

# Pacific Journal of Mathematics

**STRONGLY UNICOHERENT CONTINUA**

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## STRONGLY UNICOHERENT CONTINUA

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In this journal the author introduced the concept of strongly unicoherent continua and proved that such a continuum is aposyndetic at a point  $p$  if and only if it is connected im kleinem at  $p$ . In the present paper we obtain some new results on strongly unicoherent continua. The main theorem states that the strongly unicoherent continuum  $M$  contains a unique irreducible subcontinuum between two points  $p$  and  $q$  provided  $M$  is both aposyndetic and semi-locally-connected at  $p$ .

Throughout this paper a *continuum* is a compact connected metric space and  $M$  will denote a continuum. The continuum  $M$  is *unicoherent* if whenever  $M = A \cup B$ , with  $A$  and  $B$  subcontinua of  $M$ ,  $A \cap B$  is connected.  $M$  is *hereditarily unicoherent* if every subcontinuum of  $M$  is unicoherent. A continuum  $M$  is said to be *irreducible between a pair of points*  $p$  and  $q$  of  $M$  provided no proper subcontinuum of  $M$  contains both  $p$  and  $q$ .  $M$  is said to be *irreducible* if there exists two points so that  $M$  is irreducible between the pair of points.

If  $N$  is a subset of  $M$ , the interior of  $N$  in  $M$  will be denoted by  $\text{int } N$  and the closure of  $N$  in  $M$  by  $\bar{N}$ .

For other terms not defined herein see [3] and [5].

**DEFINITION 1.** A unicoherent continuum  $M$  is *strongly unicoherent* provided that for any pair of proper subcontinua  $H$  and  $K$  such that  $M = H \cup K$ , each of  $H$  and  $K$  is unicoherent.

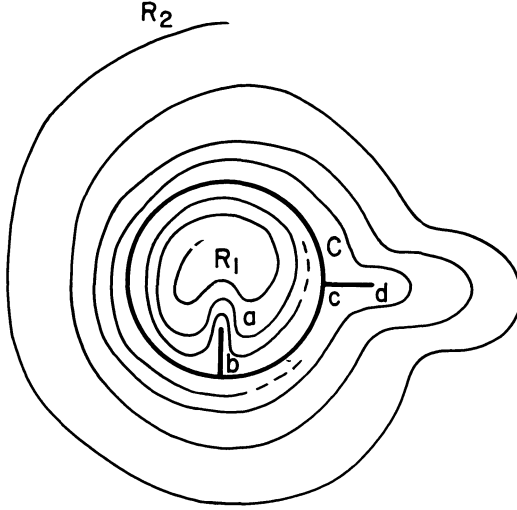
It is easily seen that this notion is a stronger form of unicoherence, but is somewhat weaker than hereditarily unicoherence. For example, a continuum which consists of a ray limiting on a circle is strongly unicoherent but fails to be hereditarily unicoherent.

In [4] Miller proved the following theorem.

**THEOREM 1.** *If the atriodic unicoherent continuum  $M$  is the sum of two proper subcontinua  $H$  and  $K$ , then  $H$  and  $K$  are unicoherent, and if  $N$  is a nonunicoherent subcontinuum of  $M$  intersecting  $H$ , it is a subset of  $H$ .*

Thus, atriodic unicoherent continua are examples of continua which are strongly unicoherent. A reasonable conjecture is that the last portion of the theorem holds for strongly unicoherent continua, but this is not the case as shown by the following example.

## EXAMPLE 1.



The strongly unicoherent continuum  $M$  consist of a circle  $C$ , two arcs  $[a, b]$  and  $[c, d]$  which intersect  $C$  at the point  $b$  and point  $c$  respectively, a ray  $R_1$  which limits on  $C \cup [a, b]$ , and a ray  $R_2$  which limits on  $C \cup [c, d]$ .

Let  $H = \bar{R}_1$  and  $K = \bar{R}_2 \cup [a, b]$ . Then  $H$  and  $K$  are proper subcontinua of  $M$  and  $M = H \cup K$ . Let

$$N = C \cup [a, b] \cup [c, d].$$

$N$  is non-unicoherent,  $N \cap H \neq \emptyset$ , but  $N \not\subset H$ .

Note that while  $M = H \cup K$  and  $K$  is unicoherent it does not follow that  $K$  is strongly unicoherent as shown by the example above.

**COROLLARY 1.** *If  $M$  is atriodic, then every unicoherent subcontinuum of  $M$  is strongly unicoherent.*

Next we investigate the relationship of a strongly unicoherent continuum  $M$  and its nonunicoherent subcontinua.

**DEFINITION 2.** A subcontinuum  $N$  of  $M$  is said to be a *continuum of condensation* if each point of  $N$  is a limit point of  $M - N$ .

The next theorem is an immediate consequence of definitions 1 and 2.

**THEOREM 2.** *If  $N$  is a nonunicoherent subcontinuum of a strongly unicoherent continuum  $M$  then either (i)  $N$  separates  $M$  or, (ii)  $N$  is a continuum of condensation.*

**COROLLARY 2.** *If  $N$  is a nonunicoherent subcontinuum of a strongly unicoherent continuum  $M$  then each subcontinuum of  $N$  either (i) separates  $M$  or (ii) is a continuum of condensation.*

*Question.* Is every nonunicoherent subcontinuum of a strongly unicoherent continuum  $M$  a continuum of condensation?

The answer to the preceding question is in the affirmative if the continuum  $M$  is also irreducible.

**THEOREM 3.** *Suppose that  $M$  is a strongly unicoherent irreducible continuum. Then every nonunicoherent subcontinuum of  $M$  is a continuum of condensation in  $M$ .*

*Proof.* Let  $a$  and  $b$  be in  $M$  such that  $M$  is irreducible between  $a$  and  $b$  and suppose  $N$  is a non-unicoherent subcontinuum of  $M$ . According to Theorem 2, we may assume that  $N$  separates  $M$ . It follows that  $\{a, b\} \cap N = \emptyset$  and  $M - N$  has exactly two components, say  $A$  and  $B$ . Without loss of generality, assume that  $a \in A$  and  $b \in B$ . If  $\bar{A} \cap \bar{B} \neq \emptyset$  then  $\bar{A} \cup \bar{B}$  is a subcontinuum of  $M$  containing  $\{a, b\}$ . Thus,  $M = \bar{A} \cup \bar{B}$  and it follows that  $N$  is a continuum of condensation.

Suppose that  $\bar{A} \cap \bar{B} = \emptyset$ . Let  $H$  and  $K$  be subcontinua of  $N$  such that  $N = H \cup K$  and  $H \cap K$  is not connected.

*Assertion.*  $\bar{A} \cap H \neq \emptyset \neq H \cap \bar{B}$ . For suppose that this is not the case. Assume that  $\bar{A} \cap H = \emptyset$ . Then  $\bar{A} \cap K \neq \emptyset$  and  $\bar{A} \cup K \cup H$  is a proper subcontinuum of  $M$ . Since  $M = (\bar{A} \cup K \cup H) \cup \bar{B}$ , it follows that  $\bar{A} \cup H \cup K$  is unicoherent. This implies that  $(\bar{A} \cup K) \cap H = H \cap K$  is connected which is a contradiction. Thus the assertion holds.

In a similar manner, it follows that  $\bar{A} \cap K \neq \emptyset \neq K \cap \bar{B}$ . Since  $M = \bar{A} \cup H \cup \bar{B}$ , then  $K - H \subset \bar{A} \cup \bar{B}$ . Also since  $M = \bar{A} \cup K \cup \bar{B}$ , it follows that  $H - K \subset \bar{A} \cup \bar{B}$ .

Finally we will show that  $H \cap K \subset \bar{A} \cup \bar{B}$ . Let  $P$  and  $Q$  be disjoint closed sets such that  $H \cap K = P \cup Q$ , let  $p \in P$  and  $C$  be the component of  $p$  in  $H - Q$ . Then  $\bar{C} \cap Q \neq \emptyset$ , but note that  $\bar{C} \cap (\text{int } Q) = \emptyset$ . Since  $\bar{C} \cup K \cup \bar{A}$  is a proper subcontinuum of  $M$  and  $M = \bar{B} \cup (\bar{C} \cup K \cup \bar{A})$ , then  $\bar{C} \cup K \cup \bar{A}$  is unicoherent. Thus  $(K \cup \bar{A}) \cap \bar{C} = (K \cap \bar{C}) \cup (\bar{C} \cap \bar{A})$  is connected. Now  $(K \cap \bar{C}) \cap P \neq \emptyset \neq (K \cap \bar{C}) \cap Q$  and  $K \cap \bar{C} \subset P \cup Q$  which implies that  $K \cap \bar{C}$  is not connected. It follows that  $\bar{C} \cap \bar{A} \neq \emptyset$ .

By interchanging the roles of  $\bar{A}$  and  $\bar{B}$  in the above argument, we have that  $\bar{C} \cap \bar{B} \neq \emptyset$ . Now  $\bar{A} \cup \bar{C} \cup \bar{B}$  is a subcontinuum of  $M$  containing  $\{a, b\}$ , so  $M = \bar{A} \cup \bar{C} \cup \bar{B}$ . Since  $(\text{int } Q) \cap \bar{C} = \emptyset$ , then  $\text{int } Q \subset \bar{A} \cup \bar{B}$ .

Suppose  $q \in Q - (\text{int } Q)$  and  $V$  is an open set containing  $q$  such that  $V \cap P = \emptyset$ . Then  $V \not\subset Q$  so there is a point  $z \in V \cap (M - Q)$ . Since  $z \notin P \cup Q$  it follows by the first portion of this proof that  $z \in \bar{A} \cup \bar{B}$ . Thus  $q$  is a limit point of  $\bar{A} \cup \bar{B}$  and hence  $q \in \bar{A} \cup \bar{B}$ . Therefore  $Q \subset \bar{A} \cup \bar{B}$ .

Since the preceding argument is symmetric with respect to  $P$  and  $Q$ , it follows that  $P \subset \bar{A} \cup \bar{B}$ .

Therefore  $N = (H - K) \cup (K - H) \cup P \cup Q \subset \bar{A} \cup \bar{B}$  which implies that  $N$  is a continuum of condensation.

The following well known characterization of hereditarily unicoherent continua was given in [4].

**THEOREM 4 (Miller).** *In order that the continuum  $M$  be hereditarily unicoherent it is necessary and sufficient that for any two points  $p$  and  $q$  of  $M$  there is only one subcontinuum of  $M$  which is irreducible between  $p$  and  $q$ .*

**DEFINITION 3.** A continuum  $M$  is *hereditarily unicoherent at the point  $p$*  of  $M$  provided for each  $q \in M$  different from  $p$ , there is a unique subcontinuum of  $M$  which is irreducible between  $p$  and  $q$ .

Thus if  $M$  is hereditarily unicoherent at  $p$  and  $q \in M - \{p\}$ , then the intersection of all subcontinuum of  $M$  which contain  $\{p, q\}$  is connected.

We shall show that strongly unicoherent continua are "hereditarily unicoherent at certain points", but first we prove the following lemma.

**LEMMA 1.** *Let  $p$  and  $q$  be points of the continuum  $M$ ,  $I_1$  and  $I_2$  be subcontinua of  $M$  which are irreducible between  $p$  and  $q$ , and  $D$  be a subcontinuum of  $M$  containing  $p$ . If the continuum  $D \cup I_1 \cup I_2$  is unicoherent, then  $I_1 \cap M - D = I_2 \cap M - D$ .*

*Proof.* Suppose that  $D \cup I_1 \cup I_2$  is unicoherent. Then  $(D \cup I_2) \cap I_1$  is connected and hence is a subcontinuum of  $I_1$  containing  $\{p, q\}$ . Since  $I_1$  is irreducible between  $p$  and  $q$ , then  $(D \cup I_2) \cap I_1 = I_1$ . Therefore  $I_1 \subset D \cup I_2$  which implies that (i)  $I_1 \cap (M - D) \subset I_2$ .

Likewise  $(D \cup I_1) \cap I_2$  is subcontinuum of  $I_2$  containing  $\{p, q\}$  so  $(D \cup I_1) \cap I_2 = I_2$ . Thus  $I_2 \subset D \cup I_1$  and it follows that (ii)  $I_2 \cap (M - D) \subset I_1$ .

The conclusion follows from (i) and (ii).

**THEOREM 5.** *Let  $M$  be a strongly unicoherent continuum and  $p \in M$ . If  $M$  is both aposyndetic at  $p$  and semi-locally-connected at  $p$ , then  $M$  is hereditarily unicoherent at  $p$ .*

*Proof.* Suppose that  $M$  is aposyndetic and semi-locally-connected

at  $p, q \in M - \{p\}$ , and  $I_1$  and  $I_2$  are subcontinua of  $M$  which are irreducible between  $p$  and  $q$ .

Since  $M$  is aposyndetic and semi-locally-connected at  $p$ , there are subcontinua  $H_1$  and  $K_1$  of  $M$  such that  $p \in H_1 - K_1, q \in K_1 - H_1$ , and  $M = H_1 \cup K_1$  (Theorem 6 of [2]).

Since  $M$  is aposyndetic at  $p$ , according to Theorem 6 of [1],  $M$  is also connected im kleinen at  $p$ . So there is a subcontinuum  $L$  in  $M - (H_1 \cap K_1)$  such that  $p \in \text{int } L$ . Since  $M$  is semi-locally-connected at  $p$ , there is an open set  $V$  such that  $p \in V \subset (\text{int } L)$  and  $M - V$  has a finite number of components. Let  $F_1, \dots, F_n$  be the components of  $M - V$  and without loss of generality assume that  $K_1 \subset F_1$ . Let  $H_2 = L \cup (F_2 \cup F_3 \cup \dots \cup F_n)$  and  $K_2 = F_1$ . Then  $H_2$  and  $K_2$  are subcontinua such that  $p \in H_2 - K_2, q \in K_2 - H_2$ , and  $M = H_2 \cup K_2$ .

Since  $M$  is strongly unicoherent, then  $K_1 \cup I_1 \cup I_2, H_2 \cup I_1 \cup I_2$ , and  $K_2 \cup I_1 \cup I_2$  are unicoherent. So by the preceding lemma,  $I_1 \cap (M - K_1) = I_2 \cap (M - K_1), I_1 \cap (M - H_2) = I_2 \cap (M - H_2)$ , and  $I_1 \cap (M - K_2) = I_2 \cap (M - K_2)$ .

Now  $H_2 \cap K_2 \subset L \subset M - K_1$  so  $I_1 \cap (H_2 \cap K_2) = I_2 \cap (H_2 \cap K_2)$ . So it follows that

$$\begin{aligned} I_1 &= [I_1 \cap (M - H_2)] \cup [I_1 \cap (H_2 \cap K_2)] \cup [I_1 \cap (M - K_2)] \\ &= [I_2 \cap (M - H_2)] \cup [I_2 \cap (H_2 \cap K_2)] \cup [I_2 \cap (M - K_2)] = I_2. \end{aligned}$$

Therefore  $M$  is hereditarily unicoherent at  $p$ .

**COROLLARY 3.** *If the strongly unicoherent  $M$  is aposyndetic at each point, then  $M$  is hereditarily unicoherent.*

*Proof.* By Theorem 4 of [2]  $M$  is semi-locally-connected at each point, so it follows from the preceding theorem that  $M$  is hereditarily unicoherent.

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Received November 1, 1974. A portion of the research reported in this paper was done while the author was a recipient of a research fellowship at the University of Kentucky during the summer of 1974.

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