A CHARACTERIZATION OF PRÜFER DOMAINS IN TERMS OF POLYNOMIALS

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Assume that $D$ is an integral domain with identity and with
quotient field $K$. Each element of $K$ is the root of a polynomial
$f$ in $D[X]$ such that the coefficients of $f$ generate $D$ if and only if
the integral closure of $D$ is a Prüfer domain.

All rings considered in this paper are assumed to be commutative
and to contain an identity element. By an overring of a ring $R$, we
mean a subring of the total quotient ring of $R$ containing $R$. The
symbol $X$ in the notation $R[X]$ denotes an indeterminate over $R$.

In the study of integral domains, Prüfer domains arise in many
different contexts. See, for example, [1; Exer. 12, p. 93] or [2; Chap.
IV] for some of the multitudinous characterizations of Prüfer
domains. Among such characterizations there are at least two in terms
of polynomials: (1) The domain $D$ is a Prüfer domain if and only if
$A_f A_g = A_h$ for all $f, g \in D[X]$, where $A_h$ denotes the ideal of $D$
generated by the coefficients of the polynomial $h \in D[X]$ ($A_h$ is called
the content of $h$) [3], [10], [2; p. 347]. (2) $D$ is a Prüfer domain if and
only if $D$ is integrally closed and for each prime ideal $P$ of $D$, the only
prime ideals of $D[X]$ contained in $P[X]$ are those of the form $P[X]$, where
$P$ is a prime ideal of $D$ contained in $P$ [2; p. 241]. In Theorem 2
we provide another characterization of Prüfer domains in terms of
polynomials: $D$ is a Prüfer domain if and only if $D$ is integrally closed
and each element of the quotient field $K$ of $D$ is a root of a polynomial
$f \in D[X]$ such that $A_f = D$. Then in Theorem 5 we obtain an extension
of this result to the case where $D$ need not be integrally closed.

Our interest in domains $D$ such that each element of $K$ is a root of
a polynomial $f \in D[X]$ with $A_f = D$ stemmed from the fact that this
property is common to both $\Delta$-domains—that is, integral domains
whose set of overrings is closed under addition [4]—and to integral
domains having property $(n)$ for some $n > 1$—that is, integral domains
$D$ with the property that $(x, y)^n = (x^n, y^n)$ for all $x, y \in D$ [9]. Thus, if
$D$ is a $\Delta$-domain with quotient field $K$ and if $t \in K$, then since
$D[t^2] + D[t^3]$ is an overring of $D$, $t^2 t^3 \in D[t^2] + D[t^3]$, whence it is
evident that $t$ is the root of a polynomial in $D[X]$ in which the
coefficient of $X^2$ is a unit. If $D$ has property $(n)$ for some $n > 1$ and if
t = a/b \in K, where $a, b \in D$ and $b \neq 0$, then from the equality $(a, b)^n =
(a^n, b^n)$ it follows that $a^{n-1} b = d_1 a^n + d_2 b^n$ for some $d_1, d_2 \in D$; divid-
ing both sides of this equation by \( b^n \) yields \( d_1 x^n - x^{n-1} + d_2 \) as a polynomial satisfied by \( t \).

We show that the condition described in the preceding paragraph is equivalent to the condition that each element of the quotient field of \( D \) satisfies a polynomial with a unit coefficient.

**Theorem** Let \( f = \sum_{i=0}^{n} f_i x^i \) be an element of \( R[X] \). Then \( A_f = (f_0, f_1, \ldots, f_n) \) is the set of coefficients of elements of the principal ideal of \( R[X] \) generated by \( f \).

**Proof.** Denote by \( E \) the set of coefficients of elements of \( (f) \); \( E \) is an ideal of \( R \) and the inclusion \( A_f \subseteq E \) is clear. Conversely, if \( t = \sum r_i f_i \) is an element of \( A_f \), then \( (\sum r_i x^{n-i})f \) is an element of \( (f) \) and the coefficient of \( x^n \) in this polynomial is \( t \). Hence \( t \in E \) and the equality \( E = A_f \) holds, as asserted.

A modification of the proof of Theorem 1 shows that the result generalizes to polynomials in an arbitrary set of indeterminates, and this observation, in turn, yields a further generalization of Theorem 1.

**Corollary 1.** Let \( \{f_\alpha \} \) be a subset of the polynomial ring \( R[\{X_\alpha \}] \), and for each \( \alpha \), let \( A_\alpha \) be the ideal of \( R \) generated by the coefficients of \( f_\alpha \). Then \( \sum_\alpha A_\alpha \) is the set of coefficients of the ideal of \( R[\{X_\alpha \}] \) generated by \( \{f_\alpha \} \).

The equivalence of the two conditions mentioned in the paragraph immediately preceding Theorem 1 also follows at once from this result. If \( S \) is a unitary extension ring of \( R \), we say that \( R \) has property \((P)\) with respect to \( S \) or that \( S \) is a \( P \)-extension of \( R \) if each element of \( S \) satisfies a polynomial in \( R[X] \) one of whose coefficients is a unit of \( R \), or, equivalently, whose coefficients generate the unit ideal of \( R \). The next result is not unexpected.

**Theorem 2.** Let \( D \) be an integrally closed domain with quotient field \( K \). Then \( D \) is a Prüfer domain if and only if \( K \) is a \( P \)-extension of \( D \).

**Proof.** If \( D \) is a Prüfer domain, then \( D \) has property \((n)\) for each positive integer \( n \) [5; Theorem 2.5 (e)], [2; Theorem 24.3], and hence, as already shown, \( D \) has property \((P)\) with respect to \( K \). Conversely, suppose that \( K \) is a \( P \)-extension of \( D \). Let \( M \) be a maximal ideal of \( D \) and let \( t \) be an element of \( K \). Then \( t \) is a root of a polynomial \( f \) in \( D[X] \) such that \( A_f = D \), and hence \( f \notin M[X] \). It then follows from [11; p. 19] that \( t \) or \( t^{-1} \) is in \( D_M \). Consequently, \( D_M \) is a valuation ring and \( D \) is a Prüfer domain, as asserted.
To obtain a characterization of domains $D$ for which $K$ is a $P$-extension of $D$, we introduce some useful notation. Let $R$ be a ring, let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal ideals of $R$, and let $N$ be the set of elements $f$ in $R[X]$ such that $A_f = R$; W. Krull [7] observed that $N$ is a regular multiplicative system in $R[X]$ and he considered properties of the ring $R[X]_N$, which M. Nagata in [8, p. 17] denotes by $R(X)$. It is clear that $N = R[X] - \bigcup_\lambda M_\lambda[X]$, and in Chapter 33 of [2] it is shown that if an ideal $E$ of $R[X]$ is contained in one of the ideals $M_\lambda[X]$, then $E$ is contained in $M_\lambda[X]$. Consequently, $\{M_\lambda[X]\}$ is the set of prime ideals of $R[X]$ maximal with respect to not meeting $N$ and $\{M_\lambda R(X)\}$ is the set of maximal ideals of $R(X)$. With these facts recorded, we state and prove our next theorem.

**Theorem 3.** Let $T$ be a unitary extension ring of the ring $R$ and let $S$ be the integral closure of $R$ in $T$.

(a) The ring $S(X)$ is integral over $R(X)$.

(b) If $T[X]$ is integrally closed, then $S(X)$ is the integral closure of $R(X)$ in $T(X)$.

**Proof.** (a): Let $\{M_\alpha\}_{\alpha \in A}$ and $\{M'_\beta\}_{\beta \in B}$ be the sets of maximal ideals of $R$ and $S$, respectively. If $N = R[X] - \bigcup_\alpha M_\alpha[X]$ and $N' = S[X] - \bigcup_\beta M'_\beta[X]$, then $R(X) = R[X]_N$ and $S(X) = S[X]_{N'}$. The ring $S[X]_N$ is integral over $R[X]_N$ and we prove (a) by showing that $N'$ is the saturation of the multiplicative system $N$ in $S[X]$. Let $N^*$ be the saturation of $N$ in $S[X]$; since $N \subseteq N'$ and since $N'$ is saturated, it follows that $N^* \subseteq N'$. The multiplicative system $N^*$ is characterized as the complement in $S[X]$ of the set $\mathcal{P}$ of prime ideals of $S[X]$ maximal with respect to not meeting $N$; hence, to prove that $N'$ is contained in $N^*$, we prove that $\mathcal{P} \subseteq \{M'_\beta[X]\}_{\beta \in B}$. Thus, let $P' \in \mathcal{P}$ and let $P' \cap R[X] = P$. Since $P' \cap N = \emptyset$, $P$ also fails to meet $N$—that is, $P \subseteq \bigcup_\alpha M_\alpha[X]$; as we remarked earlier, this inclusion implies that $P \subseteq M_\alpha[X]$ for some $\alpha \in A$. Since $S[X]$ is integral over $R[X]$, there is a prime ideal $Q'$ of $S[X]$ such that $Q'$ contains $P'$ and $Q' \cap R[X] = M_\alpha[X]$. Hence $(Q' \cap S) \cap R = (Q' \cap R[X]) \cap R = M_\alpha[X] \cap R = M_\alpha$, a maximal ideal of $R$; from the integrality of $S$ over $R$ we infer that $Q' \cap S$ is a maximal ideal of $S$, that is, $Q' \cap S = M'_\beta$ for some $\beta \in B$. It follows that $M'_\beta[X] \subseteq Q'$ and in fact, $Q' = M'_\beta[X]$ since $S[X]$ is integral over $R[X]$ and since $Q' \cap R[X] = M'_\beta[X] \cap R[X] = M_\alpha[X]$. We therefore obtain the inclusion $P' \subseteq M'_\beta[X]$. Since $M'_\beta[X]$ misses $N$ and since $P'$ is maximal with respect to missing $N$, it follows that $P' = M'_\beta[X]$ and $\mathcal{P} \subseteq \{M'_\beta[X]\}_{\beta \in B}$. This completes the proof of (a).

To prove (b) we recall that $S[X]$ is the integral closure of $R[X]$ in $T[X]$ [2, Theorem 10.7], and hence $S[X]_N = S(X)$ is the integral closure of $R[X]_N = R(X)$ in $T[X]_N$. If $T[X]$ is integrally closed, then $T[X]_N$
is also integrally closed, and since $T(X)$ is an overring of $T[X]_N$, it follows that the integral closure of $R(X)$ in $T(X)$ coincides with the integral closure of $R(X)$ in $T[X]_N$. Thus $S(X)$ is the integral closure of $R(X)$ in $T(X)$, as asserted.

**Remark 1.** The following result follows from the proof of part (a) of Theorem 3: Assume that $S$ is a unitary ring extension of the ring $R$ and that $S$ is integral over $R$. Let $N$ be a multiplicative system in $R$, let $\{P_a\}$ be the set of prime ideals of $R$ maximal with respect to not meeting $N$, and let $\{P'_b\}$ be the set of prime ideals of $S$ such that $P'_b \cap R \in \{P_a\}$. Then $S - (\cup P'_b)$ is the saturation of $N$ in $S$ (cf. [2; Proposition 11.10]). More generally, this conclusion is valid if the extension $R \subseteq S$ satisfies going up in the terminology of [6; p. 28].

**Remark 2.** We do not know if the conclusion of (b) is valid without the hypothesis that $T[X]$ is integrally closed. As the proof of part (b) of Theorem 3 shows, sufficient conditions for $S(X)$ to be the integral closure of $R(X)$ in $T(X)$ are that $T[X]_N$ is integrally closed in $T(X)$, a quotient ring of $T[X]_N$. It is easy to give examples to show that the inclusion $T[X]_N \subseteq T(X)$ may be proper; if $R$ is a $v$-domain with quotient field $T$, then a necessary condition that $T(X)$ should be $T[X]_N$ is that $R$ be a Prüfer $v$-multiplication ring (see §33 of [2] for terminology). The condition that $T[X]$ is integrally closed is not, insofar as we know, definitive in terms of $T$; it implies that $T$ is integrally closed, but the converse fails in general.

**Theorem 4.** Assume that $T$ is a unitary extension ring of the ring $R$ and that $S$ is an intermediate ring integral over $R$. If $T$ is a $P$-extension of $S$, then $T$ is a $P$-extension of $R$.

**Proof.** Let $t \in T$, let $Q' = \{f \in S[X] | f(t) = 0\}$, and let $Q = Q' \cap R[X]$. If $N$ and $N'$ are defined as in the proof of Theorem 3, so that $R(X) = R[X]_N$ and $S(X) = S[X]_{N'}$, then the hypothesis that $T$ is a $P$-extension of $S$ implies that $Q' \cap N' \neq \emptyset$. If we show that $Q \cap N \neq \emptyset$, then the proof of Theorem 4 will be complete. We first observe that $QR(X) = Q'(S[X]_N) \cap R(X)$. That the right side contains the left side is clear, and if $f/n = d/m \in Q'(S[X]_N) \cap R(X)$, where $f \in Q'$, $d \in R[X]$, and $n$, $m \in N$, then $fm = dn \in Q' \cap R[X] = Q$, so that $f/n = fm/nm \in QR(X)$ and $Q'(S[X]_N) \cap R(X) = QR(X)$. It follows from the proof of Theorem 3 that $(S[X])_N = (S[X])_{N'}$; hence

$$QR(X) = Q'S(X) \cap R(X) = S(X) \cap R(X) = R(X),$$

which means that $Q \cap N \neq \emptyset$. 

**Remark 1.** The following result follows from the proof of part (a) of Theorem 3: Assume that $S$ is a unitary ring extension of the ring $R$ and that $S$ is integral over $R$. Let $N$ be a multiplicative system in $R$, let $\{P_a\}$ be the set of prime ideals of $R$ maximal with respect to not meeting $N$, and let $\{P'_b\}$ be the set of prime ideals of $S$ such that $P'_b \cap R \in \{P_a\}$. Then $S - (\cup P'_b)$ is the saturation of $N$ in $S$ (cf. [2; Proposition 11.10]). More generally, this conclusion is valid if the extension $R \subseteq S$ satisfies going up in the terminology of [6; p. 28].

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$$QR(X) = Q'S(X) \cap R(X) = S(X) \cap R(X) = R(X),$$

which means that $Q \cap N \neq \emptyset$. 

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The characterization of Prüfer domains stated at the beginning of this paper is a direct consequence of the preceding results.

**Theorem 5.** Let $D$ be an integral domain with quotient field $K$, and let $J$ be the integral closure of $D$. Then $J$ is a Prüfer domain if and only if $K$ is a $P$-extension of $D$.

*Proof.* Suppose that $K$ is a $P$-extension of $D$. Then $K$ is, *a fortiori*, a $P$-extension of $J$. We invoke Theorem 2 to conclude that $J$ is a Prüfer domain.

If, conversely, $J$ is a Prüfer domain, then by Theorem 2, $K$ is a $P$-extension of $J$ and hence, by Theorem 4, a $P$-extension of $D$.

There is an extension of Theorem 5 to the case where $K$ is not the quotient field of $D$.

**Theorem 6.** Let $D$ be a domain with integral closure $J$, and let $L$ be an algebraic extension field of the quotient field $K$ of $D$. Then $J$ is a Prüfer domain if and only if $L$ is a $P$-extension of $D$.

*Proof.* If $L$ is a $P$-extension of $D$, then so is $K$, and hence $J$ is a Prüfer domain by Theorem 5. Conversely, if $J$ is a Prüfer domain, and if $t \in L$, then $t$ is a root of a nonzero polynomial $f \in J[X]$. The ideal $A_f$ of $J$ is finitely generated, and hence is invertible. If $A_f^{-1} = (g_0, g_1, \ldots, g_n)$, and if $g = \sum_{i=0}^{n} g_i x^i$ then $A_f g = A_f A_s = J$ so that $fg \in J[X]$ and $(fg)(t) = f(t)g(t) = 0$. It follows that $L$ is a $P$-extension of $J$, and hence by Theorem 4, $L$ is a $P$-extension of $D$.

**References**


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