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ON DOUBLY HOMOGENEOUS ALGEBRAS

LOWELL G. SWEET

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The algebras to be discussed are assumed to be finite dimensional and not necessarily associative. If A is an algebra over a field K let $\operatorname{Aut}(A)$ denote the group of algebra automorphisms of A. We define A to be doubly homogeneous if $\operatorname{Aut}(A)$ is doubly transitive on the one-dimensional subspaces of A. Also a doubly homogeneous algebra A is said to be nontrivial if $A^2 \approx 0$ and dimension A > 1. It is shown that the only nontrivial doubly homogeneous algebra is unique up to isomorphism.

An algebra A is said to be homogeneous if Aut (A) acts transitively on the one-dimensional subspaces of A. The reader is referred to the author's previous paper [1] for a discussion of homogeneous algebras and a bibliography of the related literature.

An arbitrary algebra A is said to be nonzero if $A^2 \neq 0$. If the nonzero elements of A form a quasi-group under multiplication then we say that A is a quasi-division algebra.

LEMMA. If A is a nonzero doubly homogeneous algebra over a field K then A is a quasi-division algebra.

Proof. Let dim A = n. If n = 1 then A is isomorphic to K and the result is obvious and so we assume that n > 1. Let a be any element of A. We claim that if $b \notin Ka$ then $ab \neq 0$. For if ab = 0 the doubly homogeneity condition implies that ac = 0 for all c such that $c \notin Ka$. But then in particular $b + a \notin Ka$ and so a(b + a) = 0 which implies that $a^2 = 0$ and thus aA = 0. In this case the homogeneity condition implies that $A^2 = 0$ which is a contradiction and the claim is verified.

Now suppose that $a^2 = 0$. Then the homogeneity condition implies that $x^2 = 0$ for all $x \in A$. Suppose there exists $b \notin Ka$ such that

$$ab \in Ka$$
.

Then by doubly homogeneity we would also have

$$(a+b)b \in K(a+b)$$

and $b^2 = 0$ implies that

$$ab \in Ka \cap K(a + b) = \{0\}$$

which is impossible. Fix some $b \notin Ka$. Let c be any nonzero element

of A. Then there must exist $\alpha \in Aut(A)$ such that

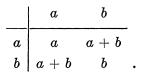
 $\alpha(ab) \in Kc$

and

$$\alpha(a) \in Ka$$
.

This implies that L_a (left multiplication by a) is a surjective map which is impossible and so $a^2 \neq 0$. Hence L_a is invertible and the homogeneity condition implies that A is a quasi-division algebra.

THEOREM. If A is a nonzero doubly homogeneous algebra over a field K then either $A \cong K$ or K = GF(2) and A is isomorphic to the following algebra



Proof. If dim A = 1 then clearly $A \cong K$. If dim A = 2 then A must be contained in the authors list of 2-dimensional homogeneous algebras [1] and it is easily checked that the only possibility is that K = GF(2) and A is isomorphic to the following algebra

Hence to prove the theorem it is sufficient to show that there exist no nonzero doubly homogeneous algebras of dimension n > 2.

Let A be a nonzero doubly homogeneous algebra of dimension n > 2. If a is any fixed nonzero element in A then the lemma implies that the equation ax = a must have a unique solution, say b and the doubly homogeneity condition now implies that $b \in Ka$. It follows that A is a nonzero, power-associative, homogeneous algebra and so Theorem 7 of the author's previous paper [1] implies that K = GF(2).

Now let a and b be any two distinct nonzero elements of A and let $A_1 = \langle a, b \rangle$ be the subalgebra of A generated by a and b. It can be shown that A_1 is also a doubly homogeneous algebra and it is generated by any two distinct nonzero elements. Hence only the identity automorphism of A_1 can fix two distinct nonzero elements of A_1 and so Aut (A_1) is sharply doubly transitive on $A_1 \setminus \{0\}$. Hence the order of Aut (A_1) must be even and so Aut (A_1) must contain at least one involution, say α . This involution α fixes at most 1 one-

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dimension subspace of A_1 . But since any involution acting on a vector space V over a field of characteristic 2 fixes vectorwise a subspace of dimension $\geq 1/2$ dim V this forces dim $A_1 = 2$ and so we may assume that

$$ab = a + b$$
.

But since A is doubly homogeneous it follows that

$$x^2 = x$$
 for all $x \in A$
 $xy = x + y$ whenever $y \notin Kx$.

Now since n > 2 we can choose three independent vectors $a, b, c \in A$. But then

$$(a+b)c=a+b+c$$

and

$$ac+bc=a+c+b+c=a+b$$

which is impossible and the proof is complete.

Reference

1. L. G. Sweet, On homogeneous algebras, (the previous paper).

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