SETS OF $p$-SPECTRAL SYNTHESIS

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Let $G$ be a Hausdorff locally compact Abelian group, $\Gamma$ its character group. Certain closed subsets of $\Gamma$ are introduced, these being closely related to sets of spectral synthesis for $L'(G)$. Some properties and examples of these sets are discussed, and then a Malliavin-type result is obtained.

In general we follow the notation used in [1]. We shall let $\lambda, \theta$ denote Haar measures on $G, \Gamma$ respectively, chosen so that Plancherel's theorem holds.

1. The definition and some properties of $S_p$- and $C_p$-sets.

**Definition 1.1.** Let $\Xi$ be a closed subset of $\Gamma$. We shall call $\Xi$ an $S_p$-set ($p \in [1, \infty)$) if, given $\epsilon > 0$ and $f \in L' \cap L^p(G)$ such that $\hat{f}$ vanishes on $\Xi$, there exists $g \in L' \cap L^p(G)$ such that $\hat{g}$ vanishes on a neighbourhood of $\Xi$ and $\|f - g\|_p < \epsilon$. If such a $g$ can be found of the form $h \ast f$, where $h \in L'(G)$ and $\hat{h}$ vanishes on a neighbourhood of $\Xi$, then $\Xi$ will be called a $C_p$-set. We also define $S_\infty$- and $C_\infty$-sets as above, with $f, g$ in $\text{V} \Pi \text{C}_\infty(G)$ (rather than $\text{V} \Pi \text{L}_\infty(G)$).

Since, by [1], (33.12), $L'(G)$ admits a bounded positive approximate identity $\{u_i\}_{i \in I}$ such that for each $i \in I$, $u_i \in L' \cap C_0(G)$ and $\text{supp}(u_i)$ is compact, it follows (see [1], (32.33) (b) and (32.48) (a)) that we can (and shall) assume in Definition 1.1 that $f, g, h \in L'(G)$ and $\hat{h}$ vanishes on a neighbourhood of $\Xi$, then we can (and shall) assume in Definition 1.1 that $f, g, h \in L'(G)$ and $\text{supp}(\hat{f})$ is compact and both $\text{supp}(\hat{g})$ and $\text{supp}(\hat{h})$ are compact and disjoint from $\Xi$ ($p \in [1, \infty)$).

Clearly every $C_p$-set is an $S_p$-set. For the case $p = 1$ we just have the familiar $S$-set and $C$-set; see [3], 7.2.5 (a) and 7.5.1 respectively.

For $f \in L'(G)$ the spectrum (written $\Sigma(f)$) will be defined as in [1], (40.21). For $f \in L^p(G)$ ($p \in [1, \infty)$), we define its spectrum by

$$\Sigma(f) = \bigcup \{\Sigma(\phi \ast f) : \phi \in C_0(G)\}$$

It is easily proved that for $f \in L'(G)$, $\Sigma(f) = \text{supp}(\hat{f})$.

Given $\Xi \subseteq \Gamma$, we write

$$L^p_\Xi(G) = \{f \in L^p(G) : \Sigma(f) \subseteq \Xi\}.$$
We now have the following characterisation of $S_p$- and $C_p$-sets:

**Theorem 1.2.** Let $p \in [1, \infty)$ and suppose $\Xi$ is a closed subset of $\Gamma$. Then

(a) $\Xi$ is an $S_p$-set if and only if for all $l \in L_p^\varepsilon(G)$ and for all $f \in L^1 \cap C_0(G)$ such that $\text{supp}(\hat{f})$ is compact and $\hat{f}$ vanishes on $\Xi$, we have $l \ast f = 0$;

(b) $\Xi$ is a $C_p$-set if and only if for all $f \in L^1 \cap C_0(G)$ such that $\text{supp}(\hat{f})$ is compact and $\hat{f}$ vanishes on $\Xi$, and for all $l \in L_p^\varepsilon(G)$ such that $l \ast f \in L_p^\varepsilon(G)$, we have $l \ast f = 0$.

This result is known for the case $p = 1$ (see [2], Chapter 7, 1.2 and 4.9). The proof is standard, and we shall not include it.

It is easy to adapt the proof of [3], Theorem 7.5.2 to give:

**Theorem 1.3.** Let $p \in [1, \infty)$. Then

(a) every one-point subset of $\Gamma$ is a $C_p$-set in $\Gamma$;

(b) finite unions of $C_p$-sets in $\Gamma$ are $C_p$-sets in $\Gamma$;

(c) if the boundary of a closed set $\Xi$ is a $C_p$-set, so is $\Xi$;

(d) if $\Xi$ is a closed subset of a closed subgroup $\Lambda$ of $\Gamma$, if $\partial_\Lambda(\Xi)$ is the boundary of $\Xi$ relative to $\Lambda$, and if $\partial_\Lambda(\Xi)$ is a $C_p$-set in $\Gamma$ then $\Xi$ is also a $C_p$-set in $\Gamma$;

(e) each closed subgroup of $\Gamma$ is a $C_p$-set in $\Gamma$.

For $p \in [1, 2)$ it is not known whether the notions of $C_p$-set and $S_p$-set are identical (it appears in Theorem 2.1 that every closed set is a $C_p$-set for $p \geq 2$). Furthermore we cannot say whether the union of two $S_p$-sets is itself an $S_p$-set. We can however obtain two partial results in this direction. Both these results (Theorem 1.4 (a), (b)) are known for the case $p = 1$ (see [2], Chapter 2, 7.5).

**Theorem 1.4.** (a) Suppose $\Xi = \Xi_1 \cup \Xi_2$, where $\Xi_1$ and $\Xi_2$ are disjoint closed subsets of $\Gamma$. Then, for $p \in [1, \infty)$, $\Xi$ is an $S_p$-set if and only if both $\Xi_1$ and $\Xi_2$ are $S_p$-sets.

(b) Let $p \in [1, \infty)$ and suppose $\Xi_1$ is an $S_p$-set and $\Xi_2$ is a $C_p$-set. Then $\Xi = \Xi_1 \cup \Xi_2$ is an $S_p$-set.

The final result of this section gives us an inclusion result between the set of $C_p$-sets (respectively $S_p$-sets) and the set of $C_q$-sets (respectively $S_q$-sets) for $1 \leq p < q \leq \infty$.

**Theorem 1.5.** Let $1 \leq p < q \leq \infty$. Then every $C_p$-set (respectively $S_p$-set) is a $C_q$-set (respectively $S_q$-set).
Proof. Assume $\Xi$ is a $C_\rho$-set. Suppose we are given $\varepsilon > 0$ and $f \in L^1 \cap C_0(G)$ with $\text{supp}(\hat{f})$ compact and $\hat{f}$ vanishing on $\Xi$. We can find $h \in L^1 \cap C_0(G)$ such that $\|f - h \ast f\|_q < \varepsilon/2$. Since $\Xi$ is a $C_\rho$-set there exists $g \in L^1(G)$ such that $\hat{g}$ has compact support disjoint from $\Xi$ and $\|h\|_r \|f - g \ast f\|_p < \varepsilon/2$, where $p^{-1} + r^{-1} - q^{-1} = 1$ (with the usual convention for the cases $p = 1$ and $q = \infty$). Now (see [1], (20.18))

$$\|f - h \ast g \ast f\|_q \leq \|f - h \ast f\|_q + \|h\|_r \|f - g \ast f\|_p < \varepsilon.$$ 

It remains only to note that $h \ast g \in L^1 \cap C_0(G)$ and $(h \ast g)^\ast$ has compact support disjoint from $\Xi$.

The proof that every $S_\rho$-set is an $S_q$-set is similar.

2. Examples of $S_\rho$- and $C_\rho$-sets.

Theorem 2.1. For $p \in [2, \infty]$ every closed subset of $\Gamma$ is a $C_\rho$-set.

Proof. In view of Theorem 1.5 we need only prove the theorem for $p = 2$.

Let $\Xi$ be a closed subset of $\Gamma$ and suppose we are given $\varepsilon > 0$ and $f \in L^1 \cap C_0(G)$ with $\text{supp}(\hat{f})$ compact, $\hat{f}$ vanishing on $\Xi$ and $\|f\|_1 \leq 1$. Now $\Omega = \{\gamma \in \Gamma: \hat{f}(\gamma) \neq 0\}$ is a relatively compact open set, and hence there exists a compact set $\bar{\Omega} \subseteq \Omega$ such that $\theta(\bar{\Omega} \setminus Y) < \varepsilon^2$. Choose an open set $\nabla$ such that $\bar{\Omega} \subset \nabla \subset \nabla^+ \subset \Omega$, and (see [3], 2.6.1) $k \in L^1 \cap C_0(G)$ such that $\xi_k \subseteq \hat{k} \subseteq \xi_\nabla$. Then, using Plancherel's theorem,

$$\|f - k \ast f\|_2 = \left(\int_{\Omega \setminus Y} |1 - \hat{k}(\gamma)|^2 |\hat{f}(\gamma)|^2 d\theta(\gamma)\right)^{\frac{1}{2}} < \theta(\Omega \setminus Y)\leq \varepsilon;$$ 

and clearly, $\hat{k}$ has compact support disjoint from $\Xi$.

Definition 2.2. Let $\Omega$ be a relatively compact open subset of $\Gamma$. We shall call $\Omega$ a $\beta$-symmetry set ($\beta > 0$) if there exist nets $\{Y_i\}_{i \in I}$ and $\{\nabla_i\}_{i \in I}$ such that each $Y_i$ is compact, $\{\nabla_i\}_{i \in I}$ is a base of symmetric open neighbourhoods of zero in $\Gamma$, partially ordered by

$$\nabla_i < \nabla_j \text{ if and only if } \nabla_i \supset \nabla_j,$$

$(Y_i + 2\nabla_i) \subseteq \Omega$ for each $i \in I$, and

$$\lim_{i \in I} \frac{\theta(\Omega \setminus Y_i)^\beta}{\theta(\nabla_i)} = 0.$$
THEOREM 2.3. Suppose we are given $\beta > 0$ and a closed subset $\Xi$ of $\Gamma$ with the property that for any relatively compact set $Y \subseteq \Xi^c$ there exists a $\beta$-symmetry set $\Omega$ such that $Y \subseteq \Omega \subseteq \Xi^c$. Then $\Xi$ is a $C_p$-set for all $p \geq (2 + \beta)^{-1}(2 + 2\beta)$.

Proof. Let $p = (2 + \beta)^{-1}(2 + 2\beta)$. Suppose we are given $\epsilon > 0$ and $f \in L^1 \cap C_0(G)$, where $\text{supp}(\hat{f})$ is compact, $\hat{f}$ vanishes on $\Xi$ and $\|f\|_p \leq 1$. Now $Y = \{\gamma \in \Gamma: \hat{f}(\gamma) \neq 0\}$ is a relatively compact open subset of $\Xi^c$ and hence, by assumption, there exists a relatively compact open set $\Omega$ such that $Y \subseteq \Omega \subseteq \Xi^c$, and nets $\{Y_i\}_{i \in I}$ and $\{\nabla_i\}_{i \in I}$ satisfying the conditions of Definition 2.2. Choose $i \in I$ such that $Y_i$ is nonvoid and

$$\left[ \frac{\theta(Y_i^c)^{\beta}}{\theta(\nabla_i^c)} \right]^{\alpha/2} < 2^{-\alpha} \theta(\Omega)^{-\alpha/2} \epsilon,$$

where $\alpha = (1 + \beta)^{-1}$. Define $k_i = \theta(\nabla_i)^{-1} g_i h_i$, where $g_i, h_i$ in $L^2(G)$ are such that $\hat{g}_i = \hat{\xi}_i$ (cf. [3], 2.6.1) $k_i \in L^1 \cap C_0(G)$, $\xi_i, \leq \hat{k}_i \leq \xi_{i+2\alpha}$, and

$$\|k_i\|_p \leq \left[ \frac{\theta(Y_i + \nabla_i)}{\theta(\nabla_i)} \right]^{1/2}.$$

It follows from Hölder's inequality that

$$\|f - k_i * f\|_p \leq \|f - k_i * f\|_p^\alpha \|f - k_i * f\|_2^{1-\alpha} \leq \|f\|_p^\alpha \left[ 1 + \left[ \frac{\theta(Y_i + \nabla_i)}{\theta(\nabla_i)} \right]^{1/2} \right]^{\alpha} \theta(\Omega)^{(1-\alpha)/2} \leq 2^\alpha \theta(Y_i + \nabla_i)^{\alpha/2} \frac{\theta(\Omega)^{(1-\alpha)/2}}{\theta(\nabla_i)^{\alpha/2}} < \epsilon$$

(recall that $\alpha = (1 + \beta)^{-1}$ and $p = (2 + \beta)^{-1}(2 + 2\beta) = 2(1 + \alpha^{-1})^{-1}$). Noting that $\hat{k}_i$ has compact support disjoint from $\Xi$ we see that $\Xi$ is a $C_p$-set, and the conclusion follows from Theorem 1.5.

We have two corollaries when $G$ is a Euclidean space.

COROLLARY 2.4. Let $m \geq 1$ and suppose $\Xi \subseteq \mathbb{R}^m$ is an open set with the property that for any relatively compact set $Y \subseteq \mathbb{R}^m$ there exists a number $\kappa_m\ (= \kappa_m(Y))$ such that

$$\theta((\partial(\Xi) \cap Y) + \nabla_n) \leq \kappa_m n^{-1}$$
for all \( n \in \{1, 2, \cdots \} \), where \( \partial(\Xi) \) denotes the boundary of \( \Xi \) and
\[
\nabla_n = \{ x \in \mathbb{R}^m : \| x \| < n^{-1} \}.
\]

Then \( \Xi, \Xi^c \) and \( \partial(\Xi) \) are \( C_p \)-sets for all \( p > (2 + m)^{-1}(2 + 2m) \).

Proof. By Theorem 1.3 (c) we need consider only \( \partial(\Xi) \).

Let \( Y \) be any relatively compact open subset of \( \partial(\Xi)^c \). We shall show that for any \( \epsilon > 0 \) there exists an \( (m + \epsilon) \)-symmetry set \( \Omega \) such that \( Y \subseteq \Omega \subseteq \partial(\Xi)^c \). Since \( Y \) is relatively compact in \( \mathbb{R}^m \) there exists an integer \( n_0 > 0 \) such that
\[
Y \subseteq \Delta_{n_0} = \{ x \in \mathbb{R}^m : \| x \| < n_0 \}.
\]

For each \( n \in \{1, 2, \cdots \} \) define
\[
Y_n = ( \partial(\Xi) + \nabla_n )^c \cap ( \Delta_{n_0} \setminus \Delta_{n_0-n^{-1}} )^c \cap \Delta_{n_0}.
\]

Clearly \( Y_n \) is compact and
\[
( Y_n + 2\nabla_3 )^c \subseteq \Delta_{n_0} \cap \partial(\Xi)^c.
\]

Putting \( \Omega = \Delta_{n_0} \cap \partial(\Xi)^c \) we have
\[
\Omega \setminus Y_n = (\Omega \cap (\partial(\Xi) + \nabla_n)) \cup (\Omega \cap (\Delta_{n_0} \setminus \Delta_{n_0-n^{-1}})) = (\Delta_{n_0} \cap \partial(\Xi)^c \cap (\partial(\Xi) + \nabla_n)) \cup (\Delta_{n_0} \cap \partial(\Xi)^c \cap (\Delta_{n_0} \setminus \Delta_{n_0-n^{-1}})) \subseteq (\Delta_{n_0} \cap (\partial(\Xi) + \nabla_n)) \cup (\Delta_{n_0} \setminus \Delta_{n_0-n^{-1}}).
\]

Hence, since \( \Delta_{n_0} + \nabla_1 \) is relatively compact,
\[
\theta(\Omega \setminus Y_n) \leq \kappa_m (\Delta_{n_0} + \nabla_1)n^{-1} + O(n^{-1}).
\]

Using the fact that
\[
\theta(\nabla_{3n}) = \kappa_m 3^{-m}n^{-m}
\]
for some constant \( \kappa_m \), we have
\[
\lim_{n \to \infty} \frac{\theta(\Omega \setminus Y_n)^{m+\epsilon}}{\theta(\nabla_{3n})} = 0,
\]
and so \( \Omega \) is an \( (m + \epsilon) \)-symmetry set for all \( \epsilon > 0 \).
Thus $\vartheta(\Xi)$ satisfies the conditions of Theorem 2.3 with $\beta = m + \epsilon$, and hence is a $C_p$-set for all $p > (2 + m)^{-1}(2 + 2m)$.

**Corollary 2.5.** Let $m \geq 1$ and put

$$\Xi = \{x \in R^m : \|x\| = 1\}.$$

Then \(\Xi\) is a $C_p$-set for all $p > (2 + m)^{-1}(2 + 2m)$.

**Proof.** Let $\nabla$ be any relatively compact set in $R^m$. Then

$$\vartheta((\Xi \cap \nabla) + \nabla_n) \leq \vartheta(\Xi + \nabla_n)$$

$$= \kappa'_m((1 + n^{-1})^m - (1 - n^{-1})^m)$$

$$= O(n^{-1}),$$

where $\kappa'_m$ is a constant. Now apply Corollary 2.4.

**Remark 2.6.** For $m \geq 3$, Corollary 2.5 gives an example of a $C_p$-set $((2 + m)^{-1}(2 + 2m) < p < 2)$ which is not an $S$-set; cf. [3], 7.3.2.

### 3. The failure of certain closed sets to be $S_p$-sets

In this section we use a proof along the lines of that of Malliavin's theorem ([3], 7.6.1) to show that every nondiscrete $\Gamma$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$. As in the proof of [3], Theorem 7.6.1, we first consider the cases:

(a) $\Gamma$ is an infinite compact group;
(b) $\Gamma = R$.

**Theorem 3.1.** Let $G$ be an infinite discrete group. Then there exists a closed set $\Xi \subseteq \Gamma$ which is not an $S_p$-set for any $p \in [1, 2)$.

**Proof.** Using the notation of [3], Theorem 7.8.6 we consider the function $\phi_1$ on $G$ defined by

$$\phi_1 : x \mapsto (D'm_x)(\xi).$$

It is easily proved from [3], 7.6.4 and Theorem 7.8.6 that $f_0 \in L'(G)$ and $\phi_1$ (as above) can be chosen so that $f_0$ and $\xi$ satisfy the hypotheses of [3], 7.6.3 (Theorem) (with $f = f_0$ and $\xi = \xi$) and $\phi_1 \in L^q(G)$ for all $q > 2$. Having thus chosen $f_0$ and $\phi_1$ we shall prove that the closed set $\Xi = \{\gamma \in \Gamma : f_0(\gamma) = \xi\}$ is not an $S_p$-set for any $p \in [1, 2)$. 


Let $p \in [1, 2)$ and put

$I = \{f \in L'(G): \hat{f}(\Xi) = \{0\}\},$

$I_1 = \text{the closed ideal of } L'(G) \text{ generated by } f_0 - \xi \xi_{(0)},$

$I_2 = \text{the closed ideal of } L'(G) \text{ generated by } (f_0 - \xi \xi_{(0)})^2,$

and $J = \{f \in L'(G): \hat{f} \text{ vanishes on a neighbourhood of } \Xi\}^-.$

Clearly

$$\Xi = Z(I) = Z(I_1) = Z(I_2) = Z(J)$$

(where $Z(I)$ denotes the zero set of the ideal $I$; see [3], 7.1.3). Since $I$ and $J$ are respectively the largest and smallest closed ideals in $L'(G)$ having $\Xi$ as their zero set, we have that $J \subseteq I_2 \subseteq I_1 \subseteq I.$

As $\phi_1 \in L^p(G)$ we can define a continuous linear functional $T$ on $(L'(G), \|\cdot\|_p)$ by

$$T(g) = \sum_{x \in G} g(-x)\phi_1(x)$$

(recall that $G$ is discrete and hence $L'(G) \subseteq L^p(G)$). By [3], 7.6.3, $T$ annihilates $I_2$ but not $I_1.$

Now suppose that $\Xi$ is an $S_p$-set and let $h \in L^1 \cap C_0(G) = L'(G)$ with $\hat{h}$ vanishing on $\Xi.$ Then, given $\epsilon > 0,$ there exists $h' \in J$ such that

$$\|h - h'\|_p < \epsilon$$

and hence, since $T(h') = 0,$ $|T(h)| = |T(h - h')| \leq \epsilon \|\phi_1\|_p.$ As this holds for all $\epsilon > 0$ we must have that $T(h) = 0;$ thus $T$ annihilates $I,$ a contradiction of the fact that $T$ does not annihilate $I_1 \subseteq I.$ It follows that $\Xi$ is not an $S_p$-set for any $p \in [1, 2).$

We shall now examine the case when $\Gamma$ contains an infinite compact open subgroup. We require two lemmas for arbitrary Hausdorff locally compact Abelian groups.

**Lemma 3.2.** Let $G$ be a Hausdorff locally compact Abelian group and suppose $H$ is a closed subgroup of $G.$ Then a continuous integrable function $f$ on $G$ is constant on cosets of $H$ if and only if

$$\text{supp}(\hat{f}) \subseteq A(\Gamma, H)$$

(the annihilator of $H$ in $\Gamma$).

**Proof.** The result follows readily from the property

$$(\kappa f)'(\gamma) = \gamma(h)\hat{f}(\gamma)$$

for all $\gamma \in \Gamma$ (where $\kappa f: x \rightarrow f(x + h)$).
LEMMA 3.3. Let $G$ be a Hausdorff locally compact Abelian group and suppose $\Lambda$ is an open subgroup of $\Gamma$. If $\Xi$ is a closed subset of $\Lambda$ which is not an $S_p$-set in $\Lambda$ then $\Xi$ is not an $S_p$-set in $\Gamma$.

Proof. Put $H = A(G, \Lambda)$. By [1], (23.24) (e), $H$ is compact. Furthermore, in view of Theorem 2.1, we can assume that $p < \infty$.

Suppose, to the contrary, that $\Xi$ is an $S_p$-set in $\Gamma$. Given $\epsilon > 0$ and $\hat{f} \in L^1 \cap C_0(G/H)$ such that $\text{supp}(\hat{f})$ is compact and $\hat{f}$ vanishes on $\Xi$, put $f = \hat{f} \circ \pi_H$, where $\pi_H$ denotes the natural homomorphism of $G$ onto $G/H$. Denoting the Haar measures on $H, G/H$ by $\lambda_H, \lambda_{G/H}$ respectively (normalised as in [2], Chapter 3, 3.3 (i) with $\lambda_H(H) = 1$) we have, by [2], Chapter 3, 4.5,

$$
\|f\|_p = \int_{G/H} \left\{ \int_H |f(x+y)|^p \, d\lambda_H(y) \right\} d\lambda_{G/H}(\hat{x})
$$

$$
= \int_{G/H} \left\{ \int_H \hat{f}(x+y)^p \, d\lambda_H(y) \right\} d\lambda_{G/H}(\hat{x})
$$

$$
= \int_{G/H} \hat{f}(\hat{x})^p \, d\lambda_{G/H}(\hat{x}),
$$

that is,

$$
(3.1) \quad \|f\|_p = \|\hat{f}\|_p.
$$

It is easily seen that

$$
\hat{f}(\hat{x}) = \int_H f(x+y) d\lambda_H(y)
$$

and, by [2], Chapter 4, 4.3 ((3.1) shows that $f \in L^1(G)$),

$$
(3.2) \quad \hat{f}(\gamma) = f(\gamma)
$$

for all $\gamma \in \Lambda$. Furthermore, since $f$ is constant on cosets of $H$, Lemma 3.2 shows that $\text{supp}(\hat{f}) \subset A(\Gamma, H) = \Lambda$. As $\text{supp}(\hat{f})$ is assumed to be compact it follows from (3.2) that $\text{supp}(\hat{f})$ is compact and hence (note that $f$ is continuous) we see that $f \in C_0(G)$.

Now $\hat{f}$ vanishes on $\Xi \cup \Lambda^c$ and, since by Theorem 1.4 (recall that $\Lambda^c$ is open and closed) $\Xi \cup \Lambda^c$ is an $S_p$-set, there exists $g \in L^1 \cap C_0(G)$ such that $\hat{g}$ has compact support disjoint from $\Xi \cup \Lambda^c$ and $\|f-g\|_p < \epsilon$. By Lemma 3.2 again $g$ is constant on cosets of $H$ and we have the existence of $\hat{g} \in L^1 \cap C_0(G/H)$ such that $g = \hat{g} \circ \pi_H (\hat{g} \in C_0(G/H))$ since, by [2], Chapter 3, 1.8 (vii), $\hat{g}$ is continuous and by (3.2), $\hat{g} \circ \pi_H$ has compact support. From (3.1) $\|\hat{f} - \hat{g}\|_p < \epsilon$, and (3.2) shows that $\hat{g}$ vanishes on a
neighbourhood of $\Xi$. Hence $\Xi$ is shown to be an $S_p$-set in $\Lambda$, contrary to assumption.

**Corollary 3.4.** Let $G$ be a Hausdorff locally compact Abelian group, $\Gamma$ its character group. If $\Gamma$ contains an infinite compact open subgroup then there exists a closed subset of $\Gamma$ which is not an $S_p$-set for any $p \in [1, 2)$.

**Proof.** Combine Theorem 3.1 and Lemma 3.3.

Before considering the case $\Gamma = R$ we need to extend the result in [3], Theorem 2.7.6.

**Theorem 3.5.** Suppose $f \in l^1(Z)$, $\delta \in (0, \pi)$ and $\hat{f}(\exp(ix)) = 0$ for $x \in [\pi - \delta, \pi + \delta]$. Let $u$ be defined on $R$ by

$$u(x) = \begin{cases} \hat{f}(\exp(ix)) & (|x| \leq \pi) \\ 0 & (|x| > \pi). \end{cases}$$

Then $u = \hat{g}$ for some $g \in L^1(R)$. Moreover, given $p \in [1, \infty]$, there exists a positive number $\kappa_p (= \kappa_p(\delta))$ such that

$$\|f\|_p \leq \kappa_p \|g\|_p.$$

**Proof.** The first part of Theorem 3.5 is proved in [3], 2.7.6. Let $p \in [1, \infty]$. Consider the linear operator $T$ from $L^1 \cap L^\alpha(R)$ to $l^1(Z)$, defined by

$$(T(k))(n) = k * \hat{h}(n),$$

where $n \in Z$, and $h \in L^1(R)$ is defined as in [3], 2.7.6. The argument at the end of the proof of [3], 2.7.6 shows that there is a constant $\kappa_1 = \kappa_1(\delta)$ such that $\|T(k)\|_1 \leq \kappa_1 \|k\|_1$. It is clear from (3.3) that $\|T(k)\|_\alpha \leq \kappa_2 \|k\|_\alpha$, where $\kappa_2 = \|\hat{h}\|_\alpha$. By the Riesz-Thorin convexity theorem $T$ is continuous as

$$(L^1 \cap L^\alpha(R), \|\cdot\|_{p_\alpha}) \rightarrow (l^1(Z), \|\cdot\|_{p_\alpha})$$

(recall that $l^1(Z) \subset l^\alpha(Z)$), where $\alpha \in (0, 1)$, $p_\alpha = (1 - \alpha)^{-1}$ and $\|T\|_{p_\alpha} \leq \kappa_1^{\frac{\alpha}{1-\alpha}} \kappa_2^\alpha$. In particular, choosing $\alpha \in [0, 1)$ such that $p_\alpha = p$ (and $\alpha = 1$ if $p = \infty$) and noting that $g \in L^1 \cap L^\alpha(R)$ and (see [3], 2.7.6, (5)) $f(n) = g * \hat{h}(n)$ for all $n \in Z$, we have

$$\|f\|_p \leq \kappa_1^{\frac{\alpha}{1-\alpha}} \kappa_2^\alpha \|g\|_p,$$

as required.
THEOREM 3.6. The real line $\mathbb{R}$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$.

Proof. It appears from Theorem 3.1 that there exists a closed set $\Xi_1 \subset T$ (the circle group) which is not an $S_p$-set for any $p \in [1, 2)$. By translation if necessary we can assume that $-1 \not\in \Xi_1$ and that $\Xi_1$ is disjoint from $\Xi_2$ for some closed arc $\Xi_2 \subset T$ containing $-1$. Put

$$Y_1 = \{x \in (-\pi, \pi): \exp(ix) \in \Xi_1\},$$

$$Y_2 = \{x \in (-\pi, \pi): \exp(ix) \in \Xi_2\} \cup [\pi, \infty) \cup (-\infty, -\pi],$$

$$\Xi = \Xi_1 \cup \Xi_2 \quad \text{and} \quad Y = Y_1 \cup Y_2.$$

Let $p \in [1, 2)$ and suppose $Y_1$ is an $S_p$-set. By Theorem 1.4, $Y$ is an $S_p$-set. Given $f \in L^1(Z)$ with $\hat{f}(\Xi) = \{0\}$ define $g \in L^1 \cap C_0(\mathbb{R})$ by

$$\hat{g}(x) = \begin{cases} \hat{f}(\exp(ix)) & (|x| \leq \pi) \\ 0 & (|x| > \pi) \end{cases}$$

(see Theorem 3.5). Clearly $\hat{g}$ vanishes on $Y$ and hence, since $Y$ is an $S_p$-set, there exists a sequence $(g_n) \subset L^1 \cap C_0(\mathbb{R})$ such that each $\hat{g}_n$ vanishes on a neighbourhood of $Y$ and

$$\|g - g_n\|_p \to 0. \quad (3.4)$$

If, for each $x \in (-\pi, \pi]$, we define $f_n \in L^1(Z)$ by

$$\hat{f}_n(\exp(ix)) = \hat{g}_n(x)$$

(see [3], Theorem 2.7.6) then Theorem 3.5 applied to (3.4) gives $\|f - f_n\|_p \to 0$ (note that each $\hat{f}_n$ vanishes on a neighbourhood of $\Xi$). Hence $\Xi$ and consequently (see Theorem 1.4) $\Xi_1$ would be an $S_p$-set, contradicting our choice of $\Xi_1$. It follows that $Y_1$ is not an $S_p$-set for any $p \in [1, 2)$.

We require two lemmas before proving the main result of this section.

LEMMA 3.7. Let $G, H$ be Hausdorff locally compact Abelian groups and suppose $k \in L^1 \cap C_0(G \times H)$ is such that $Y = \text{supp}(k)$ is compact. Then the function $y \to k(x, y)(x \to k(x, y))$ is integrable over
H for every $x \in G$ (over $G$ for every $y \in H$). Furthermore the functions

$$
\phi_1: x \mapsto \int_H k(x, y) d\lambda_H(y), \quad \phi_2: y \mapsto \int_G k(x, y) d\lambda_G(x)
$$

are continuous.

Proof. Since $k$ is continuous the function $y \mapsto k(x, y)$ is continuous, and hence measurable, for every $x \in G$.

Choose $k_1(k_2)$ in $L^1 \cap C_0(G)(L^1 \cap C_0(H))$ such that $\hat{k}_1 = 1$ ($\hat{k}_2 = 1$) on a neighbourhood $\mathcal{V}_1(\mathcal{V}_2)$ of $Y_G(Y_H)$, where $Y_G, Y_H$ are the projections of $Y$ onto $G, H$ respectively. If we define $h$ on $G \times H$ by $h[(x, y)] = k_1(x)k_2(y)$ then [1], (31.7) (b) shows that $\hat{h} = 1$ on $\mathcal{V}_1 \times \mathcal{V}_2$, a neighbourhood of $Y$. Thus $h * k = k$ 1.a.e. and, since $h * k$ and $k$ are continuous,

$$
(3.5) \quad h * k = k.
$$

Now the map $\nu_x$ on $H \times G \times H$, defined by

$$
\nu_x[(y, s, t)] = h(x - s, y - t)k(s, t),
$$

is continuous for every $x \in G$. Applying [1], (13.4) to $|\nu_x|$, considered as a function on $H \times (G \times H)$, it follows that $\nu_x$ is integrable and, using (3.5), that the function $y \mapsto k(x, y)$ is integrable over $H$ for every $x \in G$. Furthermore, since $\nu_x$ is integrable on $H \times (G \times H)$, we can use (3.5) and [1], (13.8) to deduce that

$$
\phi_1(x) = \int_H k_2(y) d\lambda_H(y) \int_{G \times H} k_1(x - s)k(s, t) d\lambda_G \times \lambda_H(s, t).
$$

As $k \in L^1(G \times H)$, $k_2 \in L^1(H)$ and $k_1$ is uniformly continuous it follows that $\phi_1$ is continuous.

The other part of the lemma is proved similarly.

**Lemma 3.8.** Suppose $G, H$ are Hausdorff locally compact Abelian groups, with character groups $\Gamma, \Lambda$ respectively. If $p \in [1, 2)$ and the closed set $\Xi' \subset \Gamma$ is not an $S_p$-set, then $\Xi = \Xi' \times \Lambda$ is not an $S_p$-set in $\Gamma \times \Lambda$.

Proof. Suppose to the contrary that $\Xi$ is an $S_p$-set in $\Gamma \times \Lambda$. Let $f \in L^1 \cap C_0(G)$ with $\text{supp}(\hat{f})$ compact and $\hat{f}$ vanishing on $\Xi$, and choose $g \in L^1 \cap C_0(H)$ such that $\text{supp}(\hat{g})$ is compact and $|g(y)| \equiv 1$ for all $y$ in...
some neighbourhood $V$ of zero in $H$. Define $h$ on $G \times H$ by 
$h[(x, y)] = f(x)g(y)$. Then, by [1], (31.7) (b), supp($\hat{h}$) is compact and

$$\hat{h}([\gamma_1, \gamma_2]) = \hat{f}(\gamma_1)\hat{g}(\gamma_2) = 0$$

for all $[\gamma_1, \gamma_2] \in \Xi$.

Let $\epsilon > 0$ be given. Since $\Xi$ is assumed to be an $S_p$-set we can find $k \in L' \cap C_0(G \times H)$ such that supp$(\hat{k})$ is compact and disjoint from $\Xi$, and

$$\|h - k\|_p < \epsilon \lambda_H(V)^{1/p}. \tag{3.6}$$

Thus, for all $\gamma_1$ in some neighbourhood $V$ of $\Xi'$ and for all $\gamma_2 \in \Lambda$, we have (see [1], (13.8))

$$\int_{\Lambda} \left\{ \int_{G} k(x, y)\tilde{\gamma}_1(x) d\lambda_G(x) \right\} \tilde{\gamma}_2(y) d\lambda_H(y)$$

$$= \int_{G \times H} k(x, y)([\gamma_1, \gamma_2])^{-1}(x, y) d\lambda_G \times \lambda_H(x, y)$$

$$= 0.$$

Since $\gamma_2 \in \Lambda$ was chosen arbitrarily

$$\int_{G} k(x, y)\tilde{\gamma}_1(x) d\lambda_G(x) = 0 \quad \lambda_H - \text{a.e.}$$

Now

$$\psi: (x, y) \rightarrow k(x, y)\tilde{\gamma}_1(x)$$

is continuous and integrable, and supp($\hat{\psi}$) is compact. Hence, by Lemma 3.7, the function $\phi$ on $H$ defined by

$$\phi(y) = \int_{G} \psi(x, y) d\lambda_G(x)$$

is continuous and so, for all $y \in H$ and $\gamma_1 \in \nabla$,

$$\int_{G} k(x, y)\tilde{\gamma}_1(x) d\lambda_G(x) = 0. \tag{3.7}$$

Using (3.6) we see that

$$W = \left\{ y \in V: \int_{G} |h(x, y) - k(x, y)|^p d\lambda_G(x) < \epsilon^p \right\}$$
has the property that $\lambda_H(V \setminus W) < \lambda_H(V)$, that is, $\lambda_H(W) > 0$. Choose any $y_0 \in W$ ($W$ is nonempty). Then

$$
(3.8) \quad \int_G |f(x) - g(y_0)^{-1}k(x, y_0)|^p d\lambda_G(x) < \epsilon^p \left| g(y_0)^{-1} \right| \leq \epsilon^p
$$

and so, defining $f_1 \in L^1 \cap C_0(G)$ by $f_1(x) = g(y_0)^{-1}k(x, y_0)$, (3.7) shows that $f_1$ vanishes on $V$ and, from (3.8), $\|f - f_1\|_p < \epsilon$; thus we have a contradiction of the assumption that $\Xi'$ is not an $S_p$-set.

**Theorem 3.9.** Let $G$ be a Hausdorff noncompact locally compact Abelian group, $\Gamma$ its character group. Then $\Gamma$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$.

**Proof.** By [1], (24.30), $\Gamma$ is topologically isomorphic with $\mathbb{R}^n \times \Gamma_0$, where $\Gamma_0$ is a Hausdorff locally compact Abelian group containing a compact open subgroup.

If $n \geq 1$ then Theorem 3.6 and Lemma 3.8 combine to show that $\mathbb{R}^n \times \Gamma_0$ contains a closed set which is not an $S_p$-set for any $p \in [1, 2)$.

If $n = 0$ then $\Gamma$ contains a compact open subgroup (with is infinite since $\Gamma$ is nondiscrete) and the result follows from Corollary 3.4.

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