies in $\mathscr{H}$ that covers $\beta(t)$, $\eta(s, 1) = \eta(1, t) = 2$, and such that $\eta(s, t)$ is less than the distance from $f(s, t)$ to $A - \alpha_k$. Let $F(s, t)$ be the parallel 2-isotopy such that $F(s, t)(K_k)$ is a half ball of radius $\eta(s, t)$ centered at $f(s, t)$. It now follows directly from Lemma 2 that $\bar{\psi}(1, \varepsilon) \varepsilon \mathscr{G}$.

Continuing in this fashion we can find a sequence of elements in $\mathscr{G}$, $\bar{\psi}_1, \cdots, \bar{\psi}_r$ such that each one fixes a neighborhood $N_r \subset N_1$, of A, $\bar{\psi}_r \cdots \bar{\psi}_1\varphi(B - N_r) \cap C - [\varphi(M_1), \varphi(M_2)]) = \varphi(B - N_r) \cap (C - [\varphi(M_1), \varphi(M_2)])$, and the disc G_r bounded by the arcs $[\varphi(M_1), \varphi(M_2)]$ (equal $[\varphi\bar{\psi}_1 \cdots \bar{\psi}_k(M_1), \varphi\bar{\psi}_1 \cdots \bar{\psi}_k(M_2)]$) and $\varphi\bar{\psi}_1 \cdots \bar{\psi}_k[M_1, M_2]$ contains none of the arcs c_k. Now we find one last A-isotopy, $\bar{\psi}_{r+1}$, supported on a small neighborhood of G_r such that $G_r \cap C \cap \bar{\psi}_{r+1} \cdots \bar{\psi}_1\varphi(B - N_r) = \varnothing$.

Thus, letting $\tilde{\psi}_1 = \bar{\psi}_{r+1} \cdots \bar{\psi}_1$, $\tilde{\psi}_1$ is in $\mathscr{G}$, $\tilde{\psi}_1$ fixes a neighborhood N_r of A, and $\varphi(B - N_r) \cap C$ has two more points in it than $\tilde{\psi}_1\varphi(B - N_r) \cap C$. By repeating this process several times we find an element ψ in $\mathscr{G}$, that fixes a neighborhood M of A, such that $\psi\varphi(B - M) \cap C = \varnothing$.

Next we would like to arrange that $\psi\varphi$ leave the set C fixed. Let H be the disc bounded by c_1 and $\varphi\psi(b_1)$. (See Figure 3.) Suppose $H \cap (C - c_1) \neq \varnothing$. That is, suppose some c_k, $k \neq 1$, is contained in the interior of H. By a procedure analogous to the process carried out in the last two paragraphs for removing the arcs c_k from the disc G, we may find an element θ_1 fixed on a neighborhood N of A such that: $\theta_1\psi\varphi(B - N) \cap C = \varnothing$, $\theta_1 \in \mathscr{G}$, and the disc H_1 bounded by

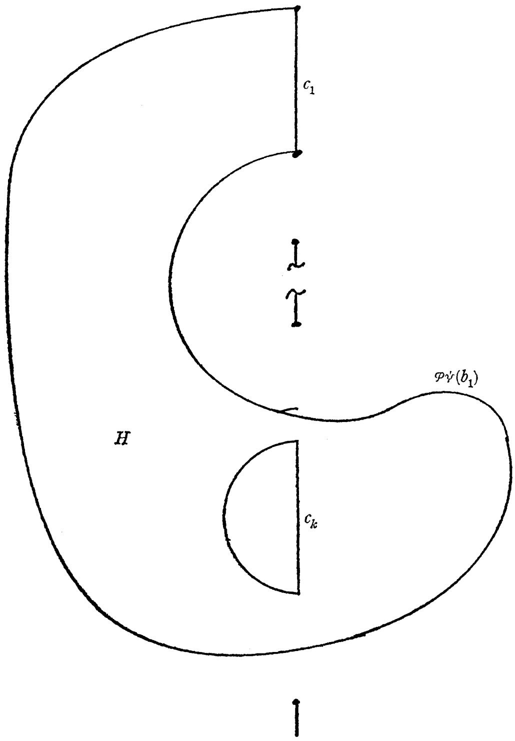

Figure 3

$\theta_1\psi\varphi(b_1)$ and c_1 satisfies $H_1 \cap (C - c_1) = \emptyset$. Since c_1 is ambient A-isotopic to b_1, $\theta_1\psi\varphi(c_1)$ is ambient A-isotopic to $\theta_1\psi\varphi(b_1)$ which is in turn ambient A-isotopic to c_1. ($\theta_1\psi\varphi(b_1)$ and c_1 bound the disc H_1.) Thus there is an A-isotopy θ_1' such that $\theta_1'\theta_1\psi\varphi$ fixes each point of c_1. Continuing in this manner we find elements of $\mathscr{G}$, $\theta_2, \cdots, \theta_n$, and $\theta_2', \cdots, \theta_n'$ such that $\theta_n'\theta_n \cdots \theta_1'\theta_1\psi\varphi$ fixes every point of C. (Note that by their construction, θ_k and θ_k' fix the points of c_j for $j < k$.) If we let $\psi_1 = \theta_n'\theta_n \cdots \theta_1'\theta_1\psi$, then $\psi_1 \in \mathscr{G}$ and $\psi_1\varphi$ fixes the points of C.

From this point to the end of the proof we shall denote the restriction of a homeomorphism of D to ∂D by placing a bar under it. $\underline{\psi_1\varphi}$ defines an element of the subgroup generated by orientation preserving homeomorphisms of the mapping class group of $P_{zy} - \{R_1, \cdots, R_n\}$. It is shown in [1] and [2] that this group is generated by $\underline{\xi_{ij}}$, $1 \leqq i, j \leqq n - 1$. Thus there is a homeomorphism ψ_2 of D that is a product of ξ_{ij}, $1 \leqq i, j \leqq n - 1$, and $\underline{\psi_2\psi_1\varphi}$ is isotopic to the identity via an isotopy fixing $R_1, \cdots, R_n$. Thus also $\psi_2\psi_1\varphi$ is isotopic to the identity via an isotopy fixing $R_1, \cdots, R_n$ (first isotopy on the boundary, then use the Alexander isotopy inside). Let θ_t be such an isotopy. Let $K(\varepsilon, k)$ be the hemisphere of radius ε centered at R_k.

If $\varepsilon > 0$ is small enough then $\theta_t(K(\varepsilon, k)) \cap H_k = \emptyset$ for $k = 1, \cdots, n$, $0 \leqq t \leqq 1$. Let $\rho(s) = \mu'_k((1-\varepsilon/2)s)\mu'_{k-1}((1-\varepsilon/2)s) \cdots \mu'_1((1-\varepsilon/2)s)$, where $\mu'_i(t)$ is an ambient isotopic extension of μ_i supported on a small neighborhood of K_i. Thus ρ pushes each half ball K_k inside $K(\varepsilon, k)$. The expression $\underline{\rho(s)}\underline{\theta}\underline{\rho}(s)$ defines an isotopy between $\underline{\theta}$ and $\underline{\rho^{-1}\theta\rho}$ in the mapping class group of $P_{zy} - \{R'_1, R''_1, R'_2, R''_2, \cdots, R''_n\}$. We note that under the isotopy $\underline{\rho^{-1}\theta_t\rho}$ the paths of R'_k and R''_k never leave $K_k \cap P_{zy}$. The mapping class group of $U(K_k \cap P_{zy}) - \{R'_k, R''_k\}$ is generated by $\underline{\tau}_k^2$ where $U(K_k \cap P_{zy})$ is a small neighborhood of $K_k \cap P_{zy}$. Thus there are integers $p_1, \cdots, p_n$ such that $\underline{\tau}_n^{2p_n} \cdots \underline{\tau}^{2p_1}[\underline{\rho^{-1}\theta\rho\theta^{-1}}]\underline{\theta}$ is isotopic to the identity via an isotopy fixing $\{R'_1, \cdots, R''_k\}$. Also θ is A-isotopic to $\rho\theta\rho^{-1}$. To see this, let

$$F(s, t) = \rho(s)^{-1}\theta_{\max(s,t)}\rho(s)$$

restricted to A and apply Lemma 2. Thus there is a homeomorphism $\psi_3 \in \mathscr{G}$ such that $\psi_3\psi_2\psi_1\varphi$ is isotopic to the identity via an isotopy ψ_4 fixing $A \cap P_{zy}$. Let $\psi_5 = \psi_4\psi_3\psi_2\psi_1$. Then $\psi_5 \in \mathscr{G}$ and $\psi_5\varphi$ fixes $A \cup \partial D$. The last step is to show $\psi_5\varphi$ is A-isotopic to 1. Let ψ_6 be an isotopy such that ψ_6 restricted to $\partial D \cup A$ is the identity, $\psi_6\psi_5\varphi(E) \cap E$ is a 1-manifold without boundary, and the number of components of $\psi_6\psi_5\varphi(E) \cap E$ is as small as possible. Then $\psi_6\psi_5\varphi(E) \cap E = \partial E$ since if C_1 were a circle in $E \cup (\psi_6\psi_5\varphi)^{-1}(E)$ not containing any other circles, then $\varphi(C_1)$ would bound a disc in the interior of E and also a disc in the interior of $\psi_6\psi_5\varphi(E)$, these two discs would form a sphere bounding a ball and we could isotopy away the intersection. Since $E_k \cup \psi_6\psi_5\varphi(E_k)$ is a sphere for each k, we can find another isotopy ψ_7 fixed on $A \cup \partial D$ such that $\psi_7\psi_6\psi_5\varphi$ is the identity on $\partial D \cup E$.

We now need only one more isotopy, ψ_8, an Alexander isotopy fixed on $\partial D \cup E$ such that $\psi_8\psi_7\psi_6\psi_5\varphi = $ identity. The homeomorphism φ equals $\psi_5^{-1}\psi_6^{-1}\psi_7^{-1}\psi_8^{-1}$, which belongs to $\mathscr{G}$ and we are done.

4. Generators for $\mathscr{A}_k$ and $\mathscr{F}_k$. Let $\mathscr{A}_k$ be the group of orientation preserving homeomorphisms of the pair (D, A) modulo the subgroup of those homeomorphisms isotopic to the identity via an isotopy leaving the set A invariant. Let $\mathscr{F}_k$ be the subgroup of $\mathscr{A}_k$ generated by homeomorphisms leaving the set A fixed pointwise.

THEOREM 4. *The group $\mathscr{F}_k$ is generated by the following set:* $\{\tau_i^2, 1 \leqq i \leqq n; \rho_{ij}, 1 \leqq i, j \leqq n; \omega_{ij}, 1 \leqq i, j \leqq n, i \neq j, j-1; \xi_{ij}, 1 \leqq i \leqq j \leqq n\}$.

Proof. If $\varphi \in \mathscr{F}_k$, we can easily find an A-isotopy ψ such that

$\psi\varphi$ fixes a neighborhood of $\{\infty\}$ and $\psi\varphi$ fixes a neighborhood of A. The theorem now follows immediately from Theorem 3.

THEOREM 5. *The group $\mathscr{A}_k$ is generated by the following set*: $\{\tau_i, 1 \leqq i \leqq n; \rho_{ij}, 1 \leqq i, j \leqq n; \omega_{ij}, 1 \leqq i, j \leqq n, i \neq j, j-1; \sigma_i, 1 \leqq i \leqq n-1\}$.

Proof. Let φ belong to $\mathscr{A}_k$. The homeomorphism φ permutes the components of A and $\sigma_i, 1 \leqq i \leqq n-1$, generates the set of permutations of the components of A. Thus there is an element ψ_1 which is a product of the $\sigma_i, 1 \leqq i \leqq n-1$, such that $\psi_1\varphi$ leaves each component of A fixed as a set. The homeomorphism $\psi_1\varphi$ may reverse the orientation of various components of A. Since the homeomorphism τ_i reverses the orientation of the i^{th} component of A while leaving the other component fixed, there is a homeomorphism ψ_2 that is a product of $\tau_i, 1 \leqq i \leqq n$, such that $\psi_2\psi_1\varphi$ leaves each component of A fixed and preserves its orientation. Since each component of A is an arc, it is easy to see that there is an isotopy ψ_3 of D, leaving the set A fixed as a set and supported on a neighborhood of A such that $\psi_3\psi_2\psi_1\varphi$ leaves the set A fixed pointwise. The proof of Theorem 5 now follows immediately from Theorem 4 and the observation that each ξ_{ij} is a product of $\sigma_i, 1 \leqq i \leqq n-1$.

It is perhaps worthwhile to remak that the sets of generators in Theorems 4 and 5 can be made much smaller by utilizing the relations in the mapping class group. For example

$$\tau_i = (\sigma_{i-1}\sigma_{1-2} \cdots \sigma_1)\tau_1(\sigma_{i-1} \cdots \sigma_1)^{-1}$$

so $\tau_2, \cdots, \tau_n$ are unnecessary in Theorem 5.

5. Concluding remarks. In [4] D. M. Dahm finds generators for the group of motions of n unlinked, unknotted circles C_n in E^3. This group is quite close to the group of orientation preserving homeomorphisms of (E^3, C_n) modulo those isotopic to the identity via an isotopy fixing C_n as a set. There are three types of generators:

1. flip the i^{th} circle;
2. pull the i^{th} circle through the j^{th} circle;
3. exchange the i^{th} and j^{th} circles.

It can be seen that these are analogous to the generators of $\mathscr{A}_k$ in Theorem 5. Dahm proves that these elements generate the motion group by defining an isomorphism from the motion group to a subgroup of $\operatorname{Aut}(\pi_1(E^3 - C_n))$. ($\pi_1(E^3 - C_n)$ is free on n generators). This method doesn't generalize to a proof of Theorem 5 as nontrivial elements of $\mathscr{A}_k$ can be constructed that induce the identity automorphism on $\pi_1(D - A)$.

In [6] Deborah L. Goldsmith proves several theorems about the group of motions of a link in S^3. She constructs certain examples of motions of a link. Theorem 5 can be used to construct examples of motions of links in the following way: Let ℓ be a k-bridge link, let A be defined as in § 2 with k arcs, let (D', A') be another copy of (D, A), and let i be the map that identifies D with D' restricted to ∂D. Then $(S^3, \ell) = (D, A) \cup_\alpha (D', A')$ for some $\alpha \in \mathscr{B}_{2k}$, the $2k$ string braid group of the sphere. Let $\mathscr{N}$ be the group $A_k \cap \alpha \mathscr{A}_k \alpha^{-1}$. Each element of m of $\mathscr{N}$ induces a homeomorphism θ_m of (S^3, ℓ) and therefore a motion of the link ℓ by the following formula: $\theta_m(x) = \bar{m}(x)$ if $x \in D$, $\theta_m(x) = \overline{i^{-1}\alpha^{-1} m \alpha i(x)}$ if $x \in D'$, where the bar indicates the homeomorphism is an extension from $(\partial D, \partial D \cap A)$ (or $(\partial D', \partial D' \cap A')$) to (D, A) (or (D', A')).

A plat presentation of a link ℓ, $(S^3, \ell) = (D, A) \bigcup_\alpha (D', A')$, can be thought of as a Heegaard splitting of the link. Equivalence classes of Heegaard splittings of links then correspond in a natural way to double cosets of $\mathscr{A}_k$ in $\mathscr{M}_{2k}$.

In [3], it is shown that there is a natural connection between Heegaard splittings of 3-manifolds and Heegaard splittings of knots and links. In particular (Theorem 8 of [3]) there is a one-to-one correspondence beetween equivalence classes of Heegaard splittings of genus two closed orientable 3-manifolds and double cosets of $\mathscr{A}_3$ in $\mathscr{M}_6$.

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