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K-SPACES, THEIR ANTISPACES AND RELATED MAPPINGS

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The purpose of this paper is to investigate various properties of a mapping between spaces X and Y by relating them to properties of the corresponding mapping between the antispace of X and that of Y. The particular properties discussed include "closed", "proper", "perfect", "compact" "reflexive compact" and "compact trace". In general the context is that of k-spaces.

Throughout the paper X and Y are topological spaces in which each compact set closed, $f: X \rightarrow Y$ is a (not necessarily continuous) surjection between the spaces X and Y and $f^*: X^* \rightarrow Y^*$ is the corresponding surjection between the antispace of X and the antispace of Y.

1. Antispaces.

All of the results in this section are due to J. deGroot and appear, in essence, in [5].

Suppose X is a topological space. The collection of all compact sets in X is closed under the formation of finite unions and arbitrary intersections. Consequently this collection together with the set X itself can be taken as the closed sets for a new (weaker) T_1 topology on X, denoted X^*, and called the *antispace* of X. Furthermore each closed subset of X is compact in X^* and, provided X is a *k-space* (a T_2 space in which a set is closed if and only if it has compact intersection with each compact set), each compact subset of X^* is in turn closed in X. The relation between the closed and the compact sets in X and in X^* is indicated in the following diagram.

$$\begin{array}{ccc} X & \Longrightarrow & X^* \\ \text{closed} & \xLeftarrow{\text{if } X \text{ is a } k\text{-space}} & \text{compact} \\ \Downarrow & & \Downarrow \\ \text{compact} & \xLeftarrow{\text{if a proper subset}} & \text{closed} \\ & \Longrightarrow & \end{array}$$

If X is compact, X and its antispace X^* coincide. If X is noncompact, X^* is a compact, connected, locally connected, T_1, non-T_2 space in which each nonempty open set is both dense and connected.

The concept of an antispace was first introduced by J. deGroot in [5]. For papers in related topics see [6], [7] and [8].

The concept of a k-space was first introduced by R. Arens in

[1]. The category of k-spaces includes all first countable T_2 spaces, spaces complete in the sense of Čech (G_δ subsets of compact T_2 spaces) and all quotient spaces of locally compact T_2 spaces. In fact Cohen [4] has shown that the latter category is all k-spaces. For further discussion the reader is referred to [2, 3, 11, 12, 13, 14, 15 and 17].

2. **Mappings.** f: $X \rightarrow Y$ is *proper* if $f^{-1}(K)$ is compact in X whenever K is compact in Y. It is easy to show that a 1-1, closed (open) mapping is proper as is a continuous mapping from a compact space onto a T_2 space. On the other hand, a proper mapping from a T_2 space onto a compact space in continuous. f is *perfect* if it is continuous, closed and has compact point inverses. f is *quotient* if a set C in Y is closed if and only if $f^{-1}(C)$ is closed in X. A continuous closed (open) mapping is quotient. Finally f is *compact* if $f(K)$ is compact in Y whenever K is compact in X. Clearly, a continuous mapping is compact.

THEOREM 1. *Let $f: X \rightarrow Y$ be surjective and $f^*: X^* \rightarrow Y^*$ the corresponding map of the antispaces.*

1. *If f is proper then f^* is continuous.*
2. *Let Y be T_2. If f^* is continuous and nonconstant then f is proper.*
3. *Let Y be a k-space. If f is continuous then f^* in proper.*
4. *Let X be a k-space. If f^* is proper then f is continuous.*
5. *Let X be a k-space. If f is closed then f^* is compact.*
6. *Let Y be a k-space. If f^* is compact then f is closed.*
7. *If f is compact then f^* is closed.*
8. *If f^* is closed then for each compact $C \subset X$, $f(C)$ is either compact or equal to Y.*

Proof. All parts (except possibly 2) are straightforward applications of the results of §1. For a proof of 2 suppose f^* is continuous and nonconstant and C is a compact set in Y. If $C \neq Y$ then C is closed in Y^*, $f^{-1}(C)$ is a proper closed subset of X^* and hence $f^{-1}(C)$ is compact in X. If $C = Y(Y$ is compact and hence $Y = Y^*)$ then $X = f^{-1}(Y)$ is also compact by the following argument: Let y_1 and y_2 be distinct elements of $Y = Y^*$. Since Y is T_2 there exists disjoint open sets U and V such that $y_1 \in U$ and $y_2 \in V$. Then $f^{*-1}(V)$ and $f^{*-1}(U)$ are nonempty open sets in X^* and hence from §1 X is compact.

Parts 1 and 6 of Theorem 1 yield the following result of Arhangelski in [2].

THEOREM 2. *Let Y be a k-space. If $f: X \to Y$ is proper then f is closed.*

If in Theorem 2 the condition "closed" is augmented by assuming the function to have compact point inverses as well, then the k-space condition on Y can be dropped to yield the following converse.

THEOREM 3. *If $f: X \to Y$ is closed and has compact point inverses then f is proper.*

Proof. Let A be a compact set in Y, let $B = f^{-1}(A)$ and suppose $\{C_\alpha \cap B\}$ is a collection of sets having the finite intersection property where each C_α is closed in X. Let $\{D_\alpha\}$ be the collection of all finite intersections of C_α's. Clearly the collection $\{D_\alpha \cap B\}$ has the finite intersection property, as does the collection $\{f(D_\alpha) \cap A\}$. Moreover each $f(D_\alpha)$ is closed in Y. Since A is compact, there exists a point $y \in \bigcap_\alpha (f(D_\alpha) \cap A)$. Now $f^{-1}(y)$ is compact and the collection $\{f^{-1}(y) \cap C_\alpha\}$ has the finite intersection property. Finally, since $f^{-1}(y) \subset B$, the collection $\{f^{-1}(y) \cap C_\alpha \cap B\} = \{f^{-1}(y) \cap C_\alpha\}$ and since $f^{-1}(y)$ is compact, the collection $\{f^{-1}(y) \cap C_\alpha \cap B\}$ has nonempty intersection. Therefore $\{C_\alpha \cap B\}$ has nonempty intersection. Hence $f^{-1}(A)$ is compact.

COROLLARY. *Let Y be a k-space. $f: X \to Y$ is proper if and only if f is closed and has compact point inverses.*

Proof. The proof is an immediate consequence of Theorems 2 and 3.

Notice that Theorem 3 implies the well-known fact that a perfect mapping is proper. Conversely, the corollary shows that a continuous, proper mapping onto a k-space is perfect. In this latter result the "continuity" condition cannot be weakened to "compact" for if X is a T_2 space which is not a k-space then the identity map from X onto $k(X)$[1] is compact and proper (X and $k(X)$ have the same compact sets) but it is not continuous (and hence not perfect).

THEOREM 4. *Let X be a k-space. $f: X \to Y$ is continuous if and only if f is compact and has closed point inverses.*

Proof. The corresponding function f^* is closed and has compact

[1] If X is a T_2 space, the k-extension of X, denoted $k(X)$, is the set X with the following topology: a set in $k(X)$ is closed if it has closed intersection in X with each compact set in X. $k(X)$ is a k-space whose topology is stronger than that of X Moreover, $k(X)$ and X have the same compact sets, and $k(X) = X$ if and only if X is a k-space.

point inverses. Thus it is proper. Since X is a k-space the original function is continuous (part 4, Theorem 1).

It is an easy consequence of Theorem 4 that a function from a k-space is perfect if and only if it is compact, closed and has compact point inverses.

THEOREM 5. *Let X and Y be k-spaces. A compact $f: X \to Y$ is perfect if and only if f is proper.*

Proof. Suppose f is compact and proper. By Theorem 2 f is closed and by Theorem 4 f is continuous. Hence f is perfect.

In Theorem 5 the k-space condition on Y cannot be weakened to T_2 for if Y is a T_2 space which is not a k-space, the identity mapping from $k(Y)$ onto Y is compact and proper but it is not closed (and hence not perfect).

COROLLARY 1. *Suppose X is T_2 and Y is compact. $f: X \to Y$ is perfect if and only if f is closed and has compact point inverses.*

Proof. If f is closed and has compact point inverses then since X is T_2 so also is Y ([10] p. 235). Hence Y is a k-space. Since f is proper (corollary to Theorem 3), X is compact T_2 and hence also a k-space. Finally, since f is compact the result follows from Theorem 5.

COROLLARY 2. *Let X be a k-space and let Y be T_2. If $f: X \to Y$ is one-to-one, compact and proper, then X is homeomorphic to $k(X)$.*

Proof. The composition of f followed by the identity mapping from Y onto $k(Y)$ is a 1-1, compact, proper mapping from X onto $k(Y)$ and (since X and $k(Y)$ are k-spaces) hence by Theorem 5 a 1-1, perfect mapping (hence a homeomorphism).

Clearly if X is a T_2 space then X is a 1-1, compact, proper image of $k(X)$. The above corollary says, in fact, that each T_2 space is a 1-1 compact, proper image of one and only one k-space. An immediate consequence is that a T_2 space is a k-space if and only if it is not a 1-1, compact proper image of another k-space.

The collection of all T_2 spaces can be partitioned into mutually disjoint classes such that each class contains a unique k-space. Morever, each k-space is maximal among the spaces of the class in the sense that each is a 1-1, compact, proper image of it.

There are numerous references in the literature to two properties of functions closely related to those considered thus far in this section. For the sake of completeness these are discussed in the next few theorems.

A function $f: X \to Y$ has *compact trace* if for each compact set K in Y there exists a compact set L in X such that $f(L) = K$. Clearly every proper mapping has compact trace. A function f is *reflexive compact* if whenever K is compact in Y, then $f^{-1}(f(K))$ is compact in X. Reflexive compactness of a mapping together with its having compact trace implies "proper": Suppose K is a compact set in Y. By the compact trace property there exists a compact set L in X such that $f(L) = K$. By reflexive compactness $f^{-1}(K) = f^{-1}(f(L))$ is compact. The converse is true provided the mapping is compact; that is, a compact proper mapping is both reflexive compact and has compact trace. These observations together with Theorem 5 yield.

THEOREM 6. *Suppose X and Y are k-spaces. $f: X \to Y$ is perfect if and only if it is compact, reflexive compact and has compact trace.*

In [9] Duda shows that a reflexive compact, quotient mapping from a k-space onto a T_2 space is proper. In view of Theorems 4, 5 and 11 this can be strengthened to the statement

THEOREM 7. *Suppose X is a k-space and Y is T_2. $f: X \to Y$ is perfect if and only if it is reflexive compact and quotient.*

Duda shows also that for a function on a locally compact T_2 space having compact and connected point inverses implies reflexive compactness. This yields the following corollary to the above theorem.

COROLLARY. *Let X be locally compact and T_2, and let Y be T_2. If $f: X \to Y$ is a quotient mapping having compact and connected point inverses then f is perfect.*

THEOREM 8. *Let Y be a k-space. If $f: X \to Y$ is a continuous mapping having compact trace then f is quotient.*

Proof. Suppose $f^{-1}(C)$ is closed. In order to show that C is closed in Y it suffices to show that $C \cap K$ is compact for an arbitrary compact set K. Now there exists a compact set L in X such that $f(L) = K$. $f^{-1}(C) \cap L$ is compact in X and hence so is $f(f^{-1}(C) \cap L) = C \cap f(L) = C \cap K$.

In [18] Whyburn proves that a continuous function from a locally compact separable metric space onto a separable metric space which has compact trace and compact, connected point inverses is proper. The corollary to Theorem 7 together with Theorem 8 yields the following improvement of Whyburn's result.

THEOREM 9. *Let X be locally compact and T_2, and let Y be a k-space. If $f: X \rightarrow Y$ is continuous and has compact trace and compact connected point inverses then f is perfect.*

The connectedness criterion in the above theorem is necessary by the following example: $f: [0, 2] \rightarrow [0, 1]$ defined by $f(x) = x$ for $x \in [0, 1]$ and $f(x) = 2 - x$ for $x \in [1, 2]$ is continuous, has compact point inverses and has compact trace but it is not perfect.

As a final comment on the compact trace property, Michael has shown in [13] that a closed, continuous mapping from a paracompact space has compact trace.

The final two results give conditions on a function sufficient to insure that the image (respectively inverse-image) of a k-space is again a k-space. The first is well-known and appears in [10]. The second is due to Arhangelski and appears in [2].

THEOREM 10. *Let X be a k-space and let Y be T_2. If $f: X \rightarrow Y$ is a quotient mapping then Y is a k-space.*

Theorem 10 is not true for 1-1, continuous, proper maps for if Y is a T_2 space which is not a k-space then the identity mapping from $k(Y)$ onto Y is 1-1 continuous and proper.

THEOREM 11. *Let X be T_2 and let Y be a k-space. If $f: X \rightarrow Y$ is continuous and proper, then X is a k-space.*

Theorem 11 is not true for continuous, closed, open mappings for if X is an arbitrary non-k-space and Y a one point space, the constant mapping from X onto Y is continuous, closed and open, Y is a k-space but X is not. Neither can the continuity condition be weakened to compactness for if X is a T_2 space which is not a k-space the identity mapping from X onto $k(Y)$ is compact and proper.

3. Convergence. In this section the properties "compact", "closed", "proper" and "perfect" of a mapping are described in terms of the convergence of sequences and filterbases.

THEOREM 12. *Let X and Y be second countable T_2 spaces. A con-*

tinuous $f: X \to Y$ *is perfect if and only if whenever* $\{x_n\}$ *has no cluster points in* X, $\{f(x_n)\}$ *has none in* Y.

Proof. Only if. Suppose $\{f(x_n)\}$ has a cluster point y_0. There exists a subsequence $\{f(y_n)\}$ which converges to y_0. Now $A = \{f(y_n)\} \cup \{y_0\}$ is compact and thus (since f if proper) $f^{-1}(A)$ is compact and contains $\{y_n\}$. Thus $\{y_n\}$, and hence $\{x_n\}$, has a cluster point. If. In order to show that f is perfect it suffices, by Theorem 5, to show that f is proper. Assume A is compact in Y and that $C = f^{-1}(A)$ is not compact in X. Then there exists a sequence $\{x_n\}$ in C having no cluster points. Hence $\{f(x_n)\}$ has no cluster points in A. Contradiction.

COROLLARY. *A continuous mapping on the real line is perfect if and only if* $\operatorname{limit}_{x \to \pm\infty} f(x)$ *is either* $+\infty$ *or* $-\infty$.

In [18] Whyburn gives the following characterization of perfect mappings.

THEOREM 13. *A continuous mapping on the real line is perfect if and only if it has compact point inverses.*

THEOREM 14. *Suppose* X *and* Y *are second countable* T_2 *spaces.* f *is proper if and only if whenever* $\{f(x_n)\}$ *has a cluster point* y_0 *in* Y *then* $\{x_n\}$ *has a cluster point in* X *and if* $\{x_n\}$ *has only one cluster point* x_0, $f(x_0) = y_0$.

Proof. Only if. Suppose $\{f(x_n)\}$ has a cluster point y_0. Then there exists a subsequence $\{f(y_n)\}$ which converges to y_0. Now $A = \{f(y_n)\} \cup \{y_0\}$ is compact and thus $f^{-1}(A)$ is compact and contains $\{y_n\}$. Thus $\{y_n\}$, and hence $\{x_n\}$, has a cluster point x_0. Suppose $\{x_n\}$ has only x_0 as a cluster point. If $f(x_0) \neq y_0$ there exists a subsequence $\{f(y_n)\}$ of $\{f(x_n)\}$ which converges to y_0 and such that $f(y_n) \neq y_0$ for each n. Now $A = \{f(y_n))\} \cup \{y_0\}$ is compact and hence $f^{-1}(A)$ is compact. Furthermore $\{y_n\} \subset f^{-1}(A)$ and thus must have a cluster point in $f^{-1}(A)$. Since $x_0 \notin f^{-1}(A)$, $\{y_n\}$ and hence $\{x_n\}$ has a cluster point other than x_0. Contradiction. If. Suppose A is compact in Y, and $\{x_n\}$ is a sequence in $f^{-1}(A)$. $\{x_n\}$ has a cluster point x_0 in X. (Otherwise $\{f(x_n)\}$ has no cluster points in A, contradiction.) Thus there exists a subsequence $\{y_n\} \to x_0$. Now since $\{f(y_n)\}$ has a cluster point in A it must be $f(x_0)$. Therefore $x_0 \in f^{-1}(A)$. Therefore $f^{-1}(A)$ is compact.

LEMMA. *Suppose* X *is a locally compact* T_2 *space. A filterbase* $\{B_\alpha\}$ *converges to* x_0 *in* X^* *if and only if* $\{B_\alpha\}$ *has at most* x_0 *as a* *cluster point in X.*

Proof. Suppose $\{B_\alpha\}$ has at most x_0 as a cluster point in X (equivalently $\bigcap_\alpha \bar{B}_\alpha$ contains at most x_0). If $\{B_\alpha\}$ does not converge to x_0 in X^* then there exists a set U, open in X^*, containing x_0, such that $B_\alpha \cap U^c \neq \varnothing$ for every α. Since $\{B_\alpha\}$ is a filterbase $\{\bar{B}_\alpha \cap U^c\}$ has the finite intersection property and since U^c is compact in X, $\bigcap_\alpha \bar{B}_\alpha \cap U^c \neq \varnothing$. Contradiction. Only if. Suppose $\{B_\alpha\}$ has a cluster point y_0 in X in addition to perhaps x_0. Since X is locally compact T_2 there exists a set U, open in X, containing x_0, such that $y_0 \notin \bar{U}$ and such that $\bar{U}$ is compact. Thus $\bar{U}^c$ contains y_0 and is open in X^* (and hence in X) and since y_0 is a cluster point of $\{B_\alpha\}$ in X, $B_\alpha \cap \bar{U}^c \neq \varnothing$ for each α. Therefore no B_α is contained completely in U which implies that $\{B_\alpha\}$ does not converge to x_0 in X^*.

THEOREM 15. *Let X and Y be locally compact T_2 spaces. A nonconstant $f\colon X \to Y$ is proper if and only if whenever $\{B_\alpha\}$ is a filterbase in X with at most x_0 as a cluster point then $\{f(B_\alpha)\}$ is a filterbase in Y with at most $f(x_0)$ as a cluster point.*

Proof. From Theorem 1 f is proper if and only if f^* is continuous. This together with the above lemma and the standard characterization of continuity with respect to filterbases proves the theorem.

THEOREM 16. *$f\colon X \to Y$ is compact if whenever x_0 is a cluster point of a filterbase $\{B_\alpha\}$ in X then $f(x_0)$ is a cluster point of $\{f(B_\alpha)\}$ in Y.*

Proof. Suppose A is compact in X and let $\mathscr{B}$ be a filterbase in $f(A)$. Now $f^{-1}(\mathscr{B}) \cap A = \{f^{-1}(D) \cap A\colon D \in \mathscr{B}\}$ is a filterbase in A and hence has a cluster point x_0 in A, since A is compact. Therefore $f(x_0)$ is a cluster point of $f(f^{-1}(\mathscr{B}) \cap A) = \{f(f^{-1}(D) \cap A)\colon D \in \mathscr{B}\} = \{D \cap f(A)\colon D \in \mathscr{B}\} = \mathscr{B}$. Hence $f(A)$ is compact.

THEOREM 17. *Suppose X and Y are locally compact T_2. $f\colon X \to Y$ is closed if whenever a filterbase $\{B_\alpha\}$ has at most x_0 as a cluster point in X, then $\{f(B_\alpha)\}$ has a subordinate filterbase with at most $f(x_0)$ as a cluster point in Y.*

Proof. By the lemma the hypothesis is equivalent to the following statement in X^* and Y^*: whenever a filterbase $\{B_\alpha\}$ converges to x_0 in X^*, then $f(x_0)$ is a cluster point of $\{f(B_\alpha)\}$ in Y^*. Hence by Theorem 16 f^* is compact. The result follows by part 6, Theorem 1.

It is immediate from Theorem 15 that a proper mapping satisfies the hypothesis of Theorem 17. Consequently this condition is weaker than "proper" and stronger than "closed". The following counter example shows in fact, that the condition is strictly stronger: Consider $f: R \to \{0, 1\}$ defined by $f(x) = \begin{cases} 0 \text{ if } x \leqq 0 \\ 1 \text{ if } x > 0 \end{cases}$ where $\{0, 1\}$ has the discrete topology. f is closed but does not satisfy the condition in Theorem 17 for the sequence $\{1/n\}$ has at most 0 as a cluster point but $\{f(1/n)\}$ is the constant sequence of 1's and thus has no subsequence with at most 0 as a cluster point.

THEOREM 18. *Suppose X and Y are first countable spaces. f: $X \to Y$ is closed if and only if whenever $\{f(x_n)\} \to y_0$, $y_0 \neq f(x_n)$ for any n, then there exists an x_0 and a subsequence $\{y_n\}$* of *$\{x_n\}$ such that $\{y_n\} \to x_0$ and $f(x_0) = y_0$.*

Proof. If. Suppose A is closed. To prove that $f(A)$ is closed it suffices to show that if y_0 is the limit of a sequence from $f(A)$ then $y_0 \in f(A)$. Suppose $\{f(x_n)\} \to y_0$ where each $x_n \in A$. If $\{f(x_n)\}$ is a finite set we are done. If $\{f(x_n)\}$ is an infinite set there is a subsequence $\{f(a_n)\} \to y_0$ such that $y_0 \neq f(a_n)$ for any n. Therefore there exists an x_0 such that $y_0 = f(x_0)$ and x_0 is the limit of some subsequence of $\{a_n\}$. Since A is closed $x_0 \in A$ and thus $y_0 \in f(A)$. Only if. Suppose $\{f(x_n)\} \to y_0$ where $y_0 \neq f(x_n)$ for any n. Let $A = \overline{\{x_n\}}$. Then $f(A)$ is closed and contains the sequence $\{f(x_n)\}$. Thus $y_0 \in f(A)$. Therefore there exists an $x_0 \in A$ such that $y_0 = f(x_0)$. Since $x_0 \notin \{x_n\}$, there exists a subsequence of $\{x_n\}$ which converges to x_0.

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