TRANSVERSALS OF LATIN SQUARES AND THEIR GENERALIZATIONS

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The main theme in this paper is the existence of a transversal with many distinct elements in an array more general than a latin square.

A transversal of a latin square of order $n$ is any set of $n$ cells such that no two come from the same column or same row. There has been a good deal of effort spent on establishing the existence of a transversal that has many distinct elements, e.g. [4, 5]. A close inspection of the argument in [5] reveals that the results there apply in a context far more general than that explicitly considered. Indeed, the assumptions that there are no duplications in a row or column can in some cases be dropped.

A variety of conjectures conclude the paper.

1. Definitions. An $n$-square is an $n$ by $n$ array of $n^2$ cells in each of which one of the symbols 1, 2, 3, ... appears. An $n$-square in which each symbol from 1 to $n$ appears $n$ times is called an equi-$n$-square.

If $m < n$, an $(m, n)$-rectangle is an $m$ by $n$ array of $mn$ cells in each of which one of the symbols 1, 2, 3, ... appears. There are $m$ rows and $n$ columns.

A transversal of an $n$-square or an $(m, n)$-rectangle is a set of cells, one from each row and no two from the same column. A partial transversal is a subset of a transversal. A transversal is latin if no two cells have the same symbol. Since a latin transversal need not contain all the symbols in the array, we do not use the traditional term, "complete". A row (or column) of an $n$-square is latin if no two of its cells contain the same symbol.

Note that a usual latin square can be described as an equi-$n$-square for which each row and each column is latin. Observe that a latin square is an equi-$n$-square.

2. Survey of results. Ryser in [10] conjectured that a latin square of odd order $n$ has a latin transversal. Koksma [4] proved that a latin square of order $n$ has a transversal with at least $(2n + 1)/3$ distinct symbols. Lindner and Perry, in a mimeographed publication [5], proved that the average number of distinct symbols in transversals of a latin $n$-square (taken over all transversals) is precisely $567$.
\[ n\left(1 - \frac{1}{2!} + \frac{1}{3!} - \cdots \pm \frac{1}{n!}\right). \]

From this it follows that there is a transversal with at least \( [(1 - 1/e)n] = .63n \) elements. Because Koksma’s result is stronger, [5] was not formally published.

This paper utilizes the technique of Lindner and Perry, which might be called “existence by averaging”, to establish, for instance, that an equi-\( n \)-square has a transversal with a least \([(1 - 1/e)n]\) distinct symbols (Corollary 3.3). Koksma’s technique, on the other hand, using all his assumptions, does not seem to be easily generalized.

Bruck proved that the Cayley table of a group of odd order has a transversal (namely the main diagonal). This follows from the fact that such a group is the union of cyclic groups of odd order and hence every element is the square of some element. Paige [7] proved that any finite abelian group that is not of the form \( C(2^n) \times H \), where \( C(2^n) \) is the cyclic group of order \( 2^n \), \( n \geq 1 \), and \( H \) has odd order, possesses a latin transversal. In [3] Hall generalized this result.

### 3 Transversals of \( n \)-squares.

For a subset \( X \) of the \( n^2 \) cells of an \( n \)-square, let \( t(X) \) denote the number of transversals that meet \( X \). This number, examined in the context of determinants of matrices with 0-entries in \( X \), has been the subject of some study (see Netto ([6], p. 73)). In the case where \( X \) is itself a transversal or a subset of a transversal a formula for \( t(X) \) is known (see [6], [9]). It is given in the following lemma, which is another version of the “hatcheck problem”.

**Lemma 3.1.** Let \( X \) be a set of \( q \) cells in an \( n \)-square such that no two lie in the same column or in the same row. Then

\[
t(X) = n! \left( \frac{q}{n} - \binom{q}{2} \frac{1}{n(n-1)} + \binom{q}{3} \frac{1}{n(n-1)(n-2)} \right.
\]

\[
\cdot \cdots \pm \left( \frac{q}{q} \frac{1}{n(n-1) \cdots (n-q+1)} \right).
\]

The next lemma implies that a set that is not a partial transversal meets at least as many transversals as does a partial transversal of the same cardinality.

**Lemma 3.2.** Let \( X \) be a set consisting of \( q \) cells, \( q \leq n \), in an \( n \)-square. Then \( t(X) \geq t(Z) \), where \( Z \) is a set of \( q \) cells in an \( n \)-
Proof. Assume that $X$ has at least two cells in the same row. (A similar argument applies if some column contains at least two cells of $X$.) Let $c$ be a cell of $X$ in the row mentioned. Let $Y$ be the set of $q$ cells obtained from $X$ by deleting cell $c$ and adjoining a cell $c'$ in a row not meeting $X$, but in the same column as $c$. Let $X'$ be the set of cells in $X$ that are not in the row containing $c$. Let $X'' = X - \{c\}$. Thus $X'' \supset X'$.

Now, $t(X)$ equals:

- the number of transversals that meet $X''$, but not $c$ or $c'$
- $t(\{c\})$
- the number of transversals that meet $X''$ and also $c'$.

On the other hand, $t(Y)$ equals:

- the number of transversals that meet $X''$, but not $c$ or $c'$
- $t(\{c'\})$
- the number of transversals that meet both $X''$ and $c$.

To compare these two sums, observe first that the first terms of each are the same and that $t(\{c\}) = t(\{c'\})$. Also,

- the number of transversals that meet both $X''$ and $c$ equals
- the number of transversals that meet both $X'$ and $c$, which equals
- the number of transversals that meet both $X'$ and $c'$.

Since $X'' \supset X'$, it follows by comparison of the third terms of the sums for $t(X)$ and $t(Y)$ that $t(X) \geq t(Y)$. Repeated application of this argument, at most $q - 1$ times, establishes the lemma.

The following theorem and its corollary generalizes the result of Lindner and Perry from latin $n$-squares to $n$-squares.

**Theorem 3.2.** In an $n$-square in which each symbol $1, 2, \cdots, s$ appears at least $q$ times, $q \leq n$, there is a transversal that contains at least

$$s \left[ \frac{q}{n} - \left( \frac{q}{2} \right) \frac{1}{n(n - 1)} + \left( \frac{q}{3} \right) \frac{1}{n(n - 1)(n - 2)} - \cdots \pm \left( \frac{q}{q} \right) \frac{1}{n(n - 1) \cdots (n - (q - 1))} \right]$$

distinct symbols.

**Proof.** Let $U$ be the set of ordered pairs $(t, i)$ where symbol $i$
is contained in transversal \( t \). The cardinality of \( U \) is equal to
\[ n! \cdot \text{(the average number of distinct symbols in all transversals of the } n\text{-square)}. \]

On the other hand, since there are \( s \) symbols, \( U \) has cardinality
\[ s \cdot \text{(the average number of transversals that contain a given symbol)}. \]

Let \( X_i \) be the set of cells occupied by the symbol \( i \). Since \( |X_i| \geq q \), \( t(X_i) \) is greater than or equal to the number of transversals that meet \( q \) diagonal elements, by Lemma 3.2. Comparison of these two expressions for the cardinality of \( U \) together with Lemma 3.1 establishes the theorem.

The case \( q = n \) is singled out in the following corollary.

**Corollary 3.3.** In an equi-\( n \)-square there is a transversal that contains at least
\[
 n \left( 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots \pm \frac{1}{n!} \right)
\]
distinct symbols.

It is not clear how much Corollary 3.3 can be strengthened. Koksma's argument for \((2n + 1)/3\) does not apply to equi-\( n \)-squares, since it makes use in several places of the assumption that each row and each column is latin. Moreover, Ryser's conjecture is not valid for equi-\( n \)-squares, where \( n \) is odd and at least 3. To see this, consider the equi-\( n \)-square whose first \( n - 1 \) rows each consist of the symbols \( 1, 2, \ldots, n \) in order, and whose \( n^{th} \) row is the same set of symbols, in the order \( 2, 3, \ldots, n, 1 \). It is a simple matter to show that it does not have a latin transversal. Note, incidentally, that each row of this equi-\( n \)-square is latin.

The proofs of the next two theorems, being similar to that of Theorem 3.2, are only sketched.

**Theorem 3.4.** Let \( n \) be even and at least 4. Let each of \( n^2/2 \) symbols appear twice in an \( n \)-square. Then there is a transversal that contains \( n \) distinct symbols.

**Proof.**
\[ n! \cdot \text{(the average number of distinct symbols in a transversal)} \]
\[ \frac{n^2}{2} \text{ (the average number of transversals that contain a given symbol).} \]

Thus

\[ n! \text{ (the average number of distinct symbols in a transversal)} \]

\[ \geq \frac{n^2}{2} \cdot n! \left( \frac{2}{n} - \frac{1}{n(n-1)} \right). \]

Hence the average number of distinct symbols

\[ \geq n - \frac{1}{2} \cdot \frac{n}{n-1}. \]

If \( n \geq 4 \), there is consequently a transversal with \( n \) distinct symbols.

The next theorem is a companion of Theorem 3.4.

**Theorem 3.5.** Let \( q \) be greater than 2 and let \( n \) be a positive multiple of \( q \). Let each of \( n^2/q \) symbols appear \( q \) times in an \( n \)-square. Then some transversal contains more than \( n-q/2 \) distinct symbols.

**Proof.** There is a transversal for which the number of distinct symbols is at least

\[ \frac{n^2}{q} \left( \frac{q}{n} \frac{1}{n} - \frac{q \cdot q - 1}{1 \cdot 2} \frac{1}{n(n-1)} \right) + \frac{q \cdot q - 1 \cdot q - 2}{1 \cdot 2 \cdot 3} \frac{1}{n} \frac{1}{n-1} \frac{1}{n-2} \]

\[ + \ldots \frac{q(q-1) \cdots 1}{1 \cdot 2 \cdots q} \frac{1}{n} \frac{1}{n-1} \frac{1}{n-q+1}. \]

Hence, there is one with more than

\[ \frac{n^2}{q} \left( \frac{q}{n} - \frac{q(q-1)}{2} \frac{1}{n(n-1)} \right) \]

distinct symbols. Since \( n \geq q \) the theorem follows.

4. Transversals of \((m, n)\)-rectangles. The method used in Section 3 also applies to \((m, n)\)-rectangles. However, this section will illustrate a different averaging process, much simpler, and only slightly weaker. It employs the notion of a "singular" pair of cells. Two cells in different rows and different columns form a singular pair if they contain the same symbol. The method is based on a count of incidences of transversals and singular pairs.
Theorem 4.1. Let $q$ divide $mn$ and let each symbol in an $(m, n)$-rectangle appear $q$ times. Then there is a transversal with at most

$$\frac{m(q - 1)}{2(n - 1)}$$

singular pairs.

Proof. Count the set of ordered pairs $(t, p)$, where $t$ is a transversal and $p$ is a singular pair in $t$. Counting in both orders yields

$$n(n - 1) \cdots (n - m + 1) \text{ (average number of singular pairs on a transversal)}$$
$$\leq \frac{mn}{q} \cdot \frac{q(q - 1)}{2} \text{ (average number of transversals on a singular pair)}$$
$$\leq \frac{mn}{q} \cdot \frac{q(q - 1)}{2} (n - 2)(n - 3) \cdots (n - m + 1).$$

The theorem follows immediately.

The following corollaries are immediate consequences.

Corollary 4.2. If each symbol in an $(m, n)$-rectangle appears $n$ times, then there is a transversal with at least $m/2$ distinct symbols.

Corollary 4.3. If each symbol in an $(m, n)$-rectangle appears $q$ times and if

$$\frac{m(q - 1)}{2(n - 1)} < 1,$$

then there is a latin transversal.

A special case of Corollary 4.3 is given by the following.

Corollary 4.4. If each symbol in an $(m, n)$-rectangle appears $m$ times, and if

$$n > \frac{m^2 - m + 2}{2},$$

then there is a latin transversal.

The method of Section 3 yields a slightly stronger result, which implies that "\(>\)" can be replaced by "\(\geq\)" in Corollary 4.4.
5. Rows or columns with many distinct symbols. The "existence by averaging" technique may also be applied to establish the existence of a row or column in an $n$-square with "many" distinct symbols.

Theorem 5.1. Let the cells of an $n$-square be occupied by the symbols $1, 2, \ldots, k$, with $i$ appearing $n_i$ times, $1 \leq i \leq k$. Then some row or column contains at least

$$\frac{1}{n}(\sqrt{n_1} + \sqrt{n_2} + \cdots + \sqrt{n_k})$$

different symbols.

Proof. Let $U$ be the set of ordered pairs $(L, i)$, where $L$ is a line (either a row or a column) that contains the symbol $i$. Since there are $2n$ such lines, $U$ has

$$2n \cdot \text{(average number of distinct symbols in a line)}.$$ 

On the other hand, $U$ has

$$k \cdot \text{(average number of lines that contain a given symbol)}.$$ 

To evaluate the second average, let $L(i)$ be the number of lines that contain the symbol $i$. Let $R(i)$ be the number of rows and $C(i)$ be the number of columns that contain $i$. Thus $L(i) = R(i) + C(i)$.

Now, the set of cells occupied by $i$ is contained in the intersection of $R(i)$ rows and $C(i)$ columns. Consequently

$$R(i) \times C(i) \geq n_i.$$ 

It follows that

$$R(i) + C(i) \geq 2\sqrt{n_i},$$

hence that

$$L(i) \geq 2\sqrt{n_i}.$$ 

Thus

$$\sum_{i=1}^{k} L(i) \geq \sum_{i=1}^{k} 2\sqrt{n_i},$$

from which the theorem follows.
The specialization of Theorem 5.1 to an equi-\(n\)-square is described in the next corollary.

**Corollary 5.2.** *In an equi-\(n\)-square there is a row or a column that contains at least \(\sqrt{n}\) distinct symbols.*

G. D. Chakerian and D. Hickerson have independently shown that Corollary 5.2 is best possible if it is not required the set of cells occupied by a given symbol be topologically connected.

6. **Conjectures.** The following conjectures, some of which are logically related, may suggest directions for further study.

1. An equi-\(n\)-square has a transversal with at least \(n - 1\) distinct symbols.
2. An \(n\)-square in which each symbol appears at most \(n - 1\) times has a latin transversal. (It is easy to show by induction, or by either averaging method that if each symbol in an \(n\)-square, \(n \geq 3\), appears at most two times, the \(n\)-square has a latin transversal.)
3. An \((n - 1, n)\)-rectangle in which each symbol appears at most \(n\) times has a latin transversal.
4. A row-latin \((n - 1, n)\)-square has a latin transversal.
5. An \((m, n)\)-rectangle in which each symbol appears at most \(n\) times has a latin transversal.

Note that Conjectures (3) and (5) are equivalent. Moreover, for \(m = 1\), Conjecture (5) is immediate. For \(m = 2\), Conjecture (5) is valid with the weaker assumption that each symbol appears at most \(2n - 1\) times.

6. An \((n - 1, n)\)-rectangle in which each symbol appears exactly \(n\) times has a latin transversal.
7. An \((m, n)\)-rectangle in which each symbol appears at most \(m + 1\) times has a latin transversal.

**References**


Received November 27, 1974. For a general survey of transversals in latin $n$-squares see J. Dénes and A. D. Keedwell, *Latin squares and their applications*, Academic Press, 1974. Incidentally, it is mentioned there that the analog of Conjecture (1) has been proposed for latin $n$-squares.

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