

# Pacific Journal of Mathematics

**ON DOUBLY HOMOGENEOUS ALGEBRAS**

LOWELL G. SWEET

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The algebras to be discussed are assumed to be finite dimensional and not necessarily associative. If  $A$  is an algebra over a field  $K$  let  $\text{Aut}(A)$  denote the group of algebra automorphisms of  $A$ . We define  $A$  to be doubly homogeneous if  $\text{Aut}(A)$  is doubly transitive on the one-dimensional subspaces of  $A$ . Also a doubly homogeneous algebra  $A$  is said to be nontrivial if  $A^2 \neq 0$  and  $\dim A > 1$ . It is shown that the only nontrivial doubly homogeneous algebra is unique up to isomorphism.

An algebra  $A$  is said to be homogeneous if  $\text{Aut}(A)$  acts transitively on the one-dimensional subspaces of  $A$ . The reader is referred to the author's previous paper [1] for a discussion of homogeneous algebras and a bibliography of the related literature.

An arbitrary algebra  $A$  is said to be nonzero if  $A^2 \neq 0$ . If the nonzero elements of  $A$  form a quasi-group under multiplication then we say that  $A$  is a quasi-division algebra.

**LEMMA.** *If  $A$  is a nonzero doubly homogeneous algebra over a field  $K$  then  $A$  is a quasi-division algebra.*

*Proof.* Let  $\dim A = n$ . If  $n = 1$  then  $A$  is isomorphic to  $K$  and the result is obvious and so we assume that  $n > 1$ . Let  $a$  be any element of  $A$ . We claim that if  $b \notin Ka$  then  $ab \neq 0$ . For if  $ab = 0$  the doubly homogeneity condition implies that  $ac = 0$  for all  $c$  such that  $c \in Ka$ . But then in particular  $b + a \in Ka$  and so  $a(b + a) = 0$  which implies that  $a^2 = 0$  and thus  $aA = 0$ . In this case the homogeneity condition implies that  $A^2 = 0$  which is a contradiction and the claim is verified.

Now suppose that  $a^2 = 0$ . Then the homogeneity condition implies that  $x^2 = 0$  for all  $x \in A$ . Suppose there exists  $b \notin Ka$  such that

$$ab \in Ka.$$

Then by doubly homogeneity we would also have

$$(a + b)b \in K(a + b)$$

and  $b^2 = 0$  implies that

$$ab \in Ka \cap K(a + b) = \{0\}$$

which is impossible. Fix some  $b \notin Ka$ . Let  $c$  be any nonzero element

of  $A$ . Then there must exist  $\alpha \in \text{Aut}(A)$  such that

$$\alpha(ab) \in Kc$$

and

$$\alpha(a) \in Ka.$$

This implies that  $L_a$  (left multiplication by  $a$ ) is a surjective map which is impossible and so  $a^2 \neq 0$ . Hence  $L_a$  is invertible and the homogeneity condition implies that  $A$  is a quasi-division algebra.

**THEOREM.** *If  $A$  is a nonzero doubly homogeneous algebra over a field  $K$  then either  $A \cong K$  or  $K = GF(2)$  and  $A$  is isomorphic to the following algebra*

$$\begin{array}{c|cc} & a & b \\ \hline a & a & a+b \\ b & a+b & b \end{array}.$$

*Proof.* If  $\dim A = 1$  then clearly  $A \cong K$ . If  $\dim A = 2$  then  $A$  must be contained in the authors list of 2-dimensional homogeneous algebras [1] and it is easily checked that the only possibility is that  $K = GF(2)$  and  $A$  is isomorphic to the following algebra

$$\begin{array}{c|cc} & a & b \\ \hline a & a & a+b \\ b & a+b & b \end{array}.$$

Hence to prove the theorem it is sufficient to show that there exist no nonzero doubly homogeneous algebras of dimension  $n > 2$ .

Let  $A$  be a nonzero doubly homogeneous algebra of dimension  $n > 2$ . If  $a$  is any fixed nonzero element in  $A$  then the lemma implies that the equation  $ax = a$  must have a unique solution, say  $b$  and the doubly homogeneity condition now implies that  $b \in Ka$ . It follows that  $A$  is a nonzero, power-associative, homogeneous algebra and so Theorem 7 of the author's previous paper [1] implies that  $K = GF(2)$ .

Now let  $a$  and  $b$  be any two distinct nonzero elements of  $A$  and let  $A_1 = \langle a, b \rangle$  be the subalgebra of  $A$  generated by  $a$  and  $b$ . It can be shown that  $A_1$  is also a doubly homogeneous algebra and it is generated by any two distinct nonzero elements. Hence only the identity automorphism of  $A_1$  can fix two distinct nonzero elements of  $A_1$  and so  $\text{Aut}(A_1)$  is sharply doubly transitive on  $A_1 \setminus \{0\}$ . Hence the order of  $\text{Aut}(A_1)$  must be even and so  $\text{Aut}(A_1)$  must contain at least one involution, say  $\alpha$ . This involution  $\alpha$  fixes at most 1 one-

dimension subspace of  $A_1$ . But since any involution acting on a vector space  $V$  over a field of characteristic 2 fixes vectorwise a subspace of dimension  $\geq 1/2 \dim V$  this forces  $\dim A_1 = 2$  and so we may assume that

$$ab = a + b.$$

But since  $A$  is doubly homogeneous it follows that

$$\begin{aligned} x^2 &= x && \text{for all } x \in A \\ xy &= x + y && \text{whenever } y \notin Kx. \end{aligned}$$

Now since  $n > 2$  we can choose three independent vectors  $a, b, c \in A$ . But then

$$(a + b)c = a + b + c$$

and

$$ac + bc = a + c + b + c = a + b$$

which is impossible and the proof is complete.

#### REFERENCE

1. L. G. Sweet, *On homogeneous algebras*, (the previous paper).

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