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Let X be a set and $T: X \to X$ be a bijection. Consider the space \mathscr{M} of pseudo-Menger maps on X which induce a compact topology on X for which T is a homeomorphism. The lattice properties of \mathscr{M} are investigated and a bivariate nonnegative function of \mathscr{M} is defined which possesses certain properties analogous to those of the usual conditional entropy function defined on the space of measurable partitions of a probability space.

- 1. Introduction. A pseudo-Menger map on a set X is, roughly speaking, an assignment of a distribution function to every pair of points in X in a manner consistent with the axioms of a pseudometric space. Each such map induces a topology on X as defined by Schweizer and Sklar [9]. Let T be a bijection of X onto itself. Let *M* denote the space of all pseudo-Menger maps which induce a compact topology on X for which T is a homeomorphism. If $\theta \in \mathcal{M}$ let $h(T, \theta)$ denote the topological entropy of T with respect to the topology on X induced by θ . In [7] it was shown that $h(T, \cdot)$ is leftcontinuous in the sense that if θ , $\theta_n \in \mathcal{M}$, $\theta_n \ge \theta$ and $\theta_n(p, q) \to \theta(p, q)$ in distribution for all $p, q \in X$ then $h(T, \theta_n) \rightarrow h(T, \theta)$. In an effort to extend this result one is led to ask the question whether *M* is closed under the operations of Max and Min and, if so, what can one say about the entropy of T acting on the topology engendered by the maps obtained as a result of such operations. We now proceed to provide precise definitions and notation.
- 2. Preliminaries. Let I denote the closed unit interval, R the real numbers and Z^+ the positive integers. Let $\mathscr D$ be the set of all left-continuous monotone increasing functions $F\colon R\to I$ satisfying F(0)=0 and $\sup_x F(x)=1$. Endowed with the Lévy metric $\mathscr D$ is a complete metric space. If F_n , $F\in \mathscr D$ then $F_n\overset{\mathscr D}\to F$ will denote convergence with respect to the Lévy topology. It is well-known that $F_n\overset{\mathscr D}\to F$ if and only if $F_n(x)\to F(x)$ for each $x\in R$ at which F is continuous. If $F,G\in \mathscr D$ then $F\ge G$ will mean $F(x)\ge G(x)$ for all $x\in R$. Let $H\in \mathscr D$ be the function defined by: H(t)=0 for $t\le 0$ and H(t)=1 for t>0.

Let X be a fixed set. Let $\mathscr{F}(X)$ denote the collection of all functions $\theta\colon X\times X\to\mathscr{D}$. For convenience we shall often write θ_{pq} in place of $\theta(p,q)$. A statistical pseudo-metric space is an ordered pair (X,θ) where $\theta\in\mathscr{F}$ satisfies:

(SM 1)
$$\theta_{pq} = \theta_{qp}$$
 for all $p, q \in X$

(SM 2)
$$\theta_{pq}(a+b)=1$$
 whenever $\theta_{pr}(a)=\theta_{rq}(b)=1$ for some $r\in X$

(SM 3)
$$\theta_{pp} = H$$
 for all $p \in X$.

If, in addition, θ satisfies:

(SM 4)
$$\theta_{pq} = H$$
 only if $p = q$

then (X, θ) is a statistical metric space. Let $\mathcal{S}(X)$ denote the collection of all θ for which (X, θ) is a statistical pseudo-metric space.

A triangular norm is a function $\Delta: I \times I \to I$ which is associative, commutative, non-decreasing in each variable and satisfies $\Delta(y, 1) = y$ for each $y \in I$. A continuous Menger space [pseudo-Menger space] is a statistical metric space [statistical pseudo-metric space] (X, θ) for which there exists a continuous triangular norm Δ satisfying:

(SM 5)
$$\theta_{rr}(a+b) \ge \Delta(\theta_{rq}(a), \theta_{qr}(b))$$
 for all $p, q, r \in X$ and $a, b \in R$.

Let $\mathscr{M}(X)$ denote the set of all θ for which (X, θ) is a continuous pseudo-Menger space. If θ_n , $\theta \in \mathscr{M}(X)$ and $\theta_n(p,q) \xrightarrow{\mathscr{D}} \theta(p,q)$ for all $p, q \in X$ then we will write $\theta_n \xrightarrow{\mathscr{D}} \theta$. Similarly, if $\theta, \Gamma \in \mathscr{M}(X)$ and $\theta_{pq} \geq \Gamma_{pq}$ for all $p, q \in X$ then we write $\theta \geq \Gamma$. Let $\Xi \in \mathscr{M}(X)$ be defined by $\Xi_{pq} = H$ for all $p, q \in X$.

If $\theta \in \mathscr{F}(X)$ let X be endowed with the topology, denoted $\tau(\theta)$, generated by all sets of the form $N(p, \varepsilon, \lambda, \theta) = \{q \in X : \theta_{pq}(\varepsilon) > 1 - \lambda\}$ where $p \in X$, $\varepsilon > 0$, $\lambda > 0$. Let $T: X \to X$ be a bijection. Let $\mathscr{M}(X, T) = \{\theta \in \mathscr{M}(X) : T \text{ is a self-homeomorphism of } (X, \tau(\theta)) \text{ and } \tau(\theta) \text{ is compact}\}.$

If $\theta \in \mathcal{M}(X, T)$ we will let $h(T, \theta)$ denote the topological entropy of T with respect to the $\tau(\theta)$ topology. We will follow the notation and definitions of topological entropy developed in [1]. The only exception is the understanding that if T is a self-homeomorphism of (X, τ) where τ is not a compact topology and $\mathcal{U} \subset \tau$ is a cover of X which possesses a finite subcover then $h(T, \mathcal{U}) = \lim_{k \to \infty} (1/k) H(\mathbf{V}_{j=0}^{k-1} T^j \mathcal{U})$. We let $\mathcal{M}_F(X, T) = \{\theta \in \mathcal{M}(X, T) \colon h(T, \theta) < \infty\}$.

Let ρ be a pseudo-metric on a set Y and let $\mathscr U$ be an open cover of (Y, ρ) . Then ρ -diam $\mathscr U$ will mean the sup $\{\rho$ -diam $A: A \in \mathscr U\}$. Let $\tau(\rho)$ denote the topology on Y determined by ρ . In addition, if $\varepsilon > 0$ and $a \in Y$ then let $B(Y, a, \rho, \varepsilon) = \{q \in Y: \rho(q, a) < \varepsilon\}$. If D is another pseudo-metric on Y then $\rho \geq D$ will mean $\rho(a, b) \geq D(a, b)$ for all $a, b \in Y$. Finally, if X is a pseudo-metric space then X^* will denote the (unique up to uniform isomorphism) completion for which $X^* \sim X$ is Hausdorff. Such an X^* will be called the *pseudo-metric space completion of* X.

3. Lattice operations. If θ , $\Psi \in \mathcal{M}(X, T)$ define $\theta \vee \Psi = \operatorname{Min}(\theta, \Psi)$ and $\theta \wedge \Psi = \operatorname{Max}(\theta, \Psi)$. It is easy to construct examples in which $\theta \wedge \Psi$ fails even to belong to $\mathcal{S}(X)$. However, we will show that $\theta \vee \Psi$ admits a canonical extension to a map belonging to $\mathcal{M}(X^*, T^*)$ where X^* is the completion of $(X, \tau(\theta \vee \Psi))$ and T^* is the extension of T to X^* .

Proposition 1. If $\theta, \Psi \in \mathcal{M}(X)$ then $\theta \vee \Psi \in \mathcal{M}(X)$.

Proof. Let Δ_1 and Δ_2 be continuous triangular norms for θ and Ψ respectively which satisfy:

$$\theta_{pq}(a+b) \geq \Delta_1(\theta_{pr}(a), \theta_{rq}(b))$$

and

$$\Psi_{pq}(a+b) \ge \Delta_2(\Psi_{pr}(a), \Psi_{rq}(b))$$
 for all $a, b \in R$ and all $p, q, r \in X$.

It is easy to check that $\Delta_3 = \text{Min}(\Delta_1, \Delta_2)$ is a continuous triangular norm. Using the monotonicity of Δ_1 and Δ_2 we verify the triangle inequality for $\theta \vee \Psi$ with respect to Δ_3 :

$$egin{aligned} (hetaee au')_{pr}(a+b) &= \operatorname{Min}\,(heta_{pr}(a+b),\, au_{pr}(a+b)) \ &\geq \operatorname{Min}\,(arDelta_1(heta_{pq}(a),\, heta_{qr}(b)),\, arDelta_2(au_{pq}(a),\, au_{qr}(b))) \ &\geq \operatorname{Min}\,(arDelta_1,\, arDelta_2)(\operatorname{Min}\,(heta_{pq}(a),\, au_{pq}(a)),\, \operatorname{Min}\,(heta_{qr}(b),\, au_{qr}(b))) \ &= arDelta_3((hetaee au')_{pq}(a),\, (hetaee au')_{qr}(b)) \;. \end{aligned}$$

LEMMA 1. Assume $\theta, \Psi \in \mathscr{M}(X)$ determine compact topologies $\tau(\theta)$ and $\tau(\Psi)$ respectively. Let d_1 and d_2 be pseudo-metrics on X which generate the topologies $\tau(\theta)$ and $\tau(\Psi)$ respectively. Then the pseudo-metric $D = d_1 + d_2$ determines the topology $\tau(\theta \vee \Psi)$.

Proof. As a consequence of Lemma 4 of [7] and the above proposition we know that $\tau(\theta \vee \Psi) \supset \tau(\theta) \cup \tau(\Psi)$. Thus it suffices to show that $\tau(\theta \vee \Psi)$ is generated by $\{A \cap C \colon A \in \tau(\theta) \text{ and } C \in \tau(\Psi)\}$. Let $p \in X$, $\varepsilon > 0$ and $\lambda > 0$ be given. Let $q \in N(p, \varepsilon, \lambda, \theta \vee \Psi)$. For each $n \in \mathbb{Z}^+$ choose $A_n = N(q, 1/n, 1/n, \theta)$ and $C_n = N(q, 1/n, 1/n, \Psi)$. Suppose for each n there exists $y_n \in A_n \cap C_n$ such that $y_n \notin N(p, \varepsilon, \lambda, \theta \vee \Psi)$. Then $\theta_{qy_n}(1/n) > 1 - (1/n)$ and $\Psi_{qy_n}(1/n) > 1 - (1/n)$ from which $(\theta \vee \Psi)_{qy_n} \to H$. Since $(\theta \vee \Psi)_{pq}$ is left-continuous and $(\theta \vee \Psi)_{pq}(\varepsilon) > 1 - \lambda$, there exists a $\delta > 0$ for which $(\theta \vee \Psi)_{pq}(\varepsilon - \delta) > 1 - \lambda$. Now $(\theta \vee \Psi)_{py_n}(\varepsilon) \ge \Delta((\theta \vee \Psi)_{pq}(\varepsilon - \delta), (\theta \vee \Psi)_{qy_n}(\delta)) \to (\theta \vee \Psi)_{pq}(\varepsilon - \delta) > 1 - \lambda$ from which one draws the contradiction that $y_n \in N(p, \varepsilon, \lambda, \theta \vee \Psi)$ for large n.

PROPOSITION 2. Let θ , $\Psi \in \mathcal{M}(X)$ and suppose $\tau(\theta)$ and $\tau(\Psi)$ are each compact. Then $\tau(\theta \vee \Psi)$ is totally bounded.

Proof. Let d_1 , d_2 be pseudo-metrics on X which determine $\tau(\theta)$ and $\tau(\Psi)$ respectively. Then $D=d_1+d_2$ is a pseudo-metric for $\tau(\theta\vee\Psi)$. Let $\varepsilon>0$ be given. Let $p_i,\,q_i\in X,\,1\leq i\leq N,\,1\leq j\leq M$, be chosen such that $\bigcup_{i=1}^N B(X,\,p_i,\,d_1,\,\varepsilon/2)=\bigcup_{j=1}^M B(X,\,q_j,\,d_2,\,\varepsilon/2)=X$. Then it is easy to verify that, for each i and $j,\,B(X,\,p_i,\,d_1,\,\varepsilon/2)\cap B(X,\,q_j,\,d_2,\,\varepsilon/2)\subset B(X,\,z_{ij},\,D,\,\varepsilon)$ for any $z_{ij}\in B(X,\,p_i,\,d_1,\,\varepsilon/2)\cap B(X,\,q_j,\,d_2,\,\varepsilon/2)$ provided this intersection is nonempty.

THEOREM 1. Let θ , $\Psi \in \mathcal{M}(X, T)$ and let (X^*, τ^*) denote the pseudo-metric space completion of $(X, \tau(\theta \vee \Psi))$. Then:

- 1. T admits a unique extension to a self-homeomorphism T^* of (X^*, τ^*) .
- 2. $\theta \lor \Psi$ admits a unique extension to a map $(\theta \lor \Psi)^*: X^* \times X^* \to \mathscr{D}$
 - 3. $(\theta \vee \Psi)^* \in \mathcal{M}(X^*, T^*)$
 - 4. $\tau^* = \tau((\theta \vee \Psi)^*)$
- *Proof.* Let D^* denote the pseudo-metric on (X^*, τ^*) which extends the pseudo-metric D on $(X, \tau(\theta \vee \Psi))$. Since $\theta \vee \Psi \colon X \times X \to \mathscr{D}$ is a uniformly continuous map [8] it can be extended (Cor. 6.2, Ch. 14 of [3]) to a continuous map $(\theta \vee \Psi)^* \colon X^* \times X^* \to \mathscr{D}$. The work of Sherwood [11, 12] implies that $(\theta \vee \Psi)^* \in \mathscr{M}(X)$ and $\tau((\theta \vee \Psi)^*) = \tau^*$. Since T is uniformly continuous on (X, D) there exists an extension T^* which is a self-homeomorphism of (X^*, τ^*) . Now since τ^* is compact, $(\theta \vee \Psi)^* \in \mathscr{M}(X^*, T^*)$.
- 4. Entropy. We begin by investigating the relation among $h(T, \theta \vee \Psi)$, $h(T, \theta)$ and $h(T, \Psi)$. Several lemmas are required.
- LEMMA 2. Let (X, ρ) be a compact pseudo-metric space and $T: X \to X$ be a homeomorphism. Let $\{\mathscr{U}_n: n \in \mathbf{Z}^+\}$ be a sequence of open covers of X satisfying ρ -diam $\mathscr{U}_n \to 0$ as $n \to \infty$. Then $h(T, \mathscr{U}_n) \to h(T)$.
- *Proof.* Let $\{\mathscr{U}_{n_j} \colon j \in \mathbf{Z}^+\}$ be a subsequence of the $\{\mathscr{U}_n\}$. Using the Lebesgue covering lemma one can select a subsequence $\{m_j\}$ of the $\{n_j\}$ such that $\mathscr{U}_{m_j} \prec \mathscr{U}_{m_{j+1}}$ for $j \geq 1$. Now applying the Corollary on page 314 of [1] the desired result is obtained.
- LEMMA 3. Suppose D and d are pseudo-metrics on X satisfying $D \ge d$, $\tau(d)$ is compact and $\tau(D)$ is totally bounded. Assume T is a self-homeomorphism of (X, d) and of (X, D). Let $\{\mathscr{V}_n : n \in Z^+\}$ be a sequence of $\tau(D)$ -open covers of X such that D-diam $\mathscr{V}_n \to 0$ as $n \to \infty$ and each \mathscr{V}_n possesses a finite subcover of X. Then sup $\{h(T, \mathscr{U}) : \mathscr{U} \subset \tau(d)\} \le \overline{\lim}_{n \to \infty} h(T, \mathscr{V}_n)$.

Proof. Let \mathcal{W}_n be a sequence of $\tau(d)$ -open covers of X such that d-diam $\mathcal{W}_n \to 0$. Then for each n > 0 there exists an $m \ge n$ such that $\mathcal{W}_n \prec \mathcal{V}_m$. Thus $\lim_n h(T, \mathcal{W}_n) \le \overline{\lim}_n h(T, \mathcal{V}_n)$. Now sup $\{h(T, \mathcal{U}): \mathcal{U} \subset \tau(d)\} = \lim_n h(T, \mathcal{W}_n)$.

LEMMA 4. Let $\theta, \Psi \in \mathcal{M}(X, T)$ and let D be a pseudo-metric on X for which $\tau(D) = \tau(\theta \vee \Psi)$. For each $\varepsilon > 0$ let $\mathcal{U}_{\varepsilon} = \{B(X, p, D, \varepsilon) : p \in X\}$. Then $h(T^*, (\theta \vee \Psi)^*) = \sup_{\varepsilon} h(T, \mathcal{U}_{\varepsilon})$.

Proof. Let $\mathscr{U}_{\varepsilon}^* = \{B(X^*, p, D^*, \varepsilon): p \in X\}$. Then $\mathscr{U}_{\varepsilon} = \{A \cap X: A \in \mathscr{U}_{\varepsilon}^*\}$ and $h(T^*, (\theta \vee \Psi)^*) = \sup_{\varepsilon} h(T^*, \mathscr{U}_{\varepsilon}^*)$. It is easy to verify that $N(\bigvee_{\varepsilon}^K T^{*i}\mathscr{U}_{\varepsilon}^*) = N(\bigvee_{\varepsilon}^K T^{i}\mathscr{U}_{\varepsilon})$ for all $K \geq 0$. Consequently $h(T^*, \mathscr{U}_{\varepsilon}^*) = h(T, \mathscr{U}_{\varepsilon})$ and the lemma is proven.

THEOREM 2. Let θ , $\Psi \in \mathcal{M}(X, T)$. Then:

$$\operatorname{Max}(h(T,\theta),h(T,\Psi)) \leq h(T^*,(\theta \vee \Psi)^*) \leq h(T,\theta) + h(T,\Psi)$$
.

Proof. Let d_1 and d_2 be pseudo-metrics on X which generate $\tau(\theta)$ and $\tau(\Psi)$ respectively. Then $D=d_1+d_2$ is a pseudo-metric for $\tau(\theta\vee\Psi)$. Let ε_n be a sequence of positive numbers such that $\varepsilon_n\to 0$. Let $\mathscr{V}_n=\{B(X,\,p,\,D,\,\varepsilon_n)\colon p\in X\}$ and $\mathscr{V}_n^*=\{B(X^*,\,p,\,D^*,\,\varepsilon_n)\colon p\in X\}$. Applying lemma 3, we have $h(T^*,\,(\theta\vee\Psi)^*)=\lim_{n\to\infty}h(T^*,\,\mathscr{V}_n^*)\geq \overline{\lim}_{n\to\infty}h(T,\,\mathscr{V}_n)\geq \sup\{h(T,\,\mathscr{U})\colon\mathscr{U}\subset\tau(d_1)\}=h(T,\,\theta)$.

Let $\mathscr{T}_n = \{B(X, p, d_1, 1/n): p \in X\}$, $\mathscr{Q}_n = \{B(X, p, d_2, 1/n): p \in X\}$ and $\mathscr{R}_n = \{B(X, p, D, 1/n): p \in X\}$. Since $\mathscr{R}_n \prec \mathscr{T}_{4n} \lor \mathscr{Q}_{4n}$ we have $h(T, \mathscr{R}_n) \leq h(T, \mathscr{T}_{4n} \lor \mathscr{Q}_{4n}) \leq h(T, \mathscr{T}_{4n}) + h(T, \mathscr{Q}_{4n})$. Lemma 4 yields $h(T^*, (\theta \lor \Psi)^*) = \lim_{n \to \infty} h(T, \mathscr{R}_n) \leq \lim_{n \to \infty} h(T, \mathscr{T}_{4n}) + \lim_{n \to \infty} h(T, \mathscr{Q}_{4n}) = h(T, \theta) + h(T, \Psi)$.

EXAMPLE 1. Let $Y = \{0, 1, 2\}$ and $X = Y^z$. Define the shift $T: X \to X$ by $T(\{y_i\}) = \{y_{i+1}\}$. Let d_1 and d_2 be pseudo-metrics on X given by:

$$d_{\scriptscriptstyle 1}(\{u_i\},\,\{z_i\}) = \sum\limits_{i=-\infty}^{\infty} rac{\mid \omega(u_i) - \omega(z_i)\mid}{2^{\mid i\mid}}$$

where

$$\omega(a) = \left\{ egin{array}{lll} 0 & ext{if} & a=2 \ 1 & ext{if} & a=0,1 \end{array}
ight.$$

and

$$d_{\mathbf{z}}(\{u_i\},\,\{z_i\}) = \sum_{i=-\infty}^{\infty} \frac{\mid lpha(u_i) - lpha(z_i)\mid}{2^{\mid i\mid}}$$

where

$$\alpha(a) = \begin{cases} 0 & \text{if} \quad a = 0 \\ 1 & \text{if} \quad a = 1, 2 \end{cases}$$

for all $\{u_i\}$ and $\{z_i\} \in X$.

Define θ , $\Psi \in \mathcal{M}(X, T)$ by:

$$\theta_{uz}(\varepsilon) = H(\varepsilon - d_1(u, z))$$

and

$$\varPsi_{uz}(arepsilon) = H(arepsilon - d_{arepsilon}(u,\,z)) \qquad ext{for all } arepsilon > 0 ext{ and all } u,\,z \in X$$
 .

Then it follows that $h(T^*, (\theta \vee \Psi)^*) = \ln 3$, Max $(h(T, \theta), h(T, \Psi)) = \ln 2$, and $h(T, \theta) + h(T, \Psi) = \ln 4$.

DEFINITION. If θ , $\Psi \in \mathscr{M}_F(X, T)$ let $h_T(\theta \mid \Psi) = h(T^*, (\theta \lor \Psi)^*) - h(T, \Psi)$

PROPOSITION 3. Assume θ , Ψ , $\Gamma \in \mathcal{M}_{F}(X, T)$. Then:

- (a) $0 \le h_T(\theta \mid \Psi) \le h(T, \theta)$
- (b) $h_T(\theta \mid \theta) = 0$
- (c) $h_{\mathit{T}}(\theta \lor \varPsi \mid \varGamma) = h_{\mathit{T}}(\theta \mid \varPsi \lor \varGamma) + h_{\mathit{T}}(\varPsi \mid \varGamma)$ provided $\theta \lor \varPsi$, $\varPsi \lor \varGamma \in \mathscr{M}_{\mathit{F}}(X, \mathit{T})$
 - (d) $h_T(\theta \mid \Xi) = h(T, \theta)$

Proof. Statement (a) is a corollary of Theorem 2. Statements (b), (c) and (d) follow quickly from the definitions.

PROPOSITION 4. Let θ , $\Gamma \in \mathscr{M}_{\mathbb{F}}(X, T)$. Suppose $\theta \vee \Gamma \in \mathscr{M}_{\mathbb{F}}(X, T)$ and that (X, Γ) is a Menger space. Then $h_{\mathbb{F}}(\theta \mid \Gamma) = 0$.

Proof. This follows from the fact that any two compact metrizable topologies on X, each of which renders T a homeomorphism, yield the same topological entropy for T.

PROPOSITION 5. Let θ , $\Psi \in \mathscr{M}_{\mathbb{F}}(X, T)$. Assume that $\theta \vee \Psi \in \mathscr{M}_{\mathbb{F}}(X, T)$ and that $\theta_{pq} = H$ if and only if $\Psi_{pq} = H$. Then $h_T(\theta|\Psi) = 0$.

Proof. For $x, y \in X$, define $x \sim y$ if and only if $\theta_{xy} = H$. This equivalence relation on X induces a self-homeomorphism \widetilde{T} of X/\sim . It is easy to verify that $h(T) = h(\widetilde{T})$. One can then apply Proposition 4.

PROPOSITION 6. Let θ , Ψ , $\Gamma \in \mathscr{M}_F(X, T)$ and suppose $\Psi \subseteq \Gamma$. Then $h_T(\Psi \mid \theta) \geq h_T(\Gamma \mid \theta)$.

Since $\tau(\theta) \subset$

Proof. Lemma 4 of [7] yields $\tau(\Gamma) \subset \tau(\Psi \vee \theta)$. $\tau(\Psi \vee \theta)$ we have $\tau(\Gamma \vee \theta) \subset \tau(\Psi \vee \theta)$. Let (X_1^*, τ_1^*) and (X_2^*, τ_2^*) denote the completions of $(X, \tau(\theta \vee \Gamma))$ and $(X, \tau(\theta \vee \Psi))$ respectively, and let T_1^* and T_2^* denote the extensions of T to X_1^* and X_2^* respectively. One may assume that $X_1^* \subset X_2^*$ and that T_2^* extends T_1^* . Then the relative topology on X_1^* induced by τ_2^* contains τ_1^* . Let D_1 and D_2 denote pseudo-metrics on X_1^* which generate τ_1^* and $\tau_2^* \mid_{X_1^*}$ (the topology induced on X_1^* by τ_2^*) respectively. By replacing D_2 with $D_1 + D_2$, if necessary, we may assume that $D_1 \leq D_2$. Let D_2^* denote the extension of D_2 to (X_2^*, τ_2^*) . Let $\mathscr{Y}_n^* = \{B(X_2^*, p, D_2^*, 1/n):$ $p \in X_1^*$ and $\mathscr{V}_n = \{A \cap X_1^* \colon A \in \mathscr{V}_n^*\}$. Applying Lemma 3 together with the fact that $h(T_1^*, \mathcal{Y}_n) = h(T_2^*, \mathcal{Y}_n^*)$, we have:

$$egin{aligned} h(T_1^*,(hetaee arGamma_{}^*)^*) & \leq \overline{\lim}_{n o\infty} h(T_1^*,\ \mathscr{V}_n) \ & \leq \sup_n h(T_2^*,\ \mathscr{V}_n^*) \ & = h(T_2^*,(hetaee arVert \mathscr{V})^*) \;. \end{aligned}$$

EXAMPLE 2. Two examples are given to show that, in general, if θ , Ψ , $\Gamma \in \mathscr{M}_{\mathbb{F}}(X, T)$ and $\Psi \subseteq \Gamma$ then $h_T(\theta \mid \Psi)$ may be less than or greater than $h_T(\theta \mid \Gamma)$.

First, note that $\Psi \leq \Xi$ and $h_T(\theta \mid \Xi) = h(T, \theta)$. Thus, if $h(T, \theta) > 0$, then $0 = h_T(\theta \mid \theta) < h_T(\theta \mid \Xi)$.

We now provide an example in which $\varPsi \leqq \varGamma$ and $h_{\scriptscriptstyle T}(\theta \,|\, \varPsi) >$ $h_T(\theta \mid \Gamma)$. Let $Y = \{1, 2, 3, 4, 5\}$ and $X = Y^z$. Define three pseudometrics on X as follows:

$$D(\{y_i\}, \, \{z_i\}) = \sum_{-\infty}^{\infty} rac{|\, c(y_i) - c(z_i)\,|}{2^{|\, i\,|}}$$

where

$$c(a) = egin{cases} 1 & ext{if} & a = 5 \ 0 & ext{otherwise} \end{cases}$$

$$d(\{y_i\}, \{z_i\}) = \sum_{-\infty}^{\infty} \frac{|s(y_i) - s(z_i)|}{2^{|i|}}$$

where

$$s(a) = egin{cases} 2 & ext{if} & a = 1 & ext{or} & 3 \ 1 & ext{if} & a = 2 & ext{or} & 4 \ 0 & ext{if} & a = 5 \end{cases}$$

$$ho(\{y_i\},\,\{z_i\}) = \sum\limits_{-\infty}^{\infty} rac{\mid t(y_i) - t(z_i)\mid}{2^{\mid i\mid}}$$

where

$$t(a) = egin{cases} 1 & ext{if} & a = 1 & ext{or} & 2 \ 0 & ext{otherwise} \end{cases}$$

for all $\{y_i\}$, $\{z_i\} \in X$.

Define θ , Ψ , $\Gamma \in \mathcal{M}(X)$ as follows:

$$egin{aligned} heta_{pq}(arepsilon) &= H(arepsilon -
ho(p,\,q)) \ arGamma_{pq}(arepsilon) &= H(arepsilon - D(p,\,q)) \ arPsilon_{pq}(arepsilon) &= H(arepsilon - d(p,\,q)) \end{aligned}$$

for all $p, q \in X$.

Let $T: X \to X$ be the shift given by $T(\{y_i\}) = \{y_{i+1}\}$. Then θ , Γ , $\Psi \in \mathscr{M}(X, T)$, $\Psi \subseteq \Gamma$, $h_T(\theta \mid \Psi) = h(T, \theta \vee \Psi) - h(T, \Psi) = \ln 5 - \ln 3 = \ln 5/3$, and $h_T(\theta \mid \Gamma) = h(T, \theta \vee \Gamma) - h(T, \Gamma) = \ln 3 - \ln 2 = \ln 3/2$.

THEOREM 3. Let θ_n , Ψ_n , θ , $\Psi \in \mathscr{M}_F(X, T)$. Suppose $\theta_n \geq \theta$, $\Psi_n \geq \Psi$, $\theta_n \xrightarrow{\mathscr{T}} \theta$, $\Psi_n \xrightarrow{\mathscr{T}} \Psi$ and $\theta \vee \Psi \in \mathscr{M}(X, T)$. Then $h_T(\theta_n \mid \Psi_n) \to h_T(\theta \mid \Psi)$ as $n \to \infty$.

The proof of this theorem is based upon the following lemma.

LEMMA 5. Assume $\Sigma \in \mathcal{M}(X, T)$, $\Omega \in \mathcal{M}(X)$, $\tau(\Omega)$ is totally bounded, $\Omega \geq \Sigma$ and T is a self-homeomorphism of $(X, \tau(\Omega))$. Then $\Omega \in \mathcal{M}(X, T)$.

Proof. We must show $\tau(\Omega)$ is compact. Let X^* be the completion of $(X, \tau(\Omega))$ and T^* be the extension of T to X^* . So $\Omega^* \in \mathscr{M}(X^*, T^*)$. Since X^* is compact, Lemma 3 of [7] is applicable. Thus, given $q \in X^*$, $\{N(q, \varepsilon, \lambda, \Omega^*): \varepsilon, \lambda > 0\}$ is a local basis for $\tau(\Omega^*)$ at q. If $p \in X$ observe that $N(p, \varepsilon, \lambda, \Omega^*) \cap X = N(p, \varepsilon, \lambda, \Omega)$. Hence, for $p \in X$, $\{N(p, \varepsilon, \lambda, \Omega): \varepsilon, \lambda > 0\}$ is a local basis for $\tau(\Omega)$ at p. Now an argument analogous to that used in lemma 4 of [7] yields $\tau(\Omega) \subset \tau(\Sigma)$.

Proof of Theorem 3. Since $\theta_n \vee \Psi_n \geq \theta \vee \Psi \in \mathscr{M}_F(X, T)$ and $\theta_n \vee \Psi_n \in \mathscr{M}(X)$ the preceding lemma yields $\theta_n \vee \Psi_n \in \mathscr{M}(X, T)$. Now, applying the main theorem of [7], $h(T, \theta_n \vee \Psi_n) \to h(T, \theta \vee \Psi)$ and $h(T, \Psi_n) \to h(T, \Psi)$. The desired result now follows.

We conclude with a brief consideration of a special class of pseudo-Menger maps.

DEFINITION. The map $\theta \in \mathcal{M}(X, T)$ is (X, T)-deterministic if $h(T, \theta) = 0$.

PROPOSITION 7. Let θ , $\Gamma \in \mathcal{M}(X, T)$ and suppose that $\theta \geq \Gamma$. If Γ is (X, T)-deterministic then so is θ .

Proof. This follows at once from Lemma 4 of [7] where it is shown that $\tau(\theta) \subset \tau(\Gamma)$.

The following two propositions are consequences of Theorem 2.

PROPOSITION 8. Let θ , $\Gamma \in \mathcal{M}(X, T)$. Then θ and Γ are (X, T)-deterministic if and only if $\theta \vee \Gamma$ is (X^*, T^*) -deterministic.

PROPOSITION 9. If $\Gamma \in \mathcal{M}(X, T)$ is (X, T)-deterministic and $\theta \in \mathcal{M}_{\mathbb{F}}(X, T)$ then $h_T(\theta \mid \Gamma) = h(T, \theta)$.

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