ON HOMOGENEOUS ALGEBRAS

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If $A$ is an algebra over a field $K$ let $\text{Aut}(A)$ denote the group of algebra automorphisms of $A$. Then $A$ is said to be extremely homogeneous if $\text{Aut}(A)$ act transitively on $A \setminus \{0\}$. Also $A$ is said to be homogeneous if $\text{Aut}(A)$ acts transitively on the one-dimensional subspaces of $A$. The purpose of this paper is to investigate some of the basic properties of homogeneous algebras. In particular, the alternative homogeneous algebras and the homogeneous algebras of dimension 2 are classified.

All algebras are assumed to be finite dimensional and not necessarily associative.

We now include a brief historical account of this topic. The concept of an extremely homogeneous algebra arose from a particular problem in the structure of certain finite $p$-groups as studied by Boen, Rothaus and Thompson [1]. Extremely homogeneous algebras have been investigated by Kostrikin [4]. Homogeneous algebras over finite fields other than $GF(2)$ have been investigated by Shult [6], [7], and his results completed the work on the related $p$-groups. The case of homogeneous algebras over $GF(2)$ was considered by Gross [3]. Swierczkowski classified all real homogeneous Lie algebras [9] and finally Dyokovic classified all real homogeneous algebras [2]. A homogeneous algebra $A$ is said to be nontrivial if $A^2 \neq 0$ and $\dim A > 1$. The author has shown that there are no nontrivial homogeneous algebras over an algebraically closed field [8].

The paper is divided into five sections: arbitrary homogeneous algebras, alternative homogeneous algebras, power-associative homogeneous algebras, homogeneous quasi-division algebras and finally homogeneous algebras of dimension 2.

I. Arbitrary homogeneous algebras. Let $A$ be an arbitrary algebra over a field $K$. Then left multiplication by a fixed element $a \in A$ induces a linear map on $A$ which is denoted by $L_a$. Similarly right multiplication by $a$ induces a linear map on $A$ denoted by $R_a$. We do not distinguish between the map $L_a$ and its matrix representation relative to some fixed basis. By $\text{End}(A)$ we indicate the vector space of all linear maps on $A$. By $L$ we indicate the subspace of $\text{End}(A)$ consisting of all $L_x$ as $x$ runs through $A$ and similarly for $R$. An algebra $A$ is said to be nonzero if $A^2 \neq 0$.

**Theorem 1.** Let $A$ be a nonzero homogeneous algebra over a
field $K$. Then

(i) $\dim L = \dim R = \dim A$

(ii) If $a, b \in A \setminus \{0\}$ then $L_a$ and $L_b$ are projectively similar and similarly for $R_a$ and $R_b$.

(iii) $\text{Aut}(A)$ acts as a transitive group of collineations on the points of the projective geometry $P(A)$.

Proof. (1) Let $a \in A \setminus \{0\}$. Then if $aA = 0$ the homogeneity condition implies that $A^2 = 0$ which is a contradiction. This fact implies that the map $\varphi: x \to L_a$ is a linear isomorphism and so $\dim L = \dim A$. Similarly it can be shown that $\dim R = \dim A$.

(2) The proof is a simple generalization of a related result found in the introduction of the paper by Boen, Rothaus, and Thompson [1].

(3) This is obvious since the points of $P(A)$ are exactly the one-dimensional subspaces of $A$.

**Theorem 2.** Let $A$ be a nontrivial homogeneous algebra over a field $K$. Then

$$tr L_x = tr R_x = 0 \quad \forall x \in A$$

Proof. Let $\dim A = n$. It is well known that $tr: \text{End}(A) \to K$ is a linear functional and that $\dim \ker (tr) = n^2 - 1$. But then since $\dim L = \dim A = n > 1$ it follows that $L \cap \ker (tr) \neq 0$ and so there must exist at least one nonzero map $L_a \in L$ such that $tr L_a = 0$. But now the second result of the previous theorem implies that $tr L_x = 0$ for all $x \in A$. Similarly $tr R_x = 0$ for all $x \in A$.

**Theorem 3.** Let $A$ be a homogeneous algebra over a field $K$ and let $a \in A \setminus \{0\}$. If $\langle a \rangle$ denotes the subalgebra of $A$ generated by $a$ then $\langle a \rangle$ is also a homogeneous algebra enjoying the property that it is generated by each of its nonzero elements. Also $A = \bigcup A_i$ where each $A_i = \langle a_i \rangle$ for some $a_i \in A \setminus \{0\}$ and $A_i \cap A_j = \{0\}$ for $i \neq j$.

Proof. Let $b \in \langle a \rangle$. Clearly $\langle b \rangle \subseteq \langle a \rangle$. But there must exist $\alpha \in \text{Aut}(A)$ such that $\alpha(a) = \lambda b$ for some nonzero $\lambda \in K$ and this implies that $\langle a \rangle \subseteq \langle b \rangle$ and so $\langle a \rangle = \langle b \rangle$. That is $\langle a \rangle$ is generated by each of its nonzero elements. Now let $c$ and $d$ be any nonzero elements in $\langle a \rangle$. Again there must exist $\beta \in \text{Aut}(A)$ such that $\beta(c) = \lambda d$ for some nonzero $\lambda \in K$. But the fact that both $c$ and $d$ generate $\langle a \rangle$ implies that $\langle a \rangle$ is invariant under $\beta$ and so the restriction of $\beta$ to $\langle a \rangle$ is in $\text{Aut}(\langle a \rangle)$. That is, $\langle a \rangle$ is also a homogeneous algebra. The final statement again follows directly from the fact that $\langle a \rangle$ is generated by each of its nonzero elements.
The above theorem implies that in some situations it is sufficient to consider the case where a homogeneous algebra $A$ is generated by each of its nonzero elements.

**Definition.** Let $V$ be a vector space over a field $K$ and suppose $H$ is a subgroup of $GL(V)$ where $GL(V)$ is the general linear group. Then $C(H)$ is defined as

$$C(H) = \{u \in \text{End}(A) \mid uv = vu \text{ for all } V \in H\}.$$ 

**Definition.** Let $A$ be an algebra over a field $K$ and suppose $S, T \in C(\text{Aut}(A))$. Then $A(S, T)$ indicates a new algebra which coincides with $A$ when considered as a vector space over $K$ but possesses a new multiplication defined by

$$a \odot b = S(a)b + T(b)a \text{ for all } a, b \in A$$

Note that the fact that $S$ and $T$ are linear maps on $A$ ensure that $\circ : A \times A \to A$ is a bilinear map. Also the algebras $A(1,1)$, $A(1,-1)$ and $A(0,1)$ are well known and are usually denoted as $A^+$, $A^-$ and $A^\text{opp}$ respectively.

**Theorem 4.** Let $A$ be a homogeneous algebra over a field $K$ and suppose $S, T \in C(\text{Aut}(A))$. Then $A(S, T)$ is also a homogeneous algebra.

**Proof.** Let $\sigma \in \text{Aut}(A)$. Then

$$\sigma(a \odot b) = \sigma(S(a)b + T(b)a)$$

$$= \sigma(S(a)b) + \sigma(T(b)a)$$

$$= (\sigma S(a))\sigma(b) + (\sigma T(b))\sigma(a)$$

$$= (S\sigma(a))\sigma(b) + (T\sigma(b))\sigma(a)$$

$$= \sigma(a) \odot \sigma(b)$$

and so the result is true since $\text{Aut}(A) \subseteq \text{Aut}(A(S, T))$.

**Definition.** Let $A$ be an algebra over a field $K$. Then $A$ is left (right) simple if $A$ possesses no nonzero proper left (right) ideals. Also $A$ is simple if $A$ possesses no nonzero, proper, two-sided ideals and $A^2 \neq 0$.

**Theorem 5.** If $A$ is a nonzero homogeneous algebra then $A$ is left simple and right simple.

**Proof.** Assume that $A$ has proper nonzero left ideals. When
B runs through minimal left ideals then the sets \( B \setminus \{0\} \) form a partition of \( A \setminus \{0\} \). Suppose \( a \in A \setminus \{0\} \) and let \( I(a) \) denote the minimal left ideal which contains \( a \). Now \( R_0 \) map \( A \rightarrow I(a) \) and since \( I(a) \neq A \) it follows that \( R_0 \) has a nonzero kernel. That is, there exists \( b \in A \setminus \{0\} \) such that \( ba = 0 \). Let \( c \) be any point in \( A \setminus I(a) \). Then \( I(c) \cap I(a) = \{0\} \) which implies that \( I(c) \cap I(c + a) = \{0\} \). But \( b(c + a) = bc \) and so \( bc \in I(c) \cap I(c + a) \) which implies that \( bc = 0 \). Now fix some nonzero \( c \in A \setminus I(a) \) and let \( d \) be any point in \( I(c) \setminus \{0\} \). Then \( c + d \in A \setminus I(a) \) and so \( b(c + d) = bd = 0 \). Hence \( bA = 0 \) which is impossible since \( A \) is a nonzero homogeneous algebra. Hence \( A \) has no proper nonzero left ideals and similarly \( A \) has no proper nonzero right ideals.

II. Alternative homogeneous algebras. The following definition is well known.

**Definition.** An algebra \( A \) over a field \( K \) is said to be alternative if

\[
a^2b = a(ab) \\
ab^2 = (ab)b
\]

for all \( a, b \in A \).

**Theorem 6.** There are no nontrivial alternative homogeneous algebras.

**Proof.** Let \( A \) be a nontrivial alternative homogeneous algebra. Then the previous theorem implies that \( A \) is simple. But it is known that a simple alternative algebra has an identity element 1 (see Corollary 3.11 of Schafer's book [5]). But then \( A \) is certainly not homogeneous since \( \alpha(1) = 1 \) for all \( \alpha \in \text{Aut}(A) \).

Note that the above theorem of course implies that there are no nontrivial associative homogeneous algebras.

III. Power-associative homogeneous algebras.

**Theorem 7.** Let \( A \) be a power-associative nontrivial homogeneous algebra over a field \( K \). Then either \( a^2 = 0 \) for all \( a \in A \) or \( a^2 = a \) for all \( a \) in \( A \) and in the latter case \( A \) is a Jordan algebra and \( K = GF(2) \).

**Proof.** Let \( a \) be some fixed element in \( A \setminus \{0\} \). Then Theorem 3 implies that \( \langle a \rangle \) is an associative homogeneous algebra and so the previous theorem implies that \( \langle a \rangle \) is a trivial homogeneous algebra.
It follows that either $a^2 = 0$ or $a^2 = \lambda a$ for some nonzero $\lambda \in K$. In the former case the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and so we may assume the latter case. The homogeneity condition implies that $x^2 = \lambda(x)x$ where $\lambda(x)$ is a nonzero scalar in $K$ possibly depending on $x$. Since $\text{dim } A > 1$ we may choose two independent vectors in $A$, say $e_1$ and $e_2$. Since $a^2 = \lambda a$ implies that $(a/\lambda)^2 = a/\lambda$ we may assume without loss of generality that both $e_1$ and $e_2$ are idempotents. It is now necessary to perform several simple calculations. First

\[
(e_1 + e_2)^2 = e_1 + e_2 + e_1e_2 + e_2e_1 = \lambda(e_1 + e_2)(e_1 + e_2)
\]
\[
(e_1 - e_2)^2 = e_1 + e_2 - e_1e_2 - e_2e_1 = \lambda(e_1 - e_2)(e_1 - e_2)
\]

Now adding and comparing coefficients gives

\[
2 = \lambda(e_1 + e_2) + \lambda(e_1 - e_2)
\]
\[
2 = \lambda(e_1 + e_2) - \lambda(e_1 - e_2)
\]

or

\[
2\lambda(e_1 - e_2) = 0
\]

which implies that $\text{char } K = 2$.

For convenience let $\mu = \lambda(e_1 + e_2)$. Then from above

\[
e_1e_2 + e_2e_1 = (\mu + 1)(e_1 + e_2)
\]

Now consider

\[
(e_1 + e_2)^2 = e_1 + \mu^2e_2 + \mu(e_1e_2 + e_2e_1)
\]
\[
= (\mu^2 + \mu + 1)e_1 + \mu e_2
\]

from which it follows that $\mu^2 + \mu + 1 = 1$ which implies with $\text{char } K = 2$ that $\mu = 1$ and so

\[
e_1e_2 + e_2e_1 = 0
\]

Now let $\delta$ be any nonzero scalar in $K$. Then

\[
(e_1 + \delta e_2)^2 = e_1 + \delta^2 e_2 + \delta(e_1e_2 + e_2e_1) = e_1 + \delta^2 e_2
\]

which implies that $\delta^2 = \delta$ and so $\delta = 1$ and indeed $K = GF(2)$. Hence $x^2 = x$ for all $x \in A$. But then

\[
(x + y)^2 = x + y + xy + yx = x + y
\]

and so $xy = yx$ for all $x, y \in A$ and thus $A$ is a commutative algebra. The second identity for a Jordan algebra is trivally satisfied and so $A$ is a Jordan algebra over $GF(2)$. 

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It is interesting to note that Dyokovic has shown that all non-trivial real homogeneous algebras are of the first type [2] and Gross has shown that some, but not all, of the known homogeneous algebras over $GF(2)$ are of the second type [3].

IV. Homogeneous quasi-division algebras.

DEFINITION. An algebra $A$ over a field $K$ is said to be a quasi-division algebra if the nonzero elements of $A$ form a quasi-group under multiplication.

One of the reasons for devoting a separate section to homogeneous quasi-division algebras is that Shult [6] and Gross [3] have shown that all nontrivial finite homogeneous algebras are in fact quasi-division algebras.

**Theorem 8.** Let $A$ be a nontrivial homogeneous quasi-division algebra with the property that $A$ is generated by each of its nonzero elements. Then

(i) $\text{Aut}(A)$ is sharply transitive on the one-dimensional subspaces of $A$

(ii) If $a$ is any element in $A\setminus\{0\}$ then $L_a$ has precisely one eigenvalue denoted by $\lambda_a \in K$ and the corresponding eigenspace is one-dimensional

(iii) Finally $\lambda_a = \lambda_b$ if and only if there exists some $\alpha \in \text{Aut}(A)$ such that $\alpha(a) = b$.

**Proof.** (1) It is sufficient to show that no automorphism of $A$, except the identity $\text{Id}$, can have an eigenvalue in $K$. Let $\alpha \in \text{Aut}(A)$ and suppose that $\alpha$ has eigenvalue $\lambda \in K$. Then there exists $a \in A\setminus\{0\}$ such that $\alpha(a) = \lambda a$.

Since $A$ is not associative by Theorem 6 we define inductively

$$a^n = L_a^{-1}(a)$$

$n = 2, 3, 4, \ldots$

But now

$$\alpha(a^n) = \lambda^n a^n$$

$n = 1, 2, 3, \ldots$

and so there must exist positive integers $m, n$ with $m > n$ such that $\lambda^m = \lambda^n$ since $\alpha$ can only have a finite number of eigenvalues. Letting $k = m - n$ we have

$$\alpha(a^k) = \lambda^k a^k = a^k \neq 0$$
and so \( \alpha = \text{Id} \) since from the hypothesis we are assuming that \( a^k \) generates \( A \).

(2) Let \( a \) and \( b \) be any two nonzero elements of \( A \). Since \( A \) is a quasi-division algebra the equation

\[ xb = b \]

must have a solution, say \( c \) and the homogeneity condition implies that there exists \( \alpha \in \text{Aut}(A) \) such that

\[ \alpha(c) = \lambda c \quad \text{for some } \lambda \in K\setminus\{0\} . \]

But then

\[ a\alpha(b) = 1/\lambda \alpha(b) \]

and so \( L_a \) has at least one eigenvalue.

Now suppose there exist nonzero elements \( b, c \in A \) such that

\[ ab = \lambda b \]
\[ ac = \mu c \]

where \( \lambda, \mu \in K \). If \( \{b, c\} \) is an independent set then there must exist \( \alpha \in \text{Aut}(A) \) such that

\[ \alpha(c) = \delta b \quad \text{for some } \delta \in K \]

But then

\[ \alpha(a)b = \mu b \]

and thus

\[ (\lambda \alpha(a) - \mu a)b = 0 \]

which implies that \( \alpha = \text{Id} \) by the previous part of this theorem. Thus \( L_a \) has precisely one eigenvector (up to a scalar multiple) which completes the proof of the second statement.

(3) Finally if \( \alpha \in \text{Aut}(A) \) then \( ax = \lambda_a x \) for some \( \lambda \in A\setminus\{0\} \) implies that

\[ \alpha(a)\alpha(x) = \lambda_a \alpha(x) \]

and so

\[ \lambda_{\alpha(a)} = \lambda_a \]

also if \( \lambda_a = \lambda_b \) then there exists \( x, y \in A\setminus\{0\} \) such that

\[ ax = \lambda_a x \]
\[ by = \lambda_b y = \lambda_a y \].
Now choose $\beta \in \text{Aut}(A)$ such that $\beta(x) = \mu y$ for some $\mu \in K \setminus \{0\}$ and applying $\beta$ we obtain

$$\beta(a)y = \lambda \beta y = by$$

and so it follows that $\beta(a) = b$ as required.

IV. On homogeneous algebras of dimension 2. We now investigate arbitrary homogeneous algebras of dimension 2.

**Theorem 9.** Let $A$ be a nonzero, 2-dimensional, homogeneous algebra over a field $K$. Then $K = GF(2)$ and $A$ has a basis $\{a, b\}$ so that $A$ is isomorphic to one of the following algebras.

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<td>$b$</td>
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Proof. Let $a \in A \setminus \{0\}$. Then there are exactly three possibilities which will be considered separately

(i) $a^2 = 0$

(ii) $a^2 = \lambda a$ for some nonzero $\lambda \in K$

(iii) $\{a, a^2\}$ is a basis of $A$

(1) If $a^2 = 0$ then the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and the linearized form of this identity implies that $A$ is anticommutative. Extend $a$ to a basis of $A$, say $\{a, b\}$. Using the fact that $\text{tr } L_a = 0$ and $L_a \neq 0$ it follows that $ab = \lambda a$ for some nonzero $\lambda \in K$. But now $ab = \lambda a$ and $b^2 = 0$ imply that $\text{tr } L_b = -\lambda \neq 0$ which is impossible. Hence this case does not occur.

(2) If $a^2 = \lambda a$ where $\lambda \neq 0$ then the homogeneity condition implies that $A$ is power-associative and so Theorem 7 implies that $K = GF(2)$. Again extend $a$ to a basis of $A$, say $\{a, b\}$. Using the fact that $\text{tr } L_a = \text{tr } L_b = 0$ and $L_a \neq 1$ and $L_b \neq 1$ it follows that $A$ must be of the form

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<td>$a$</td>
<td>$a + b$</td>
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<td>$a + b$</td>
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By direct computations it can be shown that $\text{Aut}(A) = GL(2, 2)$ and so $\text{Aut}(A)$ is in fact triply transitive on $A \setminus \{0\}$.

(3) Suppose that $\{a, a^2\}$ is a basis of $A$. First pass from $A$ to $A^-$. By Theorem 4, $A^-$ is also a homogeneous algebra and clearly $A^-$ is of type (1) as defined above and so $A^-$ must be a zero algebra
which implies that $A$ is commutative. If $aa^2 = 0$ then $\text{tr} L_{a^2} = 0$ and $L_{a^2} \neq 0$ implies that $L_a$ is nilpotent but $L_{a+a^2}$ is invertible and so $A$ is a quasi-division algebra generated by each of its nonzero elements and so we may apply Theorem 8. Assume $aa^2 = \mu a$.

Let $b$ be any fixed nonzero element of $A$. The equation $xb = b$ must have a solution and without loss of generality we may assume that $x = a$. Hence the only eigenvalue of $L_a$ is 1 and it follows that $\mu = 1$ and $\text{char } K = 2$. Also $a^2a^2 = va + a^2$ for some nonzero $v \in K$. Now since $L_a$ and $L_{a^2}$ both have eigenvalue 1 it follows from Theorem 8 that there must exist $\alpha \in \text{Aut}(A)$ such that $\alpha(a) = a^2$. But then

$$\alpha(a^2) = (v \alpha + a^2)(va + a^2) = v^2a^2 + va + a^2.$$  

It follows that $v = 1$ and so the multiplication table of $A$ is of the form

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<th>$a$</th>
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<tbody>
<tr>
<td>$a$</td>
<td>$a^2$</td>
<td>$a$</td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td>$a + a^2$</td>
</tr>
</tbody>
</table>

If $K = GF(2)$ it is easily shown that $A$ is in fact a homogeneous algebra. If $K = GF(4)$ it can be shown that $\det (L_a + \lambda L_{a^2}) = 1 + \lambda + \lambda^2 = 0$ for some $\lambda \in GF(4)$ and so $A$ is not homogeneous since it is not a quasi-division algebra. Now assume that $K \neq GF(2)$ and $K \neq GF(4)$. Then there must exist $\lambda_0 \in K$ such that $\lambda_0$ is not a root of the polynomial $x^2 + x + 1$ or of the polynomial $x^4 + x^3 + x^2 + 1$. Since $A$ is homogeneous there must exist $\alpha \in \text{Aut}(A)$ such that

$$\alpha(a) = \lambda(a + \lambda_0 a^2) \quad \text{for some nonzero } \lambda \in K.$$  

But then

$$\alpha(aa^2) = \lambda'(1 + \lambda_0 + \lambda_0^2)a + \lambda^3\lambda_0(1 + \lambda_0 + \lambda_0^2)a^2$$  

and so

$$\lambda^2 = \frac{1}{1 + \lambda_0 + \lambda_0^2}.$$  

Also

$$\alpha(a^2a^2) = \lambda_1(1 + \lambda_0) + \lambda_2 a^2$$  

and

$$(\lambda + \lambda_0 \lambda_0) a + [\lambda \lambda_0 + \lambda^2(1 + \lambda_0^2)] a^2$$
which implies using (1) that

\[ \lambda^2 = \frac{1 + \lambda^2_o + \lambda^4_o}{1 + \lambda^2_o + \lambda^6_o} \]

and together (1) and (2) imply that

\[ \lambda^4_o + \lambda^3_o + \lambda^5_o + 1 = 0 \]

which contradicts our choice of \( \lambda_o \). Hence \( A \) is a homogeneous algebra if and only if \( K = GF(2) \).

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