

# Pacific Journal of Mathematics

**DISTRIBUTION OF SQUAREFREE INTEGERS IN NON-LINEAR  
SEQUENCES**

IVAN ERNEST STUX

## DISTRIBUTION OF SQUAREFREE INTEGERS IN NON-LINEAR SEQUENCES

IVAN E. STUX

**I. This paper investigates the occurrences of the square-free integers in sequences  $s_n = [f(n)]$ ,  $n = 1, 2, 3, \dots$  where  $f(x)$  belongs to classes of functions described by 'smoothness' conditions. The result obtained is an extension of the well known fact that  $Q(x) = 6/\pi^2 x + O(x^{1/2})$ , where  $Q(x) =$  number of squarefree integers  $\leq x$ ; it states that  $Q_s(x) \sim 6/\pi^2 g(x)$  where  $Q_s(x) =$  number of squarefree integers  $\leq x$  in the sequence  $s_n$ , and  $g(x)$  is the inverse function of  $f(x)$ .**

This result relates to the deep theorem of Piatetskii-Shapiro which states that if  $1 < c < 12/11$  then the sequence  $[n^c]$  has the proper rate of primes occurring, namely,  $\pi_c(x) \sim x^{1/c}/\log x$ .

The classes of functions used is described by the following:

DEFINITION 1. for given  $1 < c < 2$ ,  $0 < \delta < 1$

- (1)  $S(c, \delta) =$  set of functions  $f(x)$  such that for some constant  $a > 0$  depending on  $f$ , and for sufficiently large  $x$ 's, depending on  $f$ ,

$$(ax^c)^{(i)} \leq (f(x))^{(i)} < (ax^{c+\delta})^{(i)}$$

holds for  $i = 0, 1, 2$ , the superscripts indicating the  $i^{\text{th}}$  derivative.

Functions like  $x^c$ ,  $1 < c < 2$ , or more generally  $\sum_{i=1}^k a_i x^{c_i} (\log x)^{d_i}$ , where the leading term has  $a > 0$ ,  $1 < c < 2$ , belong to these classes of functions.

The following theorem will be proved:

**THEOREM 1.** *Let  $1 < c < 4/3$ , then there exists a  $\delta_c = \delta(c) > 0$ , some small value depending on  $c$  such that if  $f(x) \in S(c, \delta)$ ,  $0 \leq \delta < \delta(c)$ , then*

$$(2) \quad Q_s(x) = 6/\pi^2 g(x) + O((g(x))^{1-\varepsilon})$$

*holds for some  $\varepsilon > 0$  depending on  $c$  and  $\delta$ , where  $Q_s(x) =$  number of  $\{s_n \leq x, s_n = [f(n)], s_n = \text{squarefree}, n = 1, 2, 3, \dots\}$ ,  $g(x) =$  inverse function of  $f$ ,  $[z] =$  integer part of  $z$ .*

II. Following are the lemmas that will be used in the proof:

**LEMMA 1** (Piatetskii-Shapiro, [2]). *Let  $f(x)$  be a continuously*

$g(x)$  be its inverse function. Then, for an integer  $m$  such that  $m = [f(n)]$ , either  $\{g(m)\} = 0$  or  $1 - g'(m-1) < \{g(m)\} < 1$ . Conversely, if  $\{g(m)\} = 0$  or  $1 - g'(m+1) < \{g(m)\} < 1$ , then it follows that for some  $n$ ,  $m = [f(n)]$ .

(The curly brackets indicate the fractional part of the real number, the straight brackets the integer part, as usual.)

LEMMA 2 (a theorem of Erdős-Turán, [1]). If  $\mu_1, \mu_2, \dots$  is a real sequence and if  $D_N$  denotes its discrepancy modulo one, then for each integer  $m \geq 1$  we have

$$(3) \quad ND_N \leq K \left( \frac{N}{m+1} + \sum_{t=1}^m \frac{1}{t} \left| \sum_{n=1}^N e(t\mu_n) \right| \right)$$

(where  $K$  is a constant and  $e(z) = e^{2\pi iz}$ , as usual).

LEMMA 3 (Van der Corput, pg, 64, [3]). Let  $g(x)$  be a real function with a continuous and steadily decreasing derivative  $g'(x)$  in  $(a, b)$ , and let  $g'(b) = \alpha$ ,  $g'(a) = \beta$ . Then

$$(4) \quad \sum_{\alpha < n \leq b} e(g(n)) = \sum_{\alpha - \eta < \nu < \beta + \eta} \int_a^b e(g(x) - \nu x) dx + O(\log(\beta - \alpha + 2))$$

where  $\eta$  is any positive constant less than one.

LEMMA 4 (Van der Corput, pg. 61, [3]). Let  $F(x)$  be a real function, twice differentiable, and let  $F''(x) \geq r > 0$ , or  $F''(x) \leq -r < 0$  throughout the interval  $(a, b)$ , then

$$(5) \quad \left| \int_a^b e^{iF(x)} dx \right| \leq \frac{8}{\sqrt{r}}.$$

III. The first part of the proof is aimed at establishing the uniform distribution modulo one and the discrepancy of that distribution for sequences  $g(q)$  where  $q$  are squarefree integers and  $g(x)$  is the inverse function of a function in  $S(c, \delta)$  (where  $\delta$  is usually small, depending on  $c$ ). The following is the result in this direction:

THEOREM 2. For given  $1 < c < 2$ , and  $\delta > 0$ , small enough depending on  $c$  alone, let  $f(x) \in S(c, \delta)$  and let  $g(x)$  be the inverse function of  $f(x)$ . Then the sequence  $\{g(q): q \leq K, q \text{ the squarefree integers}\}$  is uniformly distributed modulo one and

$$(6) \quad N(K, \xi) = \xi Q(K) + Q(K) D_{Q(K)}(g),$$

and

differentiable function with  $f'(x) > 0, f''(x) \geq 0$ , for  $x \geq 1$ , and let

$$(7) \quad Q(K)D_{Q(K)}(g) \ll K^{3/5+(c+2\delta)/(5c(c+\delta))} + K^{1-1/(2c(c+\delta))}$$

where  $Q(K)$  = number of squarefree integers  $\leq K$ ,  $N(K, \xi)$  = number of elements in the sequence  $g(q), q \leq K, q$  squarefree, which fall into a fixed interval of length  $\xi (< 1)$  modulo one, and  $D_{Q(K)}(g)$  is the discrepancy, modulo one, of the sequence  $g(q)$ .

Clearly, uniform distribution holds whenever  $\delta > 0$  is small enough to make the exponents in the estimate (7) less than one.

*Proof.* For  $h \geq 1$ , consider

$$(8) \quad T_h(K) = \sum_{\substack{q \leq K \\ q \text{ squarefree}}} e(hg(q)), \quad e(z) = e^{2\pi iz}.$$

Suppose that  $K_0$  is the large value from where on the estimates of  $g, g', g''$  induced by the definition 1 hold, and let  $K > K_0$ , then

$$(8') \quad T_h(K) = \sum_{\substack{K_0 < q \leq K \\ q \text{ squarefree}}} e(hg(q)) + O(K_0),$$

and

$$(9) \quad \sum_{\substack{K_0 < q \leq K \\ q \text{ squarefree}}} e(hg(q)) = \sum_{\substack{K_0 < n \leq K \\ n = \text{integer}}} e(hg(n)) \sum_{d^2 | (n, P^2)} \mu(d)$$

where  $p = \prod_{p \leq K^{1/2}} p, p = \text{primes}, (a, b) = \text{greatest common divisor}, \mu(d) = \text{Möbius function. We can further write}$

$$(10) \quad \begin{aligned} & \sum_{\substack{d \leq \sqrt{K} \\ d | P}} \mu(d) \sum_{K_0/d^2 < m \leq K/d^2} e(hg(d^2m)) \\ &= \sum_{\substack{d \leq A \\ d | P}} \mu(d) \sum_{K_0/d^2 < m \leq K/d^2} e(hg(d^2m)) \\ & \quad + \sum_{\substack{A < d \leq K^{1/2} \\ d | P}} \mu(d) \sum_{K_0/d^2 < m \leq K/d^2} e(hg(d^2m)). \end{aligned}$$

We will pick the value of  $A$  later. The second sum in (10) can be estimated trivially as

$$(11) \quad \sum_{A < d \leq K^{1/2}} \left| \sum_{K_0/d^2 < m \leq K/d^2} e(hg(d^2m)) \right| \ll \sum_{A < d \leq K^{1/2}} \frac{K}{d^2} \ll \frac{K}{A}.$$

The first sum, on the other hand, is estimated by

$$(12) \quad \sum_{d \leq A} \left| \sum_{K_0/d^2 < m \leq K/d^2} e(hg(d^2m)) \right|.$$

To estimate the inner sum, divide the interval  $K_0/d^2 < m < K/d^2$  up into pieces of type  $1/2^r K/d^2 < m \leq 1/2^{r-1} K/d^2$ , to get

$$(12') \quad \sum_{d \leq A} \sum_r \left| \sum_{\substack{1/2^r K/d^2 < m \leq 1/2^{r-1} K/d^2 \\ m \geq K_0/d^2}} e(hg(d^2 m)) \right|.$$

We will estimate the last inner sum by using Lemma 3 and then Lemma 4. The conditions in definition 1 give that

$$\left(\frac{y}{a}\right)^{(c+\delta)^{-1}} < g(y) \leq \left(\frac{y}{a}\right)^{1/c}, \quad \frac{1}{c+\delta} \left(\frac{y}{a}\right)^{(c+\delta)^{-1}-1} < g'(y) \leq \frac{1}{c} \left(\frac{y}{a}\right)^{1/c},$$

the chain rule tells us that  $(d/dx)g(d^2x) = [(d/dz)g(z)] \cdot d^2$ ,  $z = d^2x$ , and so we have, by Lemma 3, for each  $r$

$$(13) \quad \sum_{1/2^r K/d^2 < m \leq 1/2^{r-1} K/d^2} e(hg(d^2 m)) = \sum_\nu I_\nu + E,$$

where the  $\sum_\nu$  extends over  $(1/a_1(c+\delta))h((1/2^{r-1})K)^{(c+\delta)^{-1}-1}d^2 - 1/2 < \nu < (1/a_2c)h((1/2^r)K)^{c-1-1}d^2 + 1/2$ ,  $a_1 = a^{(c+\delta)^{-1}}$ ,  $a_2 = a^{1/c}$ , and

$$(14) \quad I_\nu = \int_{2^{-r}Kd^{-2}}^{2^{-r+1}Kd^{-2}} e(hg(d^2x) - \nu x)dx,$$

and

$$E = O(\log(\max \nu - \min \nu + 2)).$$

In (14), first we change variables to  $y = d^2x$ , and then apply Lemma 4

$$(14') \quad I_\nu = \frac{1}{d^2} \int_{2^{-r}K}^{2^{-r+1}K} e(hg(y) - \frac{\nu y}{d^2})dy$$

but here  $d^2/dy^2(hg(y) - \nu y/d^2) \geq (c+\delta)^{-1}((c+\delta)^{-1} - 1)(1/a)hy^{(c+\delta)^{-1}-2}$ . thus, we get, applying Lemma 4 that

$$(15) \quad I_\nu \ll \frac{1}{d^2} \left[ h\left(\frac{1}{2^{r-1}}K\right)^{(c+\delta)^{-1}-2} \right]^{-1/2}.$$

We thus have for (12) the estimate:

$$(16) \quad \ll \sum_{d \leq A} \sum_r \sum_\nu \frac{1}{d^2} h^{-1/2} \frac{2^{(r-1)/(c+\delta)c}}{2^{r-1}} K^{1-1/(2c+2\delta)} + \sum_{d \leq A} \sum_r (E)$$

(for largest  $r$  we might get a shorter range of integration in (14), but the upper bound estimates still clearly hold in (16)). where  $\sum_\nu$  is over

$$\frac{1}{a(c+\delta)} h\left(\frac{1}{2^{r-1}}K\right)^{(c+\delta)^{-1}-1} d^2 - \frac{1}{2} < \nu < \frac{1}{ac} h\left(\frac{1}{2^r}K\right)^{c-1-1} d^2 + \frac{1}{2}.$$

From here, we have that the  $\nu$  summation is bounded by

$$\ll hd^2 2^r \left(\frac{1}{2^{r-1}}\right)^{1/(c+\delta)} K^{c-1-1} + 1$$

and so we can further estimate (12) by

$$(16') \quad \ll \sum_{d \leq A} \sum_r \left( h d^2 \frac{2^r}{2^{(r-1)/(c+\delta)}} K^{(1/c)-1} + 1 \right) \left( \frac{h^{-1/2}}{d^2} \frac{2^{(r-1)/(2(c+\delta))}}{2^{r-1}} K^{1-1/(2c+2\delta)} \right) \\ + \sum_{d \leq A} \sum_r E \ll Ah^{1/2} K^{c^{-1}-1/(2c+2\delta)} + h^{-1/2} K^{1-1/(2c+2\delta)} + A \log K.$$

(The last step is because  $\sum_r$  in the first term was just a geometric sum and so it converges, while in the second part, the number of terms of the  $\sum_r$  is  $O(\log K)$ .) The estimate (16') together with (11) now gives us that

$$(17) \quad T_h(K) \ll Ah^{1/2} K^{(c+2\delta)/(c+\delta)2c} + h^{-1/2} K^{1-1/(2c+2\delta)} + \frac{K}{A} + K_0.$$

Here, for  $K$  sufficiently large the last error term absorbs into the first one if  $A \geq 1$  (which will anyway be the case). We now pick  $A$  so as to balance the 1st and 3rd terms of (17), i.e. let  $A = [K^{2^{-1}-(c+2\delta)/(c+\delta)4c} h^{-1/4}]$ . With this choice we obtain

$$(18) \quad T_h(K) \ll h^{1/4} K^{2^{-1}+(c+2\delta)/(c+\delta)4c} + h^{-1/2} K^{1-1/(2c+2\delta)}.$$

Finally we use Lemma 2 to write:

$$(19) \quad Q(K)D_{Q(K)}(g) \ll \frac{K}{m+1} + \sum_{h=1}^m \frac{1}{h} \{h^{1/4} K^{2^{-1}+(c+2\delta)/(c+\delta)4c} + h^{-1/2} K^{1-1/(2c+2\delta)}\} \\ \ll \frac{K}{m+1} + m^{1/4} K^{2^{-1}+(c+2\delta)/(c+\delta)4c} + K^{1-1/(2c+2\delta)}.$$

We pick the optimal  $m$ , i.e.  $m = [K^{4(2^{-1}-(c+2\delta)/(c+\delta)4c)/5}]$ , and thus we have

$$(20) \quad Q(K)D_{Q(K)}(g) \ll K^{3/5+(c+2\delta)/(c+\delta)4c} + K^{1-1/(2c+2\delta)}.$$

**COROLLARY 1.** *If  $1 < c < 4/3$  then there exists  $\delta_c > 0$  depending on  $c$  such that if  $f(x) \in S(c, \delta)$  for  $0 < \delta < \delta_c$ , and  $g(x)$  is its inverse function then:*

$$(21) \quad Q(K)D_{Q(K)}(g) \ll K^{(c+\delta)^{-1-\varepsilon}} \ll (g(K))^{1-\varepsilon}$$

for some  $\varepsilon > 0$ , depending on  $c$  and  $\delta$ .

*Proof.* All we need to show is that

$$\frac{3}{5} + \frac{c+2\delta}{5c(c+\delta)} < \frac{1}{c+\delta} \quad \text{and} \quad 1 - \frac{1}{2(c+\delta)} < \frac{1}{c+\delta}$$

hold for some  $\delta > 0$ . By continuity it is enough to check that  $3/5 + 1/5c < 1/c$  and  $1 - 1/2c < 1/c$  hold. But the first of these holds if

$c < 4/3$ , the second if  $c < 3/2$ .

IV. We can now prove Theorem 1. Let

$$(22) \quad T_s(x, y) = \text{number of } \{s_n = [f(n)], y < s_n \leq x, s_n = \text{squarefree}, n = 1, 2, 3, \dots\}$$

Clearly,  $T_s(1, y) = Q_s(y)$ . Lemma 1 can now be used together with expressions (6) and (21).  $\xi$  in (6) will be taken  $g'(y + 1)$  or  $g'(x - 1)$  to give upper and lower bounds on  $T(x, y)$ , where  $g(x)$  is as usual the inverse function of  $f(x)$ . We obtain:

$$(23) \quad T_s(x, y) \begin{cases} < g'(x - 1)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) \\ > g'(y + 1)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) \end{cases}$$

where  $Q(x) = \#$  squarefree integers  $\leq x$ . Or

$$(24) \quad T_s(x, y) \begin{cases} < g'(x)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) + O(x^{\epsilon-1-2}(Q(y) - Q(x))) \\ > g'(y)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) + O(y^{\epsilon-1-2}(Q(y) - Q(x))) \end{cases}.$$

Thus, for  $0 < \alpha < 1$ , using the well-known fact that  $Q(x) = 6\pi^{-2}x + O(x^{1/2})$ ,

$$(25) \quad T_s(x, (1 + \alpha)x) = \begin{cases} < \frac{6}{\pi^2} x \cdot \alpha \cdot g'(x) \\ > \frac{6}{\pi^2} x \cdot \alpha \cdot g'((1 + \alpha)x) \end{cases} + O(x^{(c+\delta)^{-1-\epsilon}} + x^{1/2}).$$

On the other hand, clearly

$$(26) \quad Q_s(x) = \sum_{k=1}^{L(x)} T_s\left(\frac{x}{(1 + \alpha)^k}, \frac{x}{(1 + \alpha)^{k-1}}\right) + O(1)$$

holds for an appropriate function  $L(x)$  which tends to  $\infty$  for  $x \rightarrow \infty$ , if  $\alpha = \alpha(x) > 0$  is some given function of  $x$  which tends to zero as  $x \rightarrow \infty$  (the relation is  $(1 + \alpha(x))^{L(x)} \cong x$ ).

Using (25) in the expression (26) we obtain

$$(27) \quad Q_s(x) \begin{cases} < \frac{6}{\pi^2} \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1 + \alpha)^k} g'\left(\frac{x}{(1 + \alpha)^k}\right) + O(L(x) \cdot x^\gamma) \\ > \frac{6}{\pi^2} \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1 + \alpha)^k} g'\left(\frac{x}{(1 + \alpha)^{k-1}}\right) + O(L(x) \cdot x^\gamma) \end{cases}$$

where  $\gamma = \max \{1/(c + \delta) - \epsilon, 1/2\}$  and so it is actually  $1/(c + \delta) - \epsilon$ . The main terms of the expressions on the right of (27) are exactly the upper and lower approximating sums of the Riemann integral

$$\int_1^x g'(y)dy = g(x) - g(1) .$$

To see how closely these sums approximate the integral, it suffices to find out how closely they are to each other, i.e. to estimate:

$$\begin{aligned} \Delta(x) &= \left| \sum_{k=1}^{L(x)} \frac{x\alpha}{(1+\alpha)^k} g'\left(\frac{x}{(1+\alpha)^{k-1}}\right) \right. \\ &\quad \left. - \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1+\alpha)^k} g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ (28) \quad &= \left| \sum_{k=0}^{L(x)-1} \frac{x \cdot \alpha}{(1+\alpha)^{k+1}} g'\left(\frac{x}{(1+\alpha)^k}\right) \right. \\ &\quad \left. - \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1+\alpha)^k} g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &\leq \left| \alpha \cdot \sum_{k=1}^{L(x)-1} \frac{x}{(1+\alpha)^k} \left(\frac{1}{1+\alpha} - 1\right) g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &\quad + |x \cdot \alpha \cdot g'(x)| + O(\alpha) \\ &\ll \left| \alpha \cdot \sum_{k=1}^{L(x)-1} \frac{x \cdot \alpha}{(1+\alpha)^{k+1}} g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &\quad + O|x \cdot \alpha \cdot g'(x)| + O(\alpha) . \end{aligned}$$

The last sum is now the lower estimating sum of the integral, so one can write for  $\alpha = \alpha(x)$

$$(29) \quad \Delta(x) \ll \alpha(x)(g(x) - g(1)) + \alpha(x) \cdot x \cdot g'(x) ,$$

so

$$(30) \quad \Delta(x) \ll \alpha(x)g(x) + O(1) .$$

Equation  $(1 + \alpha(x))^{L(x)} \cong x$  gives us that  $\alpha(x)$  and  $L(x) = [\log x/\alpha(x)]$  is a pair for which expression (26) holds; picking in particular  $\alpha(x) = (\log x(g(x))^{-\varepsilon})^{1/2}$ , gives

$$(31) \quad \Delta(x) \ll \sqrt{\log x} (g(x))^{1-\varepsilon/2} ,$$

and

$$L(x) \cdot x^\varepsilon \ll L(x)(g(x))^{1-\varepsilon} = \sqrt{\log x} (g(x))^{1-\varepsilon/2} .$$

Calling  $\varepsilon'$  some value  $0 < \varepsilon' < \varepsilon/2$  yields

$$(32) \quad Q_s(x) = \frac{6}{\pi^2} g(x) + O((g(x))^{1-\varepsilon'}) .$$

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