

# Pacific Journal of Mathematics

## **SOME MATRIX TRANSFORMATIONS ON ANALYTIC SEQUENCE SPACES**

**TOM (ROY THOMAS JR.) JACOB**

## SOME MATRIX TRANSFORMATIONS ON ANALYTIC SEQUENCE SPACES

ROY T. JACOB, JR.

Let  $A$  denote the space of all complex sequences  $a$  such that if  $z$  is a complex number and  $|z| < 1$  then  $\sum a_n z^n$  converges, and  $B$  the space of all complex sequences  $b$  for which there is a complex number  $z$  such that  $|z| > 1$  and  $\sum b_n z^n$  converges. In this paper we characterize matrix transformations from  $A$  to  $B$  and from  $B$  to  $A$ .

M. G. Haplanov [1] has described the matrix transformations from  $A$  to  $A$ , and P. C. Tonne [4] those from  $A$  to the bounded sequences, the convergent sequences and  $l$ .

A sequence space is a linear space each point of which is an infinite complex sequence. If  $\lambda$  is a sequence space, then  $\lambda^*$ , the dual of  $\lambda$ , is the collection of all infinite complex sequences  $y$  such that  $\sum |x_n y_n|$  converges for every  $x$  in  $\lambda$ . For each  $\lambda$  a dual system with  $\lambda^*$  is formed using the bilinear functional

$$Q(x, y) = \sum_{n=0}^{\infty} x_n y_n,$$

where  $x$  is in  $\lambda$  and  $y$  is in  $\lambda^*$ . Under this duality,  $\lambda$  is provided with the standard weak topology.

Theorem A is a classic result of Köthe and Toeplitz [2]:

**THEOREM A.** *Suppose  $\lambda$  is a sequence space such that  $\lambda = \lambda^{**}$ . In order that a linear transformation from  $\lambda$  to a sequence space be weakly continuous, it is necessary and sufficient that it be a matrix transformation.*

In [3] O. Toeplitz studied the topological properties of the spaces  $A$  and  $B$ . The following theorem is a summary of his basic results:

**THEOREM B.** (1)  $A^* = B$  and  $B^* = A$ .

(2) A point set  $M$  is bounded in  $A$  [ $B$ ] if and only if there exists a point  $y$  of  $A$  [ $B$ ] such that  $|x_n| < y_n$  whenever  $x$  is a point of  $M$  and  $n$  is a nonnegative integer.

(3) A point sequence is convergent in  $A$  [ $B$ ] if and only if it is bounded in  $A$  [ $B$ ] and coordinatewise convergent.

**THEOREM 1.** *If  $M$  is an infinite matrix then the following are*

equivalent:

(1)  $M$  throws  $A$  into  $B$ .

(2) Each row and each column of  $M$  is in  $B$ , and there exist numbers  $t$  and  $r$  such that  $0 < r < 1$  and  $|M_{jk}| \leq tr^{j+k}$  whenever each of  $j$  and  $k$  is a nonnegative integer.

*Proof.* (1)  $\rightarrow$  (2). Suppose statement (1) is true and statement (2) is not. In that case there exist increasing sequences  $j_0, j_1, j_2, \dots$  and  $k_0, k_1, k_2, \dots$  of positive integers such that if  $n$  is a nonnegative integer, then

$$|M_{j_n, k_n}| > \left( \frac{n}{n+1} \right)^{j_n + k_n},$$

and either (i)  $j_n \leq k_n$  for each nonnegative integer  $n$  or (ii)  $k_n \leq j_n$  for each nonnegative integer  $n$ .

Suppose case (i) holds. For each nonnegative integer  $n$ , let  $c_n$  denote a complex number such that

$$|c_n| = \frac{1}{|M_{j_n, k_n}|} \quad \text{and} \quad \left| \sum_{i=0}^n M_{j_n, k_i} c_i \right| \geq 1.$$

Each  $c_n$  has the property that  $|c_n| < (1 + 1/n)^{2k_n}$ .

For each nonnegative integer  $n$ , let  $\xi_n$  denote the point of  $A$  such that for each nonnegative integer  $m$ ,  $\xi_{nm} = c_i$  whenever there is an integer  $i$  such that  $0 \leq i \leq n$  and  $m = k_i$ , and  $\xi_{nm} = 0$  otherwise.

The point sequence  $\xi$  is bounded in  $A$ , so  $M(\xi)$  is bounded in  $B$ . However, for each positive integer  $n$ ,

$$\begin{aligned} |(M\xi_n)_{j_n}| &= \left| \sum_{i=0}^{k_n} M_{j_n, i} \xi_{ni} \right| \\ &= \left| \sum_{i=0}^n M_{j_n, k_i} c_i \right| \\ &\geq 1. \end{aligned}$$

This is a contradiction.

In case condition (ii) holds,  $M'$  is a matrix that throws  $A$  into  $B$  and satisfies condition (i). This is also a contradiction.

(2)  $\rightarrow$  (1). If  $x$  is a point of  $A$  and  $j$  is a nonnegative integer, then

$$\begin{aligned} |(Mx)_j| &= \left| \sum_{k=0}^{\infty} M_{jk} x_k \right| \\ &\leq tr^j \sum_{k=0}^{\infty} r^k |x_k|. \end{aligned}$$

Consequently,  $\limsup_j |(Mx)_j|^{1/j} \leq r$ , and  $Mx$  is a point of  $B$ .

**THEOREM 2.** *If  $M$  is an infinite matrix then the following are equivalent:*

(1)  $M$  throws  $B$  into  $A$ .

(2) *Each row and each column of  $M$  is in  $A$ , and if  $\varepsilon > 0$  there is a positive integer  $m$  such that  $|M_{jk}|^{1/(j+k)} < 1 + \varepsilon$  whenever each of  $j$  and  $k$  is a nonnegative integer and  $j + k \geq m$ .*

*Proof.* (1)  $\rightarrow$  (2). Suppose statement (1) is true and statement (2) is not. In that case, there exist a positive number  $\varepsilon$  and infinitely many nonnegative-integer pairs  $(j, k)$  such that  $|M_{jk}|^{1/(j+k)} > 1 + \varepsilon$ .

*Case (i).* Suppose there exist infinitely many such integer pairs such that  $j \leq k$ . Let  $r$  denote a number such that  $(1 + \varepsilon)r > 1 + \varepsilon/2$ . Let  $(j_0, k_0)$  denote a nonnegative integer pair such that

$$j_0 \leq k_0 \quad \text{and} \quad |M_{j_0, k_0}|^{1/(j_0+k_0)} > 1 + \varepsilon.$$

Let  $c_0 = r$ . Then

$$c_0 |M_{j_0, k_0}|^{1/k_0} \geq c_0 |M_{j_0, k_0}|^{1/(j_0+k_0)} > (1 + \varepsilon)r > 1 + \frac{\varepsilon}{2}.$$

Let  $(j_1, k_1)$  denote a nonnegative integer pair such that  $j_1 \leq k_1$ ,  $j_0 < j_1$ ,  $k_0 < k_1$ , and

$$\sum_{i=k_1}^{\infty} |M_{j_0, i}| r^i < [(1 + \varepsilon)r]^{k_0} - \left[1 + \frac{2}{\varepsilon}\right]^{j_0}.$$

Let  $c_1$  denote a complex number such that  $|c_1| = r$  and

$$|M_{j_1, k_1} c_1^{k_1}| \leq |M_{j_1, k_0} c_0^{k_0} + M_{j_1, k_1} c_1^{k_1}|.$$

Continue this process in the following way: For each positive integer  $n$ , after choosing  $j_n, k_n$ , and  $c_n$ , let  $(j_{n+1}, k_{n+1})$  denote a nonnegative integer pair such that  $j_{n+1} \leq k_{n+1}$ ,  $j_n < j_{n+1}$ ,  $k_n < k_{n+1}$ , and

$$\sum_{i=k_{n+1}}^{\infty} |M_{j_n, i}| r^i < [(1 + \varepsilon)r]^{k_n} - \left[1 + \frac{\varepsilon}{2}\right]^{j_n},$$

and then let  $c_{n+1}$  denote a complex number such that  $|c_{n+1}| = r$  and

$$|(c_{n+1})^{k_{n+1}} (M_{j_{n+1}, k_{n+1}})| \leq \left| \sum_{i=1}^{n+1} c_i^{k_i} (M_{j_{n+1}, k_i}) \right|.$$

Now, for each nonnegative integer  $n$ , let  $\xi_n$  denote the point of  $B$  such that for each nonnegative integer  $m$ ,

$$\xi_{nm} = c_i^{k_i}$$

whenever there is an integer  $i$  such that  $0 \leq i \leq n$  and  $m = k_i$ , and

$$\hat{\xi}_{nm} = 0$$

otherwise.

The point sequence  $\xi$  is bounded in  $B$ , so  $M(\xi)$  is bounded in  $A$ . However, for each positive integer  $n$ ,

$$\begin{aligned} |(M_{\xi_n})_{j_n}| &= \left| \sum_{i=0}^n c_i^{k_i}(M_{j_n, k_i}) + \sum_{i=n+1}^{\infty} c_i^{k_i}(M_{j_n, k_i}) \right| \\ &\geq \left| \sum_{i=0}^n c_i^{k_i}(M_{j_n, k_i}) \right| - \sum_{i=n+1}^{\infty} |c_i^{k_i}(M_{j_n, k_i})| \\ &\geq \left[ 1 + \frac{\varepsilon}{2} \right]^{j_n}. \end{aligned}$$

Consequently,  $|(M_{\xi_n})_{j_n}|^{1/j_n} \geq 1 + \varepsilon/2$ . This contradicts the fact that  $M(\xi)$  is a bounded subset of  $A$ .

*Case (ii).* Suppose there exist infinitely many such integer pairs  $(j, k)$  such that  $j \geq k$ . In that case,  $M'$  throws  $B$  into  $A$  and satisfies the assumption of case (i). This is also a contradiction.

(2)  $\rightarrow$  (1). Suppose  $x$  is a point of  $B$ . Let  $t$  and  $r$  denote numbers such that  $0 < r < 1$  and  $|x_n| \leq tr^n$  for each nonnegative integer  $n$ . If  $\varepsilon$  is a positive number so small that  $(1 + \varepsilon)r < 1$ , and  $m$  is a positive integer such that  $|M_{jk}|^{1/(j+k)} < 1 + \varepsilon$  whenever each of  $j$  and  $k$  is a nonnegative integer and  $j + k \geq m$ , then for each nonnegative integer  $p$ ,

$$\begin{aligned} \left| \sum_{k=0}^{\infty} M_{m+p, k} x_k \right| &\leq \sum_{k=0}^{\infty} |M_{m+p, k}| t r^k \\ &\leq (1 + \varepsilon)^{m+p} \frac{t}{1 - (1 + \varepsilon)r}. \end{aligned}$$

Therefore,

$$\limsup_p \left| \sum_{k=0}^{\infty} M_{m+p, k} x_k \right|^{1/(m+p)} \leq 1 + \varepsilon.$$

It follows that

$$\limsup_j \left| \sum_{k=0}^{\infty} M_{jk} x_k \right|^{1/j} \leq 1,$$

that is,  $Mx$  is a point of  $A$ .

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