COMPACT SUBSETS OF A TYCHONOFF SET

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The paper establishes a relation between the partial exponential law and the compactness of certain subsets of Tychonoff sets of multifunctions, and deduces consequences bearing on the Ascoli theorems established by Weston and Lin-Rose.

1. Introduction. The "Tychonoff set" is an abstraction of a class of sets arising in extensions of the classical Tychonoff theorem to multifunction context ([2],[5]). Extending the definition of the partial exponential law to multifunctions, we show that, when it is satisfied for a topology \( \tau \), certain subsets of a Tychonoff set are \( \tau \)-compact. This approach — which is a non-trivial modification of the method introduced into function Ascoli theory by Noble [7] — will yield, in particular, sufficient conditions for compactness relative to the compact open topology.

In [6] Lin and Rose introduced a multifunction extension of the Kelley-Morse notion of even continuity, and proved a multifunction Ascoli theorem of the Weston type, without, however, showing that it contains the prototype [11, p. 20]. We deduce from our criterion a generalization of the Lin-Rose theorem. We show that this generalization contains the Weston Ascoli theorem and yields corollaries equivalent to the Tychonoff theorems for point-compact and point-closed multifunctions established in [2].

2. Multifunctions. We review the established definitions for multifunctions ([1],[9],[10]): Let \( X, Y \) be nonempty sets. A multifunction is a point to set correspondence \( f: X \to Y \) such that, for all \( x \in X, fx \) is a nonempty subset of \( Y \). For \( A \subseteq X, B \subseteq Y \) it is customary to write \( f(A) = \bigcup_{x \in A} fx, f^{-1}(B) = \{x: x \in X \text{ and } fx \cap B \neq \emptyset\} \) and \( f^+(B) = \{x: x \in X \text{ and } fx \subseteq B\}. \) If \( Y \) is a topological space, a multifunction \( f: X \to Y \) is point-compact (point-closed) if \( fx \) is compact (closed) for all \( x \in X \). If \( X, Y \) are topological spaces, a multifunction \( f: X \to Y \) is continuous if \( f^{-1}(U), f^+(U) \) are open in \( X \) whenever \( U \) is open in \( Y \). Henceforth the set of all continuous multifunctions (continuous functions) on a topological space \( X \) to a topological space \( Y \) will be denoted \( C(X, Y)(C(X, Y)) \).

Let \( \{Y_x\}_{x \in X} \) be a family of nonempty sets. The \( m \)-product \( P\{Y_x: x \in X\} \) of the \( Y_x \) is the set of all multifunctions \( f: X \to \bigcup_{x \in X} Y_x \).
such that \( fx \subseteq Y_x \) for all \( x \in X \). In the case \( Y_x = Y \) for all \( x \in X \), the \( m \)-product of the \( Y_x \), denoted \( Y^{mX} \), is the set of all multifunctions on \( X \) to \( Y \). For \( x \in X \), the \( x \)-projection \( pr_x : P\{Y_x : x \in X\} \to Y_x \) is the multifunction defined by \( pr_x f = fx \). If the \( Y_x \) are topological spaces, the \textit{pointwise topology} \( \tau_p \) on \( P\{Y_x : x \in X\} \) is defined to be the topology having as open subbase the sets of the forms \( pr_x^{-1}(U_x), pr_x^+(U_x) \), where \( U_x \) is open in \( Y_x \), \( x \in X \) ([5], [8]).

For \( F \subseteq Y^{mX}, x \in X \), we write \( F[x] = \bigcup_{f \in F} fx \). Let \( Y \) be a topological space. We say that a subset \( F \) of \( Y^{mX} \) is \textit{pointwise bounded} if \( F[x] \) has compact closure in \( Y \) for all \( x \in X \). We say that a subset \( T \) of \( Y^{mX} \) is \textit{Tychonoff} if, for every pointwise bounded subset \( F \) of \( T \), \( T \cap P\{F[x] : x \in X\} \) is \( \tau_p \)-compact. The following subsets of \( Y^{mX} \) are Tychonoff:

1. \( Y^X \), by the classical Tychonoff theorem.
2. \( Y^{mX} \), by the theorem of Lin [5, p. 400].
3. The set of all point-closed members of \( Y^{mX} \), by Corollary 2 of [2].
4. The set of all point-compact members of \( Y^{mX} \), by Corollary 3 of [2].

**Lemma 2.1.** If \( F \) is a pointwise bounded subset of a Tychonoff set \( T \), then the \( \tau_p \)-closure of \( F \) is compact.

**Proof.** Let \( \bar{F} \) denote the \( \tau_p \)-closure of \( F \). Since \( P\{\bar{F}[x] : x \in X\} \cap T \) is a \( \tau_p \)-compact subset of \( T \), if suffices to show that \( \bar{F} \subseteq P\{\bar{F}[x] : x \in X\} \). Let \( f \in \bar{F} \). We must show that, for \( x \in X, y \in fx \) and an open neighbourhood \( V \) of \( y, F[x] \cap V \neq \emptyset \). Since \( M = \{h : h \in T \text{ and } hx \cap V \neq \emptyset\} \) is a \( \tau_p \)-neighborhood of \( f \), there exists \( h' \in M \cap F \). Then \( h'x \cap V \neq \emptyset \) and \( h'x \subseteq \bar{F}[x] \), so \( F[x] \cap V \neq \emptyset \).

Let \( X, Y \) be topological spaces. The multifunction \((f, x) \to fx \) on \( Y^{mX} \times X \) to \( Y \), or any restriction, will be denoted by the symbol \( \omega \). Let \( F \subseteq Y^{mX} \). A topology \( \tau \) on \( F \) is said to be \textit{jointly continuous} if \( \omega : (F, \tau) \times X \to Y \) is continuous [8, p. 48]. The \textit{compact open topology} \( \tau_c \) on \( Y^{mX} \) is defined to be the topology having as open subbase the sets of the forms \( \{f : f(K) \subseteq U\}, \{f : fx \cap U \neq \emptyset \text{ for all } x \in K\} \), where \( K \) is a compact subset of \( X \) and \( U \) is open in \( Y \) ([6, p. 742], [8, p. 47]). Obviously \( \tau_c \) is larger than \( \tau_p \).

3. **Partial exponential law.** Let \( X, Y, Z \) be topological spaces. An element \( f \in Z^{m(X \times Y)} \) determines the function \( \tilde{f} : x \to f(x, \cdot) \) on \( X \) to \( Z^{mY} \). The function \( \mu : f \to \tilde{f} \), called the \textit{exponential map}, is a bijection of \( Z^{m(X \times Y)} \) onto \( (Z^{mY})^X \). It is clear that if \( f \in \mathcal{C}(X \times Y, Z) \), then \( \tilde{f}(x) = f(x, \cdot) \in \mathcal{C}(Y, Z) \) for all \( x \in X \). When \( \tau \) is a topology on
Z^m_Y, we say that (X, Y, Z, \tau) satisfies the partial exponential law if 
\mu(\mathcal{E}(X \times Y, Z)) \subseteq C(X, (\mathcal{E}(Y, Z), \tau)).

We establish now the main theorem of the paper:

**Theorem 3.1.** Let T be a Tychonoff set of multifunctions on a
topological space X to a topological space Y, and let \tau be a topology on
\text{Y}^m_X such that (K, X, Y, \tau) satisfies the partial exponential law for all
compact spaces K. Then a subset F of T is \tau-compact if

(a) F is \tau-closed in T,
(b) F is pointwise bounded, and
(c) \tau_p is jointly continuous on the \tau_p-closure of F in T.

**Proof.** Let \bar{F} denote the \tau_p-closure of F in T and let \omega: (\bar{F}, \tau_p) \times
X \to Y. By (c), \omega is continuous, so \bar{F} \subseteq \mathcal{E}(X, Y). Since T is a
Tychonoff set, (b) implies, by Lemma 2.1, that \bar{F} is \tau_p-compact. Then
\bar{\omega}: (\bar{F}, \tau_p) \to ((\mathcal{E}(X, Y), \tau) is continuous. Since \bar{\omega} is the inclusion map,
\bar{F} = \bar{\omega}(\bar{F}) is \tau-compact. Since, by (a), F is \tau-closed in \bar{F}, it follows
that F is \tau-compact.

The application of this theorem to \tau_c depends on the following
generalization to multifunctions of Lemma 1 of R. H. Fox [3, p. 430]:

**Lemma 3.2.** (X, Y, Z, \tau_c) satisfies the partial exponential law.

**Proof.** Let f \in \mathcal{E}(X \times Y, Z). Let x \in X. Since f(x, \cdot) = f \circ j,
where j(y) = (x, y) (y \in Y), f(x, \cdot) is continuous [9, p. 35]. Thus \tilde{f}
maps X into \mathcal{E}(Y, Z). It remains to show that \tilde{f}: X \to ((\mathcal{E}(Y, Z), \tau_c) is
continuous.

Let \mathcal{M} = \{h: h \in \mathcal{E}(Y, Z) and h(K) \subseteq U\}, where K is a compact
subset of Y and U is open in Z. Let x_0 \in \tilde{f}^{-1}(\mathcal{M}). Then f(x_0, \cdot) \in \mathcal{M},
so \{x_0\} \times K \subseteq f^*(U). By the theorem of Wallace [4, p. 142], there is a
neighbourhood V of x_0 such that V \times K \subseteq f^*(U). Let x \in V. Then,
for all y \in K, \tilde{f}(x)y = f(x, y) \subseteq U, so \tilde{f}(x)(K) \subseteq U. Thus x \in \tilde{f}^{-1}(\mathcal{M}),
and we have shown that \tilde{f}^{-1}(\mathcal{M}) is open in X.

Let \mathcal{M} = \{h: h \in \mathcal{E}(Y, Z) and hy \cap U \neq \emptyset for all y \in K\}, where K
is a compact subset of Y and U is open in Z. Let x_0 \in \tilde{f}^{-1}(\mathcal{M}). Then
f(x_0, \cdot) \in \mathcal{M}, so \{x_0\} \times K \subseteq f^*(U). There is a neighbourhood V of x_0
such that V \times K \subseteq f^*(U). Let x \in V. Then, for all y \in K, \tilde{f}(x)y \cap
U \neq \emptyset, so \tilde{f}(x) \in \mathcal{M}, that is, x \in \tilde{f}^{-1}(\mathcal{M}), and we have shown that \tilde{f}^{-1}(\mathcal{M})
is open in X.

**4. Even continuity.** Let X, Y be topological spaces and let
\text{F} \subseteq \text{Y}^m_X. Following [6], we say that \text{F} is even continuity if, for each
(x, y) ∈ X × Y and each neighborhood V of y, there exist neighbourhoods U, W of x, y, respectively, such that
(a) \( f ∈ F \) and \( fx ∩ W ≠ ∅ \) imply \( U ⊆ f^−(V) \), and
(b) \( f ∈ F, fx ∩ W ≠ ∅ \) and \( fx ⊆ V \) imply \( f(U) ⊆ V \).

**Lemma 4.1.** Let \( X, Y \) be topological spaces and let \( F ⊆ Y^{mX} \). If \( F \) is evenly continuous, \( τ_p \) on \( F \) is jointly continuous.

*Proof.* Let \( ω : (F, τ_p) × X → Y \). Suppose that \( (f, x) ∈ ω^−(V) \), where \( V \) is open in \( Y \). Choose \( y ∈ fx ∩ V \). Then there exist open neighbourhoods \( U, W \) of \( x, y \), respectively, such that \( g ∈ F \) and \( gx ∩ W ≠ ∅ \) imply \( U ⊆ g^−(V) \). Write \( M = \{ h : h ∈ F \text{ and } hx ∩ W ≠ ∅ \} \). Then \( M × U \) is a neighbourhood of \( (f, x) \), which is contained in \( ω^−(V) \). Now suppose that \( (f, x) ∈ ω^+(V) \), where \( V \) is open in \( Y \). Then \( fx ⊆ V \). Choose \( y ∈ fx \). There exist open neighbourhoods \( U, W \) of \( x, y \), respectively, such that \( g ∈ F, gx ∩ W ≠ ∅ \) and \( gx ⊆ V \) imply \( g(U) ⊆ V \). Write \( M = \{ h : h ∈ F, hx ∩ W ≠ ∅ \text{ and } hx ⊆ V \} \). Then \( M × U \) is a neighbourhood of \( (f, x) \), which is contained in \( ω^+(V) \).

**Corollary 4.2.** Let \( X, Y \) be topological spaces and let \( F ⊆ Y^{mX} \). If \( F \) is evenly continuous, then each member of \( F \) is continuous.

The following result, which generalizes the Ascoli theorem of Lin and Rose [6, p. 746], contains also the Weston Ascoli theorem [11, p. 20]:

**Theorem 4.3.** Let \( T \) be a Tychonoff set of multifunctions on a topological space \( X \) to a topological space \( Y \). Then a subset \( F \) of \( T \) is \( τ_c \)-compact if

(a) \( F \) is \( τ_c \)-closed,
(b) \( F \) is pointwise bounded, and
(c) \( F \) is evenly continuous.

*Proof.* By Lemma 3.2, this theorem will follow as a corollary of Theorem 3.1 if we show that \( τ_p \) is jointly continuous on the \( τ_p \)-closure \( \bar{F} \) of \( F \). By (c) and Lemma 4.1, \( τ_p \) on \( F \) is jointly continuous. By (a), \( F = \bar{F} \), where \( \bar{F} \) is the \( τ_c \)-closure of \( F \). Finally, by (c) and Lemma 3.1 of [6, p. 744], \( \bar{F} = \bar{F} \).

**Corollary 4.4.** Let \( (Y^{mX})_0((Y^{mX})_1) \) be the set of all point-compact (point-closed) multifunctions on a topological space \( X \) to a
topological space $Y$. Then a subset $F$ of $(Y^mX)_0((Y^mX)_1)$ is $\tau_c$-compact if

(a) $F$ is $\tau_c$-closed,
(b) $F$ is pointwise bounded, and
(c) $F$ is evenly continuous.

5. REMARKS. The Lin-Rose Ascoli theorem [6, p. 746], depends, apart from Lemma 3.1 of [6], on the Tychonoff theorem of Lin [5, p. 400]. Consequently, the Corollary 4.4 can be proved by the Lin-Rose argument, using the Tychonoff theorems of [2]. We will prove the converse implication: Let $\{Y_x\}_{x \in X}$ be a family of compact spaces. We will deduce from Corollary 4.4 that $F = (P\{Y_x : x \in X\})_0$ is $\tau_p$-compact.

We may suppose the $Y_x$ disjoint. Assign to $X$ the discrete topology and let $Y = \bigcup_{x \in X} Y_x$ have the sum topology. We have $F \subseteq (Y^mX)_0$ and, since $X$ is discrete, $F$ is evenly continuous [6, p. 743]. Since $F[x] = Y_x$ and $Y_x$ is closed in $Y$, $F$ is pointwise bounded. If we show that $F$ is $\tau_p$-closed, it will follow from Corollary 4.4 that $F$ is $\tau_c$-compact and therefore $\tau_p$-compact. Let $\{f_x\}$ be a net in $F$ which is $\tau_p$-convergent to an element $f \in (Y^mX)_0$. Let $x \in X$. Let $y \in f_x$, and let $V$ be an open neighborhood of $y$. Since $\{h : h \in (Y^mX)_0$ and $hx \cap V \neq \emptyset\}$ is a $\tau_p$-neighborhood of $f, f_x \cap V \neq \emptyset$ eventually. Since $f_x \subseteq Y_x, Y_x \cap V \neq \emptyset$. This shows that $y \in Y_x = Y_s$, proving that $f \in F$.

We prove similarly the same implication for $F = (P\{Y_x : x \in X\})_1$.

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