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## **A CHARACTERIZATION OF PRÜFER DOMAINS IN TERMS OF POLYNOMIALS**

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**Assume that  $D$  is an integral domain with identity and with quotient field  $K$ . Each element of  $K$  is the root of a polynomial  $f$  in  $D[X]$  such that the coefficients of  $f$  generate  $D$  if and only if the integral closure of  $D$  is a Prüfer domain.**

All rings considered in this paper are assumed to be commutative and to contain an identity element. By an *overring* of a ring  $R$ , we mean a subring of the total quotient ring of  $R$  containing  $R$ . The symbol  $X$  in the notation  $R[X]$  denotes an indeterminate over  $R$ .

In the study of integral domains, Prüfer domains arise in many different contexts. See, for example, [1; Exer. 12, p. 93] or [2; Chap. IV] for some of the multitudinous characterizations of Prüfer domains. Among such characterizations there are at least two in terms of polynomials: (1) The domain  $D$  is a Prüfer domain if and only if  $A_f A_g = A_{fg}$  for all  $f, g \in D[X]$ , where  $A_h$  denotes the ideal of  $D$  generated by the coefficients of the polynomial  $h \in D[X]$  ( $A_h$  is called the *content* of  $h$ ) [3], [10], [2; p. 347]. (2)  $D$  is a Prüfer domain if and only if  $D$  is integrally closed and for each prime ideal  $P$  of  $D$ , the only prime ideals of  $D[X]$  contained in  $P[X]$  are those of the form  $P_1[X]$ , where  $P_1$  is a prime ideal of  $D$  contained in  $P$  [2; p. 241]. In Theorem 2 we provide another characterization of Prüfer domains in terms of polynomials:  $D$  is a Prüfer domain if and only if  $D$  is integrally closed and each element of the quotient field  $K$  of  $D$  is a root of a polynomial  $f \in D[X]$  such that  $A_f = D$ . Then in Theorem 5 we obtain an extension of this result to the case where  $D$  need not be integrally closed.

Our interest in domains  $D$  such that each element of  $K$  is a root of a polynomial  $f \in D[X]$  with  $A_f = D$  stemmed from the fact that this property is common to both  $\Delta$ -domains—that is, integral domains whose set of overrings is closed under addition [4]—and to integral domains having property (n) for some  $n > 1$ —that is, integral domains  $D$  with the property that  $(x, y)^n = (x^n, y^n)$  for all  $x, y \in D$  [9]. Thus, if  $D$  is a  $\Delta$ -domain with quotient field  $K$  and if  $t \in K$ , then since  $D[t^2] + D[t^3]$  is an overring of  $D$ ,  $t^5 = t^2 t^3 \in D[t^2] + D[t^3]$ , whence it is evident that  $t$  is the root of a polynomial in  $D[X]$  in which the coefficient of  $X^5$  is a unit. If  $D$  has property (n) for some  $n > 1$  and if  $t = a/b \in K$ , where  $a, b \in D$  and  $b \neq 0$ , then from the equality  $(a, b)^n = (a^n, b^n)$  it follows that  $a^{n-1}b = d_1 a^n + d_2 b^n$  for some  $d_1, d_2 \in D$ ; divid-

ing both sides of this equation by  $b^n$  yields  $d_1X^n - X^{n-1} + d_2$  as a polynomial satisfied by  $t$ .

We show that the condition described in the preceding paragraph is equivalent to the condition that each element of the quotient field of  $D$  satisfies a polynomial with a unit coefficient.

**THEOREM** *Let  $f = \sum_{i=0}^n f_i X^i$  be an element of  $R[X]$ . Then  $A_f = (f_0, f_1, \dots, f_n)$  is the set of coefficients of elements of the principal ideal of  $R[X]$  generated by  $f$ .*

*Proof.* Denote by  $E$  the set of coefficients of elements of  $(f)$ ;  $E$  is an ideal of  $R$  and the inclusion  $A_f \supseteq E$  is clear. Conversely, if  $t = \sum_{i=0}^n r_i f_i$  is an element of  $A_f$ , then  $(\sum_{i=0}^n r_i X^{n-i})f$  is an element of  $(f)$  and the coefficient of  $X^n$  in this polynomial is  $t$ . Hence  $t \in E$  and the equality  $E = A_f$  holds, as asserted.

A modification of the proof of Theorem 1 shows that the result generalizes to polynomials in an arbitrary set of indeterminates, and this observation, in turn, yields a further generalization of Theorem 1.

**COROLLARY 1.** *Let  $\{f_\alpha\}$  be a subset of the polynomial ring  $R[\{X_\lambda\}]$ , and for each  $\alpha$ , let  $A_{f_\alpha}$  be the ideal of  $R$  generated by the coefficients of  $f_\alpha$ . Then  $\sum_\alpha A_{f_\alpha}$  is the set of coefficients of the ideal of  $R[\{X_\lambda\}]$  generated by  $\{f_\alpha\}$ .*

The equivalence of the two conditions mentioned in the paragraph immediately preceding Theorem 1 also follows at once from this result. If  $S$  is a unitary extension ring of  $R$ , we say that  $R$  has *property (P) with respect to  $S$*  or that  $S$  is a *P-extension of  $R$*  if each element of  $S$  satisfies a polynomial in  $R[X]$  one of whose coefficients is a unit of  $R$ , or, equivalently, whose coefficients generate the unit ideal of  $R$ . The next result is not unexpected.

**THEOREM 2.** *Let  $D$  be an integrally closed domain with quotient field  $K$ . Then  $D$  is a Prüfer domain if and only if  $K$  is a P-extension of  $D$ .*

*Proof.* If  $D$  is a Prüfer domain, then  $D$  has property (n) for each positive integer  $n$  [5; Theorem 2.5 (e)], [2; Theorem 24.3], and hence, as already shown,  $D$  has property (P) with respect to  $K$ . Conversely, suppose that  $K$  is a P-extension of  $D$ . Let  $M$  be a maximal ideal of  $D$  and let  $t$  be an element of  $K$ . Then  $t$  is a root of a polynomial  $f$  in  $D[X]$  such that  $A_f = D$ , and hence  $f \notin M[X]$ . It then follows from [11; p. 19] that  $t$  or  $t^{-1}$  is in  $D_M$ . Consequently,  $D_M$  is a valuation ring and  $D$  is a Prüfer domain, as asserted.

To obtain a characterization of domains  $D$  for which  $K$  is a  $P$ -extension of  $D$ , we introduce some useful notation. Let  $R$  be a ring, let  $\{M_\lambda\}_{\lambda \in \Lambda}$  be the set of maximal ideals of  $R$ , and let  $N$  be the set of elements  $f$  in  $R[X]$  such that  $A_f = R$ ; W. Krull [7] observed that  $N$  is a regular multiplicative system in  $R[X]$  and he considered properties of the ring  $R[X]_N$ , which M. Nagata in [8; p. 17] denotes by  $R(X)$ . It is clear that  $N = R[X] - \cup_\lambda M_\lambda[X]$ , and in Chapter 33 of [2] it is shown that if an ideal  $E$  of  $R[X]$  is contained in  $\cup_\lambda M_\lambda[X]$ , then  $E$  is contained in one of the ideals  $M_\lambda[X]$ . Consequently,  $\{M_\lambda[X]\}$  is the set of prime ideals of  $R[X]$  maximal with respect to not meeting  $N$  and  $\{M_\lambda R(X)\}$  is the set of maximal ideals of  $R(X)$ . With these facts recorded, we state and prove our next theorem.

**THEOREM 3.** *Let  $T$  be a unitary extension ring of the ring  $R$  and let  $S$  be the integral closure of  $R$  in  $T$ .*

(a) *The ring  $S(X)$  is integral over  $R(X)$ .*

(b) *If  $T[X]$  is integrally closed, then  $S(X)$  is the integral closure of  $R(X)$  in  $T(X)$ .*

*Proof.* (a): Let  $\{M_\alpha\}_{\alpha \in A}$  and  $\{M'_\beta\}_{\beta \in B}$  be the sets of maximal ideals of  $R$  and  $S$ , respectively. If  $N = R[X] - \cup_\alpha M_\alpha[X]$  and  $N' = S[X] - \cup_\beta M'_\beta[X]$ , then  $R(X) = R[X]_N$  and  $S(X) = S[X]_{N'}$ . The ring  $S[X]_N$  is integral over  $R[X]_N$  and we prove (a) by showing that  $N'$  is the saturation of the multiplicative system  $N$  in  $S[X]$ . Let  $N^*$  be the saturation of  $N$  in  $S[X]$ ; since  $N \subseteq N'$  and since  $N'$  is saturated, it follows that  $N^* \subseteq N'$ . The multiplicative system  $N^*$  is characterized as the complement in  $S[X]$  of the set  $\mathcal{P}$  of prime ideals of  $S[X]$  maximal with respect to not meeting  $N$ ; hence, to prove that  $N'$  is contained in  $N^*$ , we prove that  $\mathcal{P} \subseteq \{M'_\beta[X]\}_{\beta \in B}$ . Thus, let  $P' \in \mathcal{P}$  and let  $P' \cap R[X] = P$ . Since  $P' \cap N = \emptyset$ ,  $P$  also fails to meet  $N$ —that is,  $P \subseteq \cup_{\alpha \in A} M_\alpha[X]$ ; as we remarked earlier, this inclusion implies that  $P \subseteq M_\alpha[X]$  for some  $\alpha \in A$ . Since  $S[X]$  is integral over  $R[X]$ , there is a prime ideal  $Q'$  of  $S[X]$  such that  $Q'$  contains  $P'$  and  $Q' \cap R[X] = M_\alpha[X]$ . Hence  $(Q' \cap S) \cap R = (Q' \cap R[X]) \cap R = M_\alpha[X] \cap R = M_\alpha$ , a maximal ideal of  $R$ ; from the integrality of  $S$  over  $R$  we infer that  $Q' \cap S$  is a maximal ideal of  $S$ , that is,  $Q' \cap S = M'_\beta$  for some  $\beta \in B$ . It follows that  $M'_\beta[X] \subseteq Q'$  and in fact,  $Q' = M'_\beta[X]$  since  $S[X]$  is integral over  $R[X]$  and since  $Q' \cap R[X] = M'_\beta[X] \cap R[X] = M_\alpha[X]$ . We therefore obtain the inclusion  $P' \subseteq M'_\beta[X]$ . Since  $M'_\beta[X]$  misses  $N$  and since  $P'$  is maximal with respect to missing  $N$ , it follows that  $P' = M'_\beta[X]$  and  $\mathcal{P} \subseteq \{M'_\beta[X]\}_{\beta \in B}$ . This completes the proof of (a).

To prove (b) we recall that  $S[X]$  is the integral closure of  $R[X]$  in  $T[X]$  [2, Theorem 10.7], and hence  $S[X]_N = S(X)$  is the integral closure of  $R[X]_N = R(X)$  in  $T[X]_N$ . If  $T[X]$  is integrally closed, then  $T[X]_N$

is also integrally closed, and since  $T(X)$  is an overring of  $T[X]_N$ , it follows that the integral closure of  $R(X)$  in  $T(X)$  coincides with the integral closure of  $R(X)$  in  $T[X]_N$ . Thus  $S(X)$  is the integral closure of  $R(X)$  in  $T(X)$ , as asserted.

REMARK 1. The following result follows from the proof of part (a) of Theorem 3: Assume that  $S$  is a unitary ring extension of the ring  $R$  and that  $S$  is integral over  $R$ . Let  $N$  be a multiplicative system in  $R$ , let  $\{P_\alpha\}$  be the set of prime ideals of  $R$  maximal with respect to not meeting  $N$ , and let  $\{P'_\beta\}$  be the set of prime ideals of  $S$  such that  $P'_\beta \cap R \in \{P_\alpha\}$ . Then  $S - (\cup P'_\beta)$  is the saturation of  $N$  in  $S$  (cf. [2; Proposition 11.10]). More generally, this conclusion is valid if the extension  $R \subseteq S$  satisfies *going up* in the terminology of [6; p. 28].

REMARK 2. We do not know if the conclusion of (b) is valid without the hypothesis that  $T[X]$  is integrally closed. As the proof of part (b) of Theorem 3 shows, sufficient conditions for  $S(X)$  to be the integral closure of  $R(X)$  in  $T(X)$  are that  $T[X]_N$  is integrally closed in  $T(X)$ , a quotient ring of  $T[X]_N$ . It is easy to give examples to show that the inclusion  $T[X]_N \subseteq T(X)$  may be proper; if  $R$  is a  $v$ -domain with quotient field  $T$ , then a necessary condition that  $T(X)$  should be  $T[X]_N$  is that  $R$  be a Prüfer  $v$ -multiplication ring (see §33 of [2] for terminology). The condition that  $T[X]$  is integrally closed is not, insofar as we know, definitive in terms of  $T$ ; it implies that  $T$  is integrally closed, but the converse fails in general.

THEOREM 4. *Assume that  $T$  is a unitary extension ring of the ring  $R$  and that  $S$  is an intermediate ring integral over  $R$ . If  $T$  is a  $P$ -extension of  $S$ , then  $T$  is a  $P$ -extension of  $R$ .*

*Proof.* Let  $t \in T$ , let  $Q' = \{f \in S[X] \mid f(t) = 0\}$ , and let  $Q = Q' \cap R[X]$ . If  $N$  and  $N'$  are defined as in the proof of Theorem 3, so that  $R(X) = R[X]_N$  and  $S(X) = S[X]_{N'}$ , then the hypothesis that  $T$  is a  $P$ -extension of  $S$  implies that  $Q' \cap N' \neq \emptyset$ . If we show that  $Q \cap N \neq \emptyset$ , then the proof of Theorem 4 will be complete. We first observe that  $QR(X) = Q'(S[X]_N) \cap R(X)$ . That the right side contains the left side is clear, and if  $f/n = d/m \in Q'(S[X]_N) \cap R(X)$ , where  $f \in Q'$ ,  $d \in R[X]$ , and  $n, m \in N$ , then  $fm = dn \in Q' \cap R[X] = Q$ , so that  $f/n = fm/nm \in QR(X)$  and  $Q'(S[X]_N) \cap R(X) = QR(X)$ . It follows from the proof of Theorem 3 that  $(S[X])_N = (S[X])_{N'}$ ; hence

$$QR(X) = Q'S(X) \cap R(X) = S(X) \cap R(X) = R(X),$$

which means that  $Q \cap N \neq \emptyset$ .

The characterization of Prüfer domains stated at the beginning of this paper is a direct consequence of the preceding results.

**THEOREM 5.** *Let  $D$  be an integral domain with quotient field  $K$ , and let  $J$  be the integral closure of  $D$ . Then  $J$  is a Prüfer domain if and only if  $K$  is a  $P$ -extension of  $D$ .*

*Proof.* Suppose that  $K$  is a  $P$ -extension of  $D$ . Then  $K$  is, *a fortiori*, a  $P$ -extension of  $J$ . We invoke Theorem 2 to conclude that  $J$  is a Prüfer domain.

If, conversely,  $J$  is a Prüfer domain, then by Theorem 2,  $K$  is a  $P$ -extension of  $J$  and hence, by Theorem 4, a  $P$ -extension of  $D$ .

There is an extension of Theorem 5 to the case where  $K$  is not the quotient field of  $D$ .

**THEOREM 6.** *Let  $D$  be a domain with integral closure  $J$ , and let  $L$  be an algebraic extension field of the quotient field  $K$  of  $D$ . Then  $J$  is a Prüfer domain if and only if  $L$  is a  $P$ -extension of  $D$ .*

*Proof.* If  $L$  is a  $P$ -extension of  $D$ , then so is  $K$ , and hence  $J$  is a Prüfer domain by Theorem 5. Conversely, if  $J$  is a Prüfer domain, and if  $t \in L$ , then  $t$  is a root of a nonzero polynomial  $f \in J[X]$ . The ideal  $A_f$  of  $J$  is finitely generated, and hence is invertible. If  $A_f^{-1} = (g_0, g_1, \dots, g_n)$ , and if  $g = \sum_{i=0}^n g_i X^i$  then  $A_{fg} = A_f A_g = J$  so that  $fg \in J[X]$  and  $(fg)(t) = f(t)g(t) = 0$ . It follows that  $L$  is a  $P$ -extension of  $J$ , and hence by Theorem 4,  $L$  is a  $P$ -extension of  $D$ .

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