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In order to describe a certain "lattice type" behavior of the closed subspaces in Banach spaces, with regard to the action of linear operators, we introduce the notion of closed range modulus. Some consequences for the spectral theory of commuting finite systems of linear operators are then obtained.

1. Let X and Y be two complex Banach spaces and $\mathfrak{C}(X, Y)$ ($\mathfrak{B}(X, Y)$) the family of all closed (bounded) linear operators from X to Y. For any $T \in \mathfrak{C}(X, Y)$ denote by $\mathfrak{D}(T)$ the domain of T and by $\mathfrak{R}(T)$ the range of T. If $Z \subset X$ is any closed subspace then $T \mid Z$ means the restriction of T on Z, i.e. the operator defined on $\mathfrak{D}(T \mid Z) = \mathfrak{D}(T) \cap Z$ as T. Denote also by $\mathfrak{N}(T)$ the null-space of T and by $\mathfrak{R}_T(X, Y)$ the family of all closed subspaces Z of X with the property that $\mathfrak{R}(T \mid Z)$ is a closed subspace of Y. Since $\mathfrak{N}(T \mid Z)$ is always closed, then for any $Z \in \mathfrak{R}_T(X, Y)$ the operator

$$\tau_Z\colon Tx \in \mathfrak{R}(T \mid Z) \longrightarrow x + \mathfrak{N}(T \mid Z) \in X/\mathfrak{N}(T \mid Z) \tag{1.1}$$

is everywhere defined, hence bounded by the closed graph theorem. Denote by $\|\tau_Z\|$ its norm and for any $\mathfrak{R} \subset \mathfrak{R}_T(X, Y)$ let us define

$$\kappa(T, \mathfrak{R}) = \sup_{Z \in \mathfrak{R}} \|\tau_Z\| . \tag{1.2}$$

When $\mathfrak{R} = \mathfrak{R}_T(X, Y)$ we put simply $\kappa(T) = \kappa(T, \mathfrak{R})$.

Then number $\kappa(T)(\kappa(T, \mathfrak{R}))$ will be called the *closed range modulus* of T (on $\mathfrak{R}$). It is clear that $0 \leqq \kappa(T, \mathfrak{N}) \leqq +\infty$ and $\kappa(T, \mathfrak{R}) = 0$ if and only if $T \mid Z = 0$, for any $Z \in \mathfrak{R}$.

Let us remark that the case $\mathfrak{R} = \{Z\}$ has been considered in [8, IV. 5.1].

There are simple examples which show that $\kappa(T, \mathfrak{R})$ may be infinite. However, the class of operators having a finite closed range modulus is reasonably large, as shown by the following result.

PROPOSITION 1.1. *If $T \in \mathfrak{C}(X, Y)$ has a bounded inverse then $\kappa(T) = \|T^{-1}\|$.*

Proof. Take $Z \in \mathfrak{R}_T(X, Y)$. Then the operator τ_Z, given by (1.1), has the form $\tau_Z\colon Tx \to x$, for every $x \in \mathfrak{D}(T \mid Z)$. Therefore we can write

$$\|\tau_Z\| = \sup_{\substack{x \in \mathfrak{D}(T \mid Z)\\ \|Tx\| \leqq 1}} \|x\| \leqq \sup_{\substack{x \in \mathfrak{D}(T \mid Z)\\ \|x\| \leqq \|T^{-1}\|}} \|x\| = \|T^{-1}\| .$$

When $Z = X$ we have $\tau_Z = T^{-1}$, hence $\kappa(T) = \| T^{-1} \|$.

2. In this section we shall obtain some "lattice type" consequences of the finiteness of the closed range modulus.

LEMMA 2.1. *Suppose that* $\mathfrak{R} \subset \mathfrak{R}_T(X, Y)$ *and* $\kappa(T, \mathfrak{R}) < +\infty$. *If* $M > \kappa(T, \mathfrak{R})$ *is a fixed constant, then for any* $Z \in \mathfrak{R}$ *and for any* $y \in \mathfrak{R}(T \mid Z)$ *there is an* $x \in \mathfrak{D}(T \mid Z)$ *such that* $Tx = y$ *and* $\| x \| \leqq M \| y \|$.

Proof. The assertion is a simple consequence of the open map principle.

THEOREM 2.2. *Let* T *be in* $\mathfrak{C}(X, Y)$ *and* $\mathfrak{R} = \{Z_\alpha\}_\alpha$ *an increasingly directed family in* $\mathfrak{R}_T(X, Y)$. *Suppose that* $\kappa(T, \mathfrak{R}) < \infty$ *and* $\overline{\mathfrak{R}(T \mid Z)} = \text{c.l.m.}\{\mathfrak{R}(T \mid Z_\alpha)\}_\alpha$, *where* $Z = \text{c.l.m.}\{Z_\alpha\}_\alpha$. *Then* $Z \in \mathfrak{R}_T(X, Y)$.

Proof. Let y be in $\overline{\mathfrak{R}(T \mid Z)}$ arbitrary. Then there is a sequence $y_n \in \mathfrak{R}(T \mid Z_{\alpha_n})$ such that $y_n \to y$ as $n \to \infty$. We shall use an approximation procedure, inspired by the proof of the closed graph theorem [8]. With no loss of generality we may suppose that $\sum_{n=1}^\infty \| y_{n+1} - y_n \| < \infty$. Fix a constant $M > \kappa(T, \mathfrak{R})$. We shall apply succesively Lemma 2.1. Choose $x_1 \in \mathfrak{D}(T \mid Z_{\alpha_1})$ such that $Tx_1 = y_1$ and $\| x_1 \| \leqq M \| y_1 \|$. Then, for an arbitrary $n \geqq 1$, we may find an index β_n such that $Z_{\beta_n} \supset Z_{\alpha_n} + Z_{\alpha_{n+1}}$ and an element $x_{n+1} \in Z_{\beta_n}$ with the properties $Tx_{n+1} = y_{n+1} - y_n$ and $\| x_{n+1} \| \leqq M \| y_{n+1} - y_n \|$. According to our choice of x_n we have

$$\sum_{n=1}^{\infty} \| x_n \| \leqq M\Big(\| y_1 \| + \sum_{n=1}^{\infty} \| y_{n+1} - y_n \| \Big),$$

hence $x = \sum_{n=1}^\infty x_n \in X$. Moreover, as T is a closed operator and $T(\sum_{n=1}^k x_n) = y_k$ we infer that $x \in \mathfrak{D}(T)$ and $Tx = y$.

COROLLARY 2.3. *Suppose that* $T \in \mathfrak{B}(X, Y)$ *and let* $\mathfrak{R} = \{Z_\alpha\}_\alpha$ *be an increasingly directed family in* $\mathfrak{R}_T(X, Y)$. *If* $\kappa(T, \mathfrak{R}) < +\infty$ *then* $Z \in \mathfrak{R}_T(X, Y)$, *where* $Z = \text{c.l.m.}\{Z_\alpha\}_\alpha$.

Proof. We have only to notice that the assumption $T \in \mathfrak{B}(X, Y)$ implies $\overline{\mathfrak{R}(T \mid Z)} = \text{c.l.m.}\{\mathfrak{R}(T \mid Z_\alpha)\}_\alpha$.

THEOREM 2.4. *Suppose that* X, Y *are the duals of the Banach spaces* X_*, Y_* *respectively, and* $T \in \mathfrak{C}(X, Y)$ *is the adjoint of a densely defined operator* $T_* \in \mathfrak{C}(Y_*, X_*)$. *If* $\{Z_\alpha\}_\alpha$ *is a decreasingly directed* *family of w^*-closed subspaces in $\mathfrak{R}_T(X, Y)$ and $\kappa(T, \mathfrak{R}) < \infty$ then $Z = \bigcap_\alpha Z_\alpha$ belongs to $\mathfrak{R}_T(X, Y)$.*

Proof. Assume that $y \in \overline{\mathfrak{R}(T \mid Z)}$, hence $y = \lim_n y_n$, with $y_n \in \mathfrak{R}(T \mid Z)$, for any natural n. We may suppose that $\sum_{n=1}^\infty \| y_{n+1} - y_n \| < +\infty$ and proceed as in the proof of Theorem 2.2. Namely, for any α we can choose a sequence $x_{n,\alpha} \in \mathfrak{D}(T \mid Z_\alpha)$ such that $Tx_{1,\alpha} = y_1$, $Tx_{n+1,\alpha} = y_{n+1} - y_n (n \geqq 1)$, $\| x_{1,\alpha} \| \leqq M \| y_1 \|$ and $\| x_{n+1,\alpha} \| \leqq M \| y_{n+1} - y_n \| (n \geqq 1)$. Consequently, $x_\alpha + \sum_{n=1}^\infty x_{n,\alpha} \in Z_\alpha$ Since T is closed, we infer $x_\alpha \in \mathfrak{D}(T \mid Z_\alpha)$ and $Tx_\alpha = y$. Moreover,

$$\| x_\alpha \| \leqq M(\| y_1 \| + \sum_{n=1}^{\infty} \| y_{n+1} - y_n \| ,$$

therefore $\{x_\alpha\}_\alpha$ is uniformly bounded. Let x be a cluster point of $\{x_\alpha\}_\alpha$ in the w^*-topology of X. Since $\{Z_\alpha\}_\alpha$ is a decreasingly directed family of w^*-closed subspaces, we get $x \in Z$. Let us show that $x \in \mathfrak{D}(T)$ and $Tx = y$. Indeed, if $\varepsilon > 0$ and $f \in \mathfrak{D}(T_*)$ are arbitrary, there is an index α such that

$$| x(T_* f) - x_\alpha(T_* f) | < \varepsilon .$$

Since $Tx_\alpha = y$ and $\varepsilon > 0$ is arbitrary, we obtain $x(T_* f) = y(f)$ for any $f \in \mathfrak{D}(T_*)$, whence $x \in \mathfrak{D}(T)$ and $Tx = y$.

COROLLARY 2.5. *Suppose that X, Y are reflexive Banach spaces and let T be a densely defined operator in $\mathfrak{C}(X, Y)$. If $\mathfrak{R} = \{Z_\alpha\}_\alpha$ is a decreasingly directed family in $\mathfrak{R}_T(X, Y)$ such that $\kappa(T, \mathfrak{R}) < \infty$ then $Z = \bigcap_\alpha Z_\alpha$ belongs to $\mathfrak{R}_T(X, Y)$.*

Proof. The result follows from the previous theorem, since $T = T^{**}$ and the w^*-topology of X coincides with its w-topology.

3. From now on we restrict ourselves to the case $X = Y$ and consider only bounded operators. We shall put $\mathfrak{B}(X)$ instead of $\mathfrak{B}(X, X)$.

For the sake of simplicity, an operator $T \in \mathfrak{B}(X)$ with the property $T^2 = 0$ will be called a *2-nilpotent*. These operators are related to the definition of the joint spectrum of a commuting system of linear operators, as described in the next section.

A 2-nilpotent operator $T \in \mathfrak{B}(X)$ will be called *exact* (on X) if $\mathfrak{R}(T) = \mathfrak{N}(T)$.

THEOREM 3.1. *Let T be a 2-nilpotent operator on X and $\mathfrak{R} = \{Z_\alpha\}_\alpha$*

an increasingly directed family of closed subspaces of X, invariant under T. If $T \mid Z$ is exact for any α and $\kappa(T, \mathfrak{R}) < \infty$ then $T \mid Z$ is exact, where $Z =$ c.l.m. $\{Z_\alpha\}$.

Proof. Consider $y \in \mathfrak{R}(T \mid Z)$. Then there is a sequence $y_n \in Z_{\alpha_n}$ such that $y = \lim_n y_n$. With no loss of generality we may suppose that $Ty_n = 0$ for any n. Indeed, if $y'_n \to y$, $y'_n \in Z_{\alpha_n}$, then $Ty'_n \to Ty = 0$, therefore if $M > \kappa(T, \mathfrak{R})$ is fixed, we may choose $v_n \in Z_{\alpha_n}$ such that $\| v_n \| \leqq M \| Ty_n \|$ and $Tv_n = Ty'_n$. In particular, $v_n \to 0$ as $n \to \infty$. If we put $y_n = y'_n - v_n$, we have $Ty_n = 0$ and $y_n \to y$ as $n \to \infty$.

We can proceed now as in the proof of Theorem 2.2. Namely, assuming $\sum_{n=1}^\infty \| y_{n+1} - y_n \| < \infty$ we can find a sequence $\{x_n\}$ in Z such that $Tx_1 = y_1$, $Tx_{n+1} = y_{n+1} - y_n$ and the series $x = \sum_{n=1}^\infty x_n$ is convergent in Z. Then $Tx = y$, hence $T \mid Z$ is exact.

THEOREM 3.2. *Assume that X is the dual of the Banach space X_* and T is the adjoint of $T_* \in \mathfrak{B}(X_*)$. Let $\mathfrak{R} = \{Z_\alpha\}_\alpha$ be a decreasingly directed family of w^*-closed subspaces of X, invariant under T. If $T \mid Z$ is exact for any α and $\kappa(T, \mathfrak{R}) < \infty$ then $T \mid Z$ is exact, where $Z = \bigcap_\alpha Z_\alpha$.*

Proof. Let y be in $\mathfrak{R}(T \mid Z)$, therefore for any α, $y = Tx_\alpha$ with $x_\alpha \in Z_\alpha$. Since $\{x_\alpha\}$ can be chosen uniformly bounded, we may find, as in the proof of Theorem 2.4, a vector $x \in Z$ such that $y = Tx$.

4. First of all we recall the definition of the joint spectrum of a commuting system $T = (T_1, \cdots, T_n) \subset \mathfrak{B}(X)$, in the sense of Taylor's [9] (see also [12]).

Denote by $\Lambda^p(n, X)$ the set of all antisymmetric functions defined on $\{1, \cdots, n\}^p$, with values in X, for any natural p. Let us denote by $\delta_T^p \colon \Lambda^p(n, X) \to \Lambda^{p+1}(n, X)$, the operator defined as

$$\delta_T^p \xi(\nu_1, \cdots, \nu_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} T\nu_j \xi(v_1, \cdots, \hat{\nu}_j \cdots, \nu_{p+1}) ,$$

where the symbol "$\wedge$" means that the corresponding index is omitted. One can easily prove that $\delta_T^{p+1}\delta_T^p = 0$. We define also $\Lambda^0(n, X) = X$ and $\delta_T^0 x(\nu) = T_\nu x$, for any $x \in X$ and $\nu = 1, \cdots, n$.

The system T is called *nonsingular* (singular) if $\mathfrak{R}(\delta_T^p) = \mathfrak{N}(\delta_T^{p+1})$ for every nonnegative integer p (there is a p such that $\mathfrak{R}(\delta_T^p) \neq \mathfrak{N}(\delta_T^{p+1})$).

The *spectrum* of T on X, denoted by $\delta(T, X)$, is defined as the set of all points $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$ such that $z - T = (z_1 - T_1, \cdots, z_n - T_n)$ is singular on X.

These things can be considered in a slightly different manner. Namely, it is clear that we can identify $\Lambda^p(n, X)(1 \leqq p \leqq n)$ with the direct sum

$$\bigoplus_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} X .$$

Let us set then

$$X^{(n)} = \bigoplus_{p=0}^{n} \Lambda^p(n, X) \tag{4.1}$$

and

$$\delta_T = \bigoplus_{p=0}^{n} \delta_T^p .$$

The space $X^{(n)}$ is a direct sum of 2^n copies of X and δ_T is 2-nilpotent on $X^{(n)}$ (provided that $\delta_T^n = 0 \colon \Lambda^n(n, X) \to \Lambda^0(n, X)$). Since $\Lambda^p(n, X)$ is null for $p > n$, therefore it has no contribution in the definition of the spectrum it is easy to see that a system $T = (T_1, \cdots, T_n)$ is nonsingular on X if and only if the 2-nilpotent operator δ_T is exact on $X^{(n)}$.

The definition of the joint spectrum allows to recapture many spectral properties of the one-dimensional case, including the functional calculus with analytic functions [9], [10]. However, there are some simple properties which have not direct variants in several dimensions. One of them is given by the next result.

PROPOSITION 4.1. *Suppose that $T \in \mathfrak{B}(X)$ is invertible and let $\{Z_\alpha\}_\alpha$ be an increasingly directed family of closed subspaces of X, invariant under T, such that $T \mid Z_\alpha$ is invertible for any α. Then $T \mid Z$ is invertible, where $Z =$ c.l.m. $\{Z_\alpha\}_\alpha$.*

Proof. It is sufficient to show that $T \mid Z$ is surjective. Indeed, if $y \in Z$ then $y = \lim_n y_n$, where $y_n \in Z_{\alpha_n}$, therefore $y_n = Tx_n$, with $x_n \in Z_{\alpha_n}$. Moreover,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (T \mid Z_{\alpha_n})^{-1} y_n = \lim_{n\to\infty} T^{-1} y_n ,$$

consequently $x = \lim_n x_n \in Z$ and $Tx = y$.

A direct version of Proposition 4.1 is not possible for more than one dimension.

EXAMPLE. Let Z be a separable Hilbert space and set $Z_k = Z$ for $k = 1, 2, 3, \cdots$. Suppose that every Z_k has an orthonormal basis of the form $\{e_j^k\}_{j=-\infty}^{+\infty}$. Define $X = \bigoplus_{k=1}^{\infty} Z_k$. Let T_k be the bilateral translation on Z_k, namely

$$T_k e_j^k = e_{j+1}^k\,, \qquad (j \in \boldsymbol{Z})$$

and denote by T the direct sum $\bigoplus_{k+1}^{\infty} T_k$. Let now $\{\lambda_k\}_{k=1}^{\infty}$ a sequence of complex numbers, $\lambda_k \neq 0$ for any k, $\lambda_k \to 0$ as $k \to \infty$. We define on every Z_k the operator $S_k = \lambda_k$ and let us put $S = \bigoplus_{k=1}^{\infty} S_k$. It is obvious that $TS = ST$. Since the operator T is unitary on X, thus invertible, it follows that the system (T, S) is nonsingular on X ([9]; see also the proof of Theorem 4.5).

Consider now the subspaces

$$Z_k^+ = \text{c.l.m.}\,\{e_j^k\}_{j=0}^{\infty}\,, \qquad k = 1, 2, \cdots$$

and define $X_n^+ = \bigoplus_{k=1}^{n} Z_k^+$, $X^+ = \bigoplus_{k=1}^{\infty} Z_k$. It is clear that the spaces X_n^+ and X^+ are invariant under T and S, for any n. Furthermore, the system $(T \mid X_n^+, S \mid X_n^+)$ is nonsingular on X_n^+ since the operator $S \mid X_n^+$ is invertible (because of the choice of the numbers λ_k, $1 \leqq k \leqq n$). On the other hand, $(T \mid X^+, S \mid X^+)$ is singular. Indeed, it is sufficient to show that $X^+ \neq TX^+ + SX^+$. Let ξ be in X^+ of the form $\xi = \sum_{k=1}^{\infty} \xi_k e_0^k$. If $\xi = T\zeta + S\eta$, we would have, for the coefficients η_k of η corresponding to e_0^k, the relations

$$\lambda_k \eta_k = \xi_k\,, \qquad k = 1, 2, 3, \cdots$$

which is not possible for any choice of ξ of the given form, because $\lambda_k \to 0$ as $k \to \infty$. In this way we have shown that the nonsingularity of (T, S) on X and on any X_n^+ does not imply the nonsingularity of (T, S) on $X^+ = \text{c.l.m.}\,\{X_n^+\}_n$.

There is a result which is "symmmetrical" to Proposition 4.1.

PROPOSITION 4.2. *Suppose that $T \in \mathfrak{B}(X)$ is invertible and let $\{Z_\alpha\}_\alpha$ be a decreasingly directed family of closed subspaces of X, invariant under T, such that $T \mid Z_\alpha$ is invertible for each α. Then $T \mid Z$ is invertible, where $Z = \bigcap_\alpha Z_\alpha$.*

Proof. It is easy to see that $T \mid Z$ is injective and surjective.

According to expectation, Proposition 4.2 does not have a direct variant in several dimensions. An example in this sense can be easily obtained from the above Example. Indeed, with the same notations, consider the system (T^*, S^*) and the family of subspaces $\{(X_n^+)^\perp\}_{n=1}^{\infty}$. Since the nonsingularity is preserved by the duality and the space $(X_n^+)^\perp$ is isomorphic to the dual space $(X/X_n)^*$, we obtain that (T^*, S^*) is nonsingular on $(X_n^+)^\perp$ because (T, S) is nonsingular on both X and X_n^+ (see [9, Lemma 1.2]). Analogously, (T^*, S^*) cannot be nonsingular on $\bigcap_n (X_n^+)^\perp = (X^+)^\perp$ since (T, S) is singular on X^+.

In spite of these rather disappointing examples, there are cases when both Propositions 4.1 and 4.2 have extensions in several variables. These cases are consequences of the finiteness of a certain closed range modulus.

Consider again a system $T = (T_1, \cdots, T_n)$ of commuting operators on the Banach space X. If $Z \subset X$ is a closed subspace invariant under T (i.e. invariant under T_j for $j = 1, \cdots, n$), define $Z^{(n)}$ by the formula (4.1) and notice that $Z^{(n)}$ is a subspace of $X^{(n)}$, invariant under δ_T. If $\mathfrak{R} = \{Z_\alpha\}_\alpha$ is a family of such subspaces, then denote by $\mathfrak{R}^{(n)}$ the corresponding family $\{Z_\alpha^{(n)}\}_\alpha$. We define also the *closed range modulus* of T on $\mathfrak{R}$ by the formula

$$\kappa(T, \mathfrak{R}) = \kappa(\delta_T, \mathfrak{R}^{(n)}) . \tag{4.2}$$

For any subspace $Z \subset X$ invariant under $T = (T_1, \cdots, T_n)$, the notation $T \mid Z$ stands for the system $(T_1 \mid Z, \cdots, T_n \mid Z)$.

THEOREM 4.3. *Let $T = (T_1, \cdots, T_n)$ be a commuting system of operators in $\mathfrak{B}(X)$ and $\mathfrak{R} = \{Z_\alpha\}_\alpha$ an increasingly directed family of closed subspaces of X, invariant under T. If $T \mid Z_\alpha$ in nonsingular for any α and $\kappa(T, \mathfrak{R}) < \infty$ then $T \mid Z$ is nonsingular, where $Z =$* c.l.m. $\{Z_\alpha\}_\alpha$.

Proof. Since $Z^{(n)} =$ c.l.m. $\{Z_\alpha^{(n)}\}_\alpha$, this theorem is Theorem 3.1 rewritten.

Analogously, we get from Theorem 3.2 the following

THEOREM 4.4. *Assume that X is the dual of X_* and $T = (T_1, \cdots, T_n)$ is the adjoint of the system $T_* = (T_{1*}, \cdots, T_{n*})$, T_* acting in X_*. If $\mathfrak{R} = \{Z_\alpha\}_\alpha$ is a decreasingly directed family of w^*-closed subspaces of X, invariant under T, such that $T \mid Z_\alpha$ is nonsingular for any α and $\kappa(T, \mathfrak{R}) < \infty$, then $T \mid Z$ is nonsingular, where $Z = \bigcap_\alpha Z_\alpha$.*

We shall end this section with a result of finiteness for the closed range modulus of a commuting system of operators, in a special case.

THEOREM 4.5. *Let $T = (T_1, \cdots, T_n)$ be a commuting system of operators on X and assume that $T_1 V_1 + \cdots + T_n V_n = 1$, where $V_1, \cdots, V_n$ are operators in the commutant of T in $\mathfrak{B}(X)$. If $\mathfrak{R}$ is any family of closed subspaces of X, invariant for the action of T_j and $V_j (j = 1, \cdots, n)$, then $\kappa(T, \mathfrak{R}) < \infty$.*

Proof. We have to show that the maps

$$\mathfrak{R}(\delta_{T|Z}) \longrightarrow Z^{(n)}/\mathfrak{N}(\delta_{T|Z}) , \tag{4.3}$$

defined as in (1.1), are uniformly bounded for $Z \in \mathfrak{R}$. We shall use an idea from [9, Lemma 1.1]. Fix $Z \in \mathfrak{R}$ and take $\xi^p \in \Lambda^p(n, Z)$ with the property $\delta_T^p \xi^p = 0$. Let us define the element $\eta^p \in \Lambda^{p-1}(n, Z)$ by the formula

$$\eta^p(\nu_1, \cdots, \nu_{p-1}) = \sum_{j=1}^{n} V_j \xi(j, \nu_1, \cdots, \nu_{p-1}) . \tag{4.4}$$

Then we can write

$$\begin{aligned}
\delta_T^{p-1}\eta^p(\nu_1, \cdots, \nu_p) &= \sum_{j=1}^{p} (-1)^{j+1} T_{\nu_j} \sum_{k=1}^{n} V_k \xi^p(k, \nu_1, \cdots, \hat{\nu}_j, \cdots, \nu_p) \\
&= \sum_{k=1}^{n} V_k \sum_{j=1}^{p} (-1)^{j+1} T_{\nu_j} \xi^p(k, \nu_1, \cdots, \hat{\nu}_j, \cdots, \nu_p) \\
&= \Big(\sum_{k=1}^{n} V_k T_k\Big) \xi^p(\nu_1, \cdots, \nu^p) = \xi^p(v_1, \cdots, \nu_p) ,
\end{aligned}$$

where we have used the relation

$$T_k \xi^p(\nu_1, \cdots, \nu_p) = \sum_{j=1}^{p} (-1)^{j+1} T_{\nu_j} \xi^p(k, \nu_1, \cdots, \hat{\nu}_j, \cdots, \nu_p) ,$$

obtained from $\delta_T^p \xi^p = 0$. (In particular, we have obtained the non-singularity of $T \mid Z$.) Consequently the norm of the map $\xi^p \longrightarrow \eta^p + \mathfrak{N}(\delta_{T|Z}^p)$ has the estimation

$$\| \eta^p + \mathfrak{N}(\delta_{T|Z}^p) \| \leqq \| \eta^p \| \leqq C \| \xi^p \| ,$$

where $C > 0$ does not depend on Z, as it follows from (4.4). More general, in order to estimate the norm of (4.3), if $\xi = \bigoplus_{p=0}^{n} \xi^p \in \mathfrak{R}(\delta_{T|Z})$ we can choose a solution $\eta = \bigoplus_{p=0}^{n} Z^{(n)}$ of the equation $\delta_T \eta = \xi$ such that

$$\| \eta + \mathfrak{N}(\delta_{T|Z}) \| \leqq \| \eta \| \leqq C \| \xi \| ,$$

where $C > 0$ does not depend on Z. According to the definition (4.2) we have then

$$\kappa(T, \mathfrak{R}) = \kappa(\delta_T, \mathfrak{R}^{(n)}) < +\infty .$$

and the proof is complete.

5. We shall deal in what follows with some special problems of spectral theory of commuting systems of operators on a Banach space X.

We recall that a *spectral capacity* on $\boldsymbol{C}^n$ [3], [2] is a map assigning to every closed set $F \subset \boldsymbol{C}^n$ a closed subspace $X(F) \subset X$, with the properties:

(1) $X(\emptyset) = \{0\}$; $X(\boldsymbol{C}^n) = X$.

(2) $X(\bigcap_{k=1}^{\infty} F_k) = \bigcap_{k=1}^{\infty} X(F_k)$, for any sequence of closed sets $\{F_k\}_{k=1}^{\infty}$ in $\boldsymbol{C}^n$;

(3) $\sum_{k=1}^{p} X(\bar{G}_k) = X$ for any open covering $\{G_k\}_{k=1}^{p}$ of $\boldsymbol{C}^n$.

If the property (3) is true only for $p \leqq m$, with a fixed $m \geqq 2$, then the map $F \to X(F)$ is an *m-spectral capacity* [2], [7].

A commuting system of operators $T = (T_1, \cdots, T_n)$ is called *m-decomposable* (decomposable) if there exists an m-spectral (a spectral) capacity $F \to X(F)$ (F closed in $\boldsymbol{C}^n$) with the properties:

(1) each $X(F)$ is invariant under T;

(2) $\delta(T; X(F)) \subset F$, for every F.

It is known that each m-decomposable system T has only one m-spectral capacity [7], [11].

Let $T = (T_1, \cdots, T_n)$ be an m-decomposable system of operators on X and $F \to X(F)$ its m-spectral capacity.

We shall say that $F \to X(F)$ is *tempered* on a set $A \subset \boldsymbol{C}^n$ if for every $z \notin \bar{A}$ we have

$$\kappa(z - T, \mathfrak{R}_A) < +\infty \ , \tag{5.1}$$

where $\mathfrak{R}_A = \{X(F); F \subset A, F \text{ closed}\}$.

We shall say that $F \to X(F)$ is *$*$-tempered* on the closed set $K \subset \boldsymbol{C}^n$ if for every $z \notin K$ there is an open set $H \ni z$, $\bar{H} \cap K = \emptyset$, such that

$$\kappa(z - T^*, \mathfrak{R}^*_{K,z}) < +\infty \ , \tag{5.2}$$

where $\mathfrak{R}^*_{K,z} = \{(X/X(\bar{G}))^*; \bar{G} \cap K = \emptyset, G \supset H, G \text{ open}\}$ (the spaces $(X/X(\bar{G}))^*$ are identified here with subspaces of X^*).

For any set $A \subset \boldsymbol{C}^n$ we define

$$X(A) = \bigcup \{X(F); F \subset A, F \text{ closed}\} \ .$$

Since every spectral capacity is monotone, it is clear that $X(A)$ is a subspace, not necessarily closed, of X.

When a spectral capacity is tempered ($*$-tempered) on every (closed) set, we shall call it simply *tempered* ($*$-tempered).

PROPOSITION 5.1. *Let $T = (T_1, \cdots, T_n)$ be an m-decomposable system of operators on X and $F \to X(F)$ its m-spectral capacity.*

(a) *If $F \to X(F)$ is tempered on $A \subset \boldsymbol{C}^n$ then $\delta(T; \overline{X(A)}) \subset \bar{A}$.*

(b) *If $F \to X(F)$ is $*$-tempered on the closed set $K \subset \boldsymbol{C}^n$ then*

$$\delta(T^*; (X/\overline{X(\boldsymbol{C}^n \backslash K)})^*) \subset K \ .$$

Proof. Fix $z \notin A$ and consider the family of closed subspaces

$\mathfrak{R}_A$ which appears in (5.1). It is clear that $\mathfrak{R}_A$ is increasingly directed. According to (5.1) and Theorem 4.3, $z - T$ is nonsingular on c.l.m. $\{X(F)\}_{F\subset A} = \overline{X(A)}$. In this way we have (a).

In order to prove (b), first of all let us show that

$$\delta(T; X/X(\bar{G})) \subset C^n \backslash G , \tag{5.3}$$

for any open set $G \subset C^n$. We shall use an argument from [11]. Let H be another open set such that $G \cup H = C^n$. We have then $X = X(\bar{G}) + X(\bar{H})$, because T is m-decomposable. Notice that $X/X(\bar{G})$ is isomorphic to $X(\bar{H})/X(\bar{G} \cap \bar{H})$, hence

$$\delta(T; X/X(\bar{G})) = \delta(T; X(\bar{H})/X(\bar{G} \cap \bar{H})) \subset \bar{H} \cup (\bar{G} \cap \bar{H}) = \bar{H} ,$$

according to a Taylor's result on spectral inclusions [9, Lemma 1.2]. As H is arbitrary with the property $G \cup H = C^n$, we obtain (5.3).

Secondly, note that if $z \notin K$ is fixed, we have

$$(X/\overline{X(C^n \backslash K)})^* = \bigcap \{X/X(\bar{G}))^*; \bar{G} \cap K = \varnothing, G \supset H, G \text{ open}\} ,$$

where H is chosen such that the relation (5.2) is fulfild. On account of Theorem 4.4 and the relations (5.2) and (5.3) we obtain that $z - T^*$ is nonsingular on $(X/\overline{X(C^n \backslash K)})^*$.

Proposition 5.1(b) suggests the definition of a "dual capacity" by the formula

$$X^*(F) = (X/\overline{X(C^n \backslash F)})^* , \tag{5.4}$$

where $F \subset C^n$ is arbitrary, as a natural extension of the one-dimensional case [6]. In order to apply Proposition 5.1, we must assume that the spectral capacity $F \to X(F)$ is $*$-tempered on each closed set. It is beyond our scope to develop here a theory of duality for spectral capacities in several dimensions. We have only illustrated some of the difficulties of such an attempt.

For $n = 1$ the formulas (5.1) and (5.2) are automatically fulfiled.

PROPOSITION 5.2. *Let T be an m-decomposable operator. Then its m-spectral capacity $F \to X(F)(F \subset C$, F* closed) *is tempered and $*$-tempered.*

Proof. Fix an arbitrary set $A \subset C$ and take $z \notin \bar{A}$. According to Proposition 1.1 we have then

$$\sup_{\substack{F\subset A \\ F \text{ closed}}} \|((z - T) \mid X(F))^{-1}\| \leq \|((z - T) \mid X(\bar{A}))^{-1}\| ,$$

hence $F \to X(F)$ is tempered on A.

Now, it is known that the formula (5.4) defines the 2-spectral capacity of T [6]. From the formula (5.3) we obtain

$$\delta(T^*; (X/X(\bar{G}))^*) = \delta(T; X/X(\bar{G})) \subset C\backslash G ,$$

hence $(X/X(\bar{G}))^* \subset X^*(C\backslash G)$, the spaces $X^*(F)$ being spectra maximal (see [6] for details). If $K \cup C$ is a closed set and $z \notin K$ is arbitrary, if we define

$$H = \{w \in C; \operatorname{dist}(w, K)\} > \frac{1}{2} \operatorname{dist}(z, K) ,$$

we have for any $G \supset H$, $\bar{G} \cap K = \emptyset$,

$$\begin{aligned} \|(z - T)(X/X(\bar{G}))^*)^{-1}\| &\leqq \|((z - T^*) \mid X^*(C\backslash G))^{-1}\| \\ &\leqq \|((z - T^*) \mid X^*(C\backslash H))^{-1}\| , \end{aligned}$$

whence $F \to X(F)$ follows $*$-tempered.

In several dimensions, the existence of a result similar with Proposition 5.2 is not certain; it becomes yet less certain in virtue of our examples in the fourth section. However, the answer is positive for classes of systems satisfying the conditions of Theorem 4.5. Such examples are, for instance, the classes of systems having functional calculi [4], [1]. For the sake of simplicity we shall consider only a particular case (see [1] for details).

Let $\mathscr{E}^\infty(C^n)$ be the algebra of all indefinitely differentiable complex function in $C^n = R^{2n}$.

A *functional calculus* is a continuous homomorphism U of the unital algebra $\mathscr{E}^\infty(C^n)$ into the unital algebra $\mathfrak{B}(X)$. Denote by $T_j = U(z_j)$, where $z \to z_j$ stand for the coordinate functions ($j = 1, \cdots, n$). The system $T = (T_1, \cdots, T_n)$ will be called a *generalized scalar system*. Such a system is decomposable and its spectral capacity is given by

$$X(F) = \bigcap \{\mathfrak{N}(U(f)); \text{ supp } f \cap F = \emptyset\} ,$$

where supp f denotes the support of f in C^n, and F is an arbitrary closed set in C^n. It is known that $T \mid X(F)$ is again a generalized scalar system, for any closed F.

PROPOSITION 5.3. *Let $T = (T_1, \cdots, T_n)$ be a generalized scalar system. Then its spectral capacity is tempered and $*$-tempered.*

Proof. Let A be an arbitrary set in C^n and $z \notin \bar{A}$ a fixed point, $z = (z_1, \cdots, z_n)$. Then the functions

$$\varphi_j(w) = \overline{(z_j - w_j)} / \sum_{k=1}^{n} |z_k - w_k|^2$$

are defined in a neighborhood of $\bar{A}$, therefore the operators $V_j = (U \mid X(\bar{A}))(\varphi_j)$ are correctly defined and we have $\sum_{j=1}^{n}(z_j - T_j)V_j = 1$ on $X(\bar{A})$. Since the spaces $X(F)$ are invariant under T_j and V_j for any $F \subset A$, F closed, we may apply Theorem 4.5 and get that the spectral capacity $F \to X(F)$ is tempered on A.

Since the system $T^* = (T_1^*, \cdots, T_n^*)$ is generalized scalar on X^*, the proof that $F \to X(F)$ is $*$-tempered is similar and will be omitted.

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