

Pacific Journal of Mathematics

**DISTRIBUTION OF SQUAREFREE INTEGERS IN NON-LINEAR
SEQUENCES**

IVAN ERNEST STUX

DISTRIBUTION OF SQUAREFREE INTEGERS IN NON-LINEAR SEQUENCES

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I. This paper investigates the occurrences of the squarefree integers in sequences $s_n = [f(n)]$, $n = 1, 2, 3, \dots$ where $f(x)$ belongs to classes of functions described by 'smoothness' conditions. The result obtained is an extension of the well known fact that $Q(x) = 6/\pi^2 x + O(x^{1/2})$, where $Q(x) =$ number of squarefree integers $\leq x$; it states that $Q_s(x) \sim 6/\pi^2 g(x)$ where $Q_s(x) =$ number of squarefree integers $\leq x$ in the sequence s_n , and $g(x)$ is the inverse function of $f(x)$.

This result relates to the deep theorem of Piatetskii-Shapiro which states that if $1 < c < 12/11$ then the sequence $[n^c]$ has the proper rate of primes occurring, namely, $\pi_c(x) \sim x^{1/c}/\log x$.

The classes of functions used is described by the following:

DEFINITION 1. for given $1 < c < 2$, $0 < \delta < 1$

- (1) $S(c, \delta) =$ set of functions $f(x)$ such that for some constant $a > 0$ depending on f , and for sufficiently large x 's, depending on f ,

$$(ax^c)^{(i)} \leq (f(x))^{(i)} < (ax^{c+\delta})^{(i)}$$

holds for $i = 0, 1, 2$, the superscripts indicating the i^{th} derivative.

Functions like x^c , $1 < c < 2$, or more generally $\sum_{i=1}^k a_i x^{c_i} (\log x)^{d_i}$, where the leading term has $a > 0$, $1 < c < 2$, belong to these classes of functions.

The following theorem will be proved:

THEOREM 1. *Let $1 < c < 4/3$, then there exists a $\delta_c = \delta(c) > 0$, some small value depending on c such that if $f(x) \in S(c, \delta)$, $0 \leq \delta < \delta(c)$, then*

$$(2) \quad Q_s(x) = 6/\pi^2 g(x) + O((g(x))^{1-\varepsilon})$$

holds for some $\varepsilon > 0$ depending on c and δ , where $Q_s(x) =$ number of $\{s_n \leq x, s_n = [f(n)], s_n =$ squarefree, $n = 1, 2, 3, \dots\}$, $g(x) =$ inverse function of f , $[z] =$ integer part of z .

II. Following are the lemmas that will be used in the proof:

LEMMA 1 (Piatetskii-Shapiro, [2]). *Let $f(x)$ be a continuously*

$g(x)$ be its inverse function. Then, for an integer m such that $m = [f(n)]$, either $\{g(m)\} = 0$ or $1 - g'(m - 1) < \{g(m)\} < 1$. Conversely, if $\{g(m)\} = 0$ or $1 - g'(m + 1) < \{g(m)\} < 1$, then it follows that for some n , $m = [f(n)]$.

(The curly brackets indicate the fractional part of the real number, the straight brackets the integer part, as usual.)

LEMMA 2 (a theorem of Erdős-Turán, [1]). If μ_1, μ_2, \dots is a real sequence and if D_N denotes its discrepancy modulo one, then for each integer $m \geq 1$ we have

$$(3) \quad ND_N \leq K \left(\frac{N}{m+1} + \sum_{t=1}^m \frac{1}{t} \left| \sum_{n=1}^N e(t\mu_n) \right| \right)$$

(where K is a constant and $e(z) = e^{2\pi iz}$, as usual).

LEMMA 3 (Van der Corput, pg. 64, [3]). Let $g(x)$ be a real function with a continuous and steadily decreasing derivative $g'(x)$ in (a, b) , and let $g'(b) = \alpha$, $g'(a) = \beta$. Then

$$(4) \quad \sum_{a < n \leq b} e(g(n)) = \sum_{\alpha - \eta < \nu < \beta + \eta} \int_a^b e(g(x) - \nu x) dx + O(\log(\beta - \alpha + 2))$$

where η is any positive constant less than one.

LEMMA 4 (Van der Corput, pg. 61, [3]). Let $F(x)$ be a real function, twice differentiable, and let $F''(x) \geq r > 0$, or $F''(x) \leq -r < 0$ throughout the interval (a, b) , then

$$(5) \quad \left| \int_a^b e^{iF(x)} dx \right| \leq \frac{8}{\sqrt{r}}.$$

III. The first part of the proof is aimed at establishing the uniform distribution modulo one and the discrepancy of that distribution for sequences $g(q)$ where q are squarefree integers and $g(x)$ is the inverse function of a function in $S(c, \delta)$ (where δ is usually small, depending on c). The following is the result in this direction:

THEOREM 2. For given $1 < c < 2$, and $\delta > 0$, small enough depending on c alone, let $f(x) \in S(c, \delta)$ and let $g(x)$ be the inverse function of $f(x)$. Then the sequence $\{g(q): q \leq K, q \text{ the squarefree integers}\}$ is uniformly distributed modulo one and

$$(6) \quad N(K, \xi) = \xi Q(K) + Q(K) D_{Q(K)}(g),$$

and

differentiable function with $f'(x) > 0$, $f''(x) \geq 0$, for $x \geq 1$, and let

$$(7) \quad Q(K)D_{Q(K)}(g) \ll K^{8/5+(c+2\delta)/(5c(c+\delta))} + K^{1-1/(2c(c+\delta))}$$

where $Q(K)$ = number of squarefree integers $\leq K$, $N(K, \xi)$ = number of elements in the sequence $g(q)$, $q \leq K$, q squarefree, which fall into a fixed interval of length ξ (< 1) modulo one, and $D_{Q(K)}(g)$ is the discrepancy, modulo one, of the sequence $g(q)$.

Clearly, uniform distribution holds whenever $\delta > 0$ is small enough to make the exponents in the estimate (7) less than one.

Proof. For $h \geq 1$, consider

$$(8) \quad T_h(K) = \sum_{\substack{q \leq K \\ q \text{ squarefree}}} e(hg(q)), \quad e(z) = e^{2\pi iz}.$$

Suppose that K_0 is the large value from where on the estimates of g, g', g'' induced by the definition 1 hold, and let $K > K_0$, then

$$(8') \quad T_h(K) = \sum_{\substack{K_0 < q \leq K \\ q \text{ squarefree}}} e(hg(q)) + O(K_0),$$

and

$$(9) \quad \sum_{\substack{K_0 < q \leq K \\ q \text{ squarefree}}} e(hg(q)) = \sum_{\substack{K_0 < n \leq K \\ n = \text{integer}}} e(hg(n)) \sum_{d^2 | (n, P^2)} \mu(d)$$

where $p = \prod_{p \leq K^{1/2}} p$, p = primes, (a, b) = greatest common divisor, $\mu(d)$ = Möbius function. We can further write

$$(10) \quad \begin{aligned} & \sum_{\substack{d \leq \sqrt{K} \\ d | P}} \mu(d) \sum_{K_0 | d^2 < m \leq K | d^2} e(hg(d^2 m)) \\ &= \sum_{\substack{d \leq A \\ d | P}} \mu(d) \sum_{K_0 | d^2 < m \leq K | d^2} e(hg(d^2 m)) \\ & \quad + \sum_{\substack{A < d \leq K^{1/2} \\ d | P}} \mu(d) \sum_{K_0 | d^2 < m \leq K | d^2} e(hg(d^2 m)). \end{aligned}$$

We will pick the value of A later. The second sum in (10) can be estimated trivially as

$$(11) \quad \sum_{A < d \leq K^{1/2}} \left| \sum_{K_0 | d^2 < m \leq K | d^2} e(hg(d^2 m)) \right| \ll \sum_{A < d \leq K^{1/2}} \frac{K}{d^2} \ll \frac{K}{A}.$$

The first sum, on the other hand, is estimated by

$$(12) \quad \sum_{d \leq A} \left| \sum_{K_0 | d^2 < m \leq K | d^2} e(hg(d^2 m)) \right|.$$

To estimate the inner sum, divide the interval $K_0/d^2 < m < K/d^2$ up into pieces of type $1/2^r K/d^2 < m \leq 1/2^{r-1} K/d^2$, to get

$$(12') \quad \sum_{d \leq A} \sum_r \left| \sum_{\substack{1/2^r K/d^2 < m \leq 1/2^{r-1} K/d^2 \\ m \equiv K_0/d^2}} e(hg(d^2 m)) \right|.$$

We will estimate the last inner sum by using Lemma 3 and then Lemma 4. The conditions in definition 1 give that

$$\left(\frac{y}{a}\right)^{(c+\delta)^{-1}} < g(y) \leq \left(\frac{y}{a}\right)^{1/c}, \quad \frac{1}{c+\delta} \left(\frac{y}{a}\right)^{(c+\delta)^{-1-1}} < g'(y) \leq \frac{1}{c} \left(\frac{y}{a}\right)^{1/c},$$

the chain rule tells us that $(d/dx)g(d^2x) = [(d/dz)g(z)] \cdot d^2$, $z = d^2x$, and so we have, by Lemma 3, for each r

$$(13) \quad \sum_{1/2^r K/d^2 < m \leq 1/2^{r-1} K/d^2} e(hg(d^2 m)) = \sum_\nu I_\nu + E,$$

where the \sum_ν extends over $(1/a_1(c+\delta))h((1/2^{r-1}K)^{(c+\delta)^{-1-1}}d^2 - 1/2 < \nu < (1/a_2c)h((1/2^r)K)^{c-1-1}d^2 + 1/2$, $a_1 = a^{(c+\delta)^{-1}}$, $a_2 = a^{1/c}$, and

$$(14) \quad I_\nu = \int_{2^{-r}Kd^{-2}}^{2^{-r+1}Kd^{-2}} e(hg(d^2x) - \nu x)dx,$$

and

$$E = O(\log(\max \nu - \min \nu + 2)).$$

In (14), first we change variables to $y = d^2x$, and then apply Lemma 4

$$(14') \quad I_\nu = \frac{1}{d^2} \int_{2^{-r}K}^{2^{-r+1}K} e(hg(y) - \frac{\nu y}{d^2})dy$$

but here $d^2/dy^2(hg(y) - \nu y/d^2) \geq (c+\delta)^{-1}((c+\delta)^{-1} - 1)(1/a)hy^{(c+\delta)^{-1-2}}$. thus, we get, applying Lemma 4 that

$$(15) \quad I_\nu \ll \frac{1}{d^2} \left[h\left(\frac{1}{2^{r-1}}K\right)^{(c+\delta)^{-1-2}} \right]^{-1/2}.$$

We thus have for (12) the estimate:

$$(16) \quad \ll \sum_{d \leq A} \sum_r \sum_\nu \frac{1}{d^2} h^{-1/2} \frac{2^{(r-1)/(c+\delta)c}}{2^{r-1}} K^{1-1/(2c+2\delta)} + \sum_{d \leq A} \sum_r (E)$$

(for largest r we might get a shorter range of integration in (14), but the upper bound estimates still clearly hold in (16)). where \sum_ν is over

$$\frac{1}{a(c+\delta)} h\left(\frac{1}{2^{r-1}}K\right)^{(c+\delta)^{-1-1}} d^2 - \frac{1}{2} < \nu < \frac{1}{ac} h\left(\frac{1}{2^r}K\right)^{c-1-1} d^2 + \frac{1}{2}.$$

From here, we have that the ν summation is bounded by

$$\ll hd^2 2^r \left(\frac{1}{2^{r-1}}\right)^{1/(c+\delta)} K^{c-1-1} + 1$$

and so we can further estimate (12) by

$$(16') \quad \ll \sum_{d \leq A} \sum_r \left(h d^2 \frac{2^r}{2^{(r-1)/(c+\delta)}} K^{(1/c)-1} + 1 \right) \left(\frac{h^{-1/2}}{d^2} \frac{2^{(r-1)/(2(c+\delta))}}{2^{r-1}} K^{1-1/(2c+2\delta)} \right) \\ + \sum_{d \leq A} \sum_r E \ll Ah^{1/2} K^{e^{-1-1/(2c+2\delta)}} + h^{-1/2} K^{1-1/(2c+2\delta)} + A \log K.$$

(The last step is because \sum_r in the first term was just a geometric sum and so it converges, while in the second part, the number of terms of the \sum_r is $O(\log K)$.) The estimate (16') together with (11) now gives us that

$$(17) \quad T_h(K) \ll Ah^{1/2} K^{(c+2\delta)/(c+\delta)2c} + h^{-1/2} K^{1-1/(2c+2\delta)} + \frac{K}{A} + K_0.$$

Here, for K sufficiently large the last error term absorbs into the first one if $A \geq 1$ (which will anyway be the case). We now pick A so as to balance the 1st and 3rd terms of (17), i.e. let $A = [K^{2^{-1-(c+2\delta)/(c+\delta)4c}} h^{-1/4}]$. With this choice we obtain

$$(18) \quad T_h(K) \ll h^{1/4} K^{2^{-1+(c+2\delta)/(c+\delta)4c}} + h^{-1/2} K^{1-1/(2c+2\delta)}.$$

Finally we use Lemma 2 to write:

$$(19) \quad Q(K)D_{Q(K)}(g) \ll \frac{K}{m+1} + \sum_{h=1}^m \frac{1}{h} \{h^{1/4} K^{2^{-1+(c+2\delta)/(c+\delta)4c}} + h^{-1/2} K^{1-1/(2c+2\delta)}\} \\ \ll \frac{K}{m+1} + m^{1/4} K^{2^{-1+(c+2\delta)/(c+\delta)4c}} + K^{1-1/(2c+2\delta)}.$$

We pick the optimal m , i.e. $m = [K^{4(2^{-1-(c+2\delta)/(c+\delta)4c})/5}]$, and thus we have

$$(20) \quad Q(K)D_{Q(K)}(g) \ll K^{3/5+(c+2\delta)/(c+\delta)4c} + K^{1-1/(2c+2\delta)}.$$

COROLLARY 1. *If $1 < c < 4/3$ then there exists $\delta_c > 0$ depending on c such that if $f(x) \in S(c, \delta)$ for $0 < \delta < \delta_c$, and $g(x)$ is its inverse function then:*

$$(21) \quad Q(K)D_{Q(K)}(g) \ll K^{(c+\delta)^{-1-\varepsilon}} \ll (g(K))^{1-\varepsilon}$$

for some $\varepsilon > 0$, depending on c and δ .

Proof. All we need to show is that

$$\frac{3}{5} + \frac{c+2\delta}{5c(c+\delta)} < \frac{1}{c+\delta} \quad \text{and} \quad 1 - \frac{1}{2(c+\delta)} < \frac{1}{c+\delta}$$

hold for some $\delta > 0$. By continuity it is enough to check that $3/5 + 1/5c < 1/c$ and $1 - 1/2c < 1/c$ hold. But the first of these holds if

$c < 4/3$, the second if $c < 3/2$.

IV. We can now prove Theorem 1. Let

$$(22) \quad T_s(x, y) = \text{number of } \{s_n = [f(n)], y < s_n \leq x, s_n = \text{squarefree}, n = 1, 2, 3, \dots\}$$

Clearly, $T_s(1, y) = Q_s(y)$. Lemma 1 can now be used together with expressions (6) and (21). ξ in (6) will be taken $g'(y + 1)$ or $g'(x - 1)$ to give upper and lower bounds on $T(x, y)$, where $g(x)$ is as usual the inverse function of $f(x)$. We obtain:

$$(23) \quad T_s(x, y) \begin{cases} < g'(x - 1)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) \\ > g'(y + 1)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) \end{cases}$$

where $Q(x) = \# \text{ squarefree integers } \leq x$. Or

$$(24) \quad T_s(x, y) \begin{cases} < g'(x)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) + O(x^{c-1-2}(Q(y) - Q(x))) \\ > g'(y)(Q(y) - Q(x)) + O(y^{(c+\delta)^{-1-\epsilon}}) + O(y^{c-1-2}(Q(y) - Q(x))) \end{cases}.$$

Thus, for $0 < \alpha < 1$, using the well-known fact that $Q(x) = 6\pi^{-2}x + O(x^{1/2})$,

$$(25) \quad T_s(x, (1 + \alpha)x) = \begin{cases} < \frac{6}{\pi^2} x \cdot \alpha \cdot g'(x) \\ > \frac{6}{\pi^2} x \cdot \alpha \cdot g'((1 + \alpha)x) \end{cases} + O(x^{(c+\delta)^{-1-\epsilon}} + x^{1/2}).$$

On the other hand, clearly

$$(26) \quad Q_s(x) = \sum_{k=1}^{L(x)} T_s\left(\frac{x}{(1 + \alpha)^k}, \frac{x}{(1 + \alpha)^{k-1}}\right) + O(1)$$

holds for an appropriate function $L(x)$ which tends to ∞ for $x \rightarrow \infty$, if $\alpha = \alpha(x) > 0$ is some given function of x which tends to zero as $x \rightarrow \infty$ (the relation is $(1 + \alpha(x))^{L(x)} \cong x$).

Using (25) in the expression (26) we obtain

$$(27) \quad Q_s(x) \begin{cases} < \frac{6}{\pi^2} \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1 + \alpha)^k} g'\left(\frac{x}{(1 + \alpha)^k}\right) + O(L(x) \cdot x^\gamma) \\ > \frac{6}{\pi^2} \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1 + \alpha)^k} g'\left(\frac{x}{(1 + \alpha)^{k-1}}\right) + O(L(x) \cdot x^\gamma) \end{cases}$$

where $\gamma = \max \{1/(c + \delta) - \epsilon, 1/2\}$ and so it is actually $1/(c + \delta) - \epsilon$. The main terms of the expressions on the right of (27) are exactly the upper and lower approximating sums of the Riemann integral

$$\int_1^x g'(y)dy = g(x) - g(1) .$$

To see how closely these sums approximate the integral, it suffices to find out how closely they are to each other, i.e. to estimate:

$$\begin{aligned} \Delta(x) &= \left| \sum_{k=1}^{L(x)} \frac{x\alpha}{(1+\alpha)^k} g'\left(\frac{x}{(1+\alpha)^{k-1}}\right) \right. \\ &\quad \left. - \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1+\alpha)^k} g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &= \left| \sum_{k=0}^{L(x)-1} \frac{x \cdot \alpha}{(1+\alpha)^{k+1}} g'\left(\frac{x}{(1+\alpha)^k}\right) \right. \\ (28) \quad &\quad \left. - \sum_{k=1}^{L(x)} \frac{x \cdot \alpha}{(1+\alpha)^k} g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &\leq \left| \alpha \cdot \sum_{k=1}^{L(x)-1} \frac{x}{(1+\alpha)^k} \left(\frac{1}{1+\alpha} - 1\right) g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &\quad + |x \cdot \alpha \cdot g'(x)| + O(\alpha) \\ &\ll \left| \alpha \cdot \sum_{k=1}^{L(x)-1} \frac{x \cdot \alpha}{(1+\alpha)^{k+1}} g'\left(\frac{x}{(1+\alpha)^k}\right) \right| \\ &\quad + O|x \cdot \alpha \cdot g'(x)| + O(\alpha) . \end{aligned}$$

The last sum is now the lower estimating sum of the integral, so one can write for $\alpha = \alpha(x)$

$$(29) \quad \Delta(x) \ll \alpha(x)(g(x) - g(1)) + \alpha(x) \cdot x \cdot g'(x) ,$$

so

$$(30) \quad \Delta(x) \ll \alpha(x)g(x) + O(1) .$$

Equation $(1 + \alpha(x))^{L(x)} \cong x$ gives us that $\alpha(x)$ and $L(x) = [\log x/\alpha(x)]$ is a pair for which expression (26) holds; picking in particular $\alpha(x) = (\log x(g(x))^{-\varepsilon})^{1/2}$, gives

$$(31) \quad \Delta(x) \ll \sqrt{\log x} (g(x))^{1-\varepsilon/2} ,$$

and

$$L(x) \cdot x^\varepsilon \ll L(x)(g(x))^{1-\varepsilon} = \sqrt{\log x} (g(x))^{1-\varepsilon/2} .$$

Calling ε' some value $0 < \varepsilon' < \varepsilon/2$ yields

$$(32) \quad Q_s(x) = \frac{6}{\pi^2} g(x) + O((g(x))^{1-\varepsilon'}) .$$

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