RATIONAL VALUED SERIES OF EXPONENTIALS AND DIVISOR FUNCTIONS

E. Grosswald

Recently A. Terras established some (soon to be published) relations between the values of Riemann’s zeta function at consecutive positive integral argument and values of certain modified Bessel functions. By combining these relations with some previous results concerning the values of \( \zeta(s) \) at odd, positive integers (Grosswald–Nachrichten Akad. Wiss. Göttingen, II Math.–Phys. Klasse 1970, pp. 9–13) it follows that certain infinite series of exponentials and divisor functions (somewhat reminiscent of Lambert series) are rational valued.

Specifically, A. Terras proved [6] that for complex \( \rho \), for \( a,b \) natural integers and with \( K_u(z) \) the modified Bessel function (notation of Watson; see [1], especially 10.2.15, page 444),

\[
\zeta(2\rho)\Gamma(\rho + 1) + (1 - \rho)\zeta(2\rho - 1)\Gamma(1/2)\Gamma(\rho - 1/2) = 2\pi^\rho \sum_{a,b \geq 1} (b/a)^{\rho-1/2\{2\pi ab(K_{1,5-p}(2\pi ab) + K_{0,5+\rho}(2\pi ab)) - K_{0,5-p}(2\pi ab)\}}
\]

holds, provided that \( \text{Re } \rho > 1 \). Formula (1) seems related to results of Berndt [2], especially his formula (30), but does not seem to follow trivially from it.

If in (1) we take for \( \rho \) a natural integer \( m > 1 \), replace the Bessel functions according to classical formulae (see [1], p. 444) and perform some routine transformations, (1) is seen to imply

\[
\zeta(2m - 1) = \frac{(m - 2)!}{(2m - 2)!}(4\pi)^{2m-1}m!(B_{2m})2(2m)! - \sum_{n=1}^{\infty} e^{-2\pi m} \sigma_{-(2m-1)}(n) \sum_{k=0}^{m} \frac{(m + k - 2)! (m - 1) + k(k - 1)}{k!(m - k)!} (4\pi n)^{m-k}.\]

If we equate these representations of \( \zeta(2m - 1) \) to those established in [3], then we obtain some rather curious formulae, that involve the divisor functions \( \sigma_k(n) = \sum_{d|n} d^k \) for odd, negative \( k < -1 \). The first few of them read
\[
\sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-3}(n) \{(4\pi n)^2 + 2(4\pi n)\} = \pi^3/90,
\]
(3)
\[
\sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-3}(n) \{(4\pi n)^3 + 6(4\pi n)^2 + 12(4\pi n)\} = 2\pi^5/105,
\]
\[
\sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-3}(n) \{(4\pi n)^4 + 12(4\pi n)^3 + 84(4\pi n)^2 + 360(4\pi n)\} = 22\pi^7/525.
\]

One may wish to complete these formulae with one involving \(\sigma_{-3}(n)\), corresponding to \(m = 1\). Direct substitution of \(m = 1\) in (2) is, of course, meaningless and the correct, well known formula is indeed of a slightly different structure, namely (see [5] vol. 1, p. 257; see also [4] and [6])

(3') \[
\sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-3}(n)(4\pi n) = (\pi - 3)/6.
\]

A glance at (3) seems to indicate that the “natural” variable is \(\nu_n = 2\pi n\). If we make the corresponding change of the summation variable and set also \(\bar{\sigma}_k(n) = \sum_{d|n}(2\pi d)^k\), then we obtain the somewhat simpler formulae

\[
\sum_{n=1}^{\infty} e^{-\nu_n} \bar{\sigma}_{-1}(n)\nu_n = \frac{1}{24} - \frac{1}{8\pi}
\]
\[
\sum_{n=1}^{\infty} e^{-\nu_n} \bar{\sigma}_{-3}(n)(\nu_n^2 + \nu_n) = 1/2^6 \cdot 3^2 \cdot 5
\]
\[
\sum_{n=1}^{\infty} e^{-\nu_n} \bar{\sigma}_{-3}(n)(\nu_n^3 + 3\nu_n^2 + 3\nu_n) = 1/2^7 \cdot 3 \cdot 5 \cdot 7
\]
\[
\sum_{n=1}^{\infty} e^{-\nu_n} \bar{\sigma}_{-3}(n)(\nu_n^4 + 6\nu_n^3 + 21\nu_n^2 + 45\nu_n) = 11/2^{10} \cdot 3 \cdot 5^2 \cdot 7.
\]

Here all second members (except, naturally, in the first identity) are rational (but, as the last one shows, not necessarily the reciprocal of an integer).

2. General result and proofs. According to [3], for odd \(m > 1\),

\[
\zeta(2m - 1) = \frac{(2\pi)^{2m-1}}{(m-1)(2m)!} \sum_{k=0}^{(m-1)/2} (-1)^k (m-2k) \binom{2m}{2k} B_{2k} B_{2m-2k} - 2 \sum_{n=1}^{\infty} e^{-2\pi n} \sigma_{-(2m-1)}(n) \left(\frac{2\pi n}{m-1} + 1\right).
\]

If we set this equal to (2), we obtain, after routine simplifications
For even \( m \), according to [3],

\[
\zeta(2m - 1) = \frac{(2\pi)^{2m-1}}{2(2m)!} \sum_{k=0}^{m} (-1)^{k-1} \left(\frac{2m}{2k}\right) B_{2k} B_{2m-2k} - 2 \sum_{n=1}^{m} e^{-2\pi n \sigma_{-(2m-1)}(n)}. 
\]

We now set this equal to (2), simplify and obtain:

\[
\zeta(2m - 1) = \frac{2\pi}{2(2m)!} \sum_{k=0}^{m} (-1)^{k-1} \left(\frac{2m}{2k}\right) B_{2k} B_{2m-2k} - 2 \sum_{n=1}^{m} e^{-2\pi n \sigma_{-(2m-1)}(n)}. 
\]

Formulae (3) are the particular cases \( m = 2, 3 \) and 4 of (5') and (5''), respectively.

Finally, in terms of \( \nu_n = 2\pi n \) and \( \tilde{\sigma}_x(n) \), (5'), (5'') become

\[
\sum_{n=1}^{\infty} e^{-2\pi n \sigma_{-(2m-1)}(n)} \left\{ \sum_{k=0}^{\infty} \frac{(m + k - 2)!(m + 2k)!}{(m - k)!(m + 1)!} \nu_n^{m-k} + \frac{2(m - 3)!}{(m - 3)!} 2^{2m-1} \nu_n \right\} 
\]

for odd \( m \); and

\[
\sum_{n=1}^{\infty} e^{-2\pi n \sigma_{-(2m-1)}(n)} \left\{ \sum_{k=0}^{\infty} \frac{(m + k - 2)!(m + 2k)!}{(m - k)!(m + 1)!} \nu_n^{m-k} \right\} 
\]

for even \( m \).

Formulae (4) are the particular cases \( m = 2, 3 \) and 4 of (6'), (6''), respectively, to which has been added the formula obtained from (3') that involves \( \tilde{\sigma}_x(n) \).
It is, of course, easy to consolidate the formulae (5'), (5'') into a single formula and similarly for (6'), (6''); however, the corresponding single formulae (each valid now both for even and for odd \( m \)), while formally simpler, are somewhat artificial and not very revealing and are, therefore, not given here.

**REFERENCES**


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**TEMPLE UNIVERSITY**
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>D. E. Bennett, Strongly unicoherent continua</td>
<td>1</td>
</tr>
<tr>
<td>Walter R. Bloom, Sets of p-spectral synthesis</td>
<td>7</td>
</tr>
<tr>
<td>R. T. Bumby and D. E. Dobbs, Amitsur cohomology of quadratic extensions: Formulas and number-theoretic examples</td>
<td>21</td>
</tr>
<tr>
<td>W. W. Comfort, Compactness-like properties for generalized weak topological sums</td>
<td>31</td>
</tr>
<tr>
<td>D. R. Dunninger and J. Locker, Monotone operators and nonlinear biharmonic boundary value problems</td>
<td>39</td>
</tr>
<tr>
<td>T. S. Erickson, W. S. Martindale, 3rd and J. M. Osborn, Prime nonassociative algebras</td>
<td>49</td>
</tr>
<tr>
<td>P. Fischer, On the inequality $\sum_{p} \frac{f(p)}{f(q)} \geq 1$</td>
<td>65</td>
</tr>
<tr>
<td>G. Fox and P. Morales, Compact subsets of a Tychonoff set</td>
<td>75</td>
</tr>
<tr>
<td>R. Gilmer and J. F. Hoffmann, A characterization of Prüfer domains in terms of polynomials</td>
<td>81</td>
</tr>
<tr>
<td>L. C. Glaser, On tame Cantor sets in spheres having the same projection in each direction</td>
<td>87</td>
</tr>
<tr>
<td>Z. Goseki, On semigroups in which $X = XYX = XZX$ if and only if $X = XYZX$</td>
<td>103</td>
</tr>
<tr>
<td>E. Grosswald, Rational valued series of exponentials and divisor functions</td>
<td>111</td>
</tr>
<tr>
<td>D. Handelman, Strongly semiprime rings</td>
<td>115</td>
</tr>
<tr>
<td>J. N. Henry and D. C. Taylor, The $\beta$ topology for $w^*$-algebras</td>
<td>123</td>
</tr>
<tr>
<td>M. J. Hodel, Enumeration of weighted $p$-line arrays</td>
<td>141</td>
</tr>
<tr>
<td>S. K. Jain and S. Singh, Rings with quasiprojective left ideals</td>
<td>169</td>
</tr>
<tr>
<td>S. Jeyaratnam, The diophantine equation $Y(Y + m)(Y + 2m)(Y + 3m)$ = $2X(X + m)(X + 2m)(X + 3m)$</td>
<td>183</td>
</tr>
<tr>
<td>R. Kane, On loop spaces without $p$ torsion</td>
<td>189</td>
</tr>
<tr>
<td>Alvin J. Kay, Nonlinear integral equations and product integrals</td>
<td>203</td>
</tr>
<tr>
<td>A. S. Kechris, Countable ordinals and the analytic hierarchy, I</td>
<td>223</td>
</tr>
<tr>
<td>Ka-Sing Lau, A representation theorem for isometries of $C(X, E)$</td>
<td>229</td>
</tr>
<tr>
<td>R. C. Metzler, Positive linear functions, integration, and Choquet's theorem</td>
<td>277</td>
</tr>
<tr>
<td>A. Nobile, Some properties of the Nash blowing-up</td>
<td>297</td>
</tr>
<tr>
<td>G. E. Petersen and G. V. Welland, Plessner's theorem for Riesz conjugates</td>
<td>307</td>
</tr>
</tbody>
</table>
Donald Earl Bennett, *Strongly unicoherent continua* .................................. 1
Walter Russell Bloom, *Sets of p-spectral synthesis* ....................................... 7
Richard Thomas Bumby and David Earl Dobbs, *Amitsur cohomology of quadratic extensions: formulas and number-theoretic examples* ........... 21
W. Wistar (William) Comfort, *Compactness-like properties for generalized weak topological sums* .......................................................... 31
Dennis Robert Dunninger and John Stewart Locker, *Monotone operators and nonlinear biharmonic boundary value problems* .............................. 39
Pál Fischer, *On the inequality \( \sum_{i=0}^{n} [f(p_i)/f(q_i)] p_i \geq i \).................. 65
Geoffrey Fox and Pedro Morales, *Compact subsets of a Tychonoff set* ........... 75
Robert William Gilmer, Jr. and Joseph F. Hoffmann, *A characterization of Prüfer domains in terms of polynomials* ....................................... 81
Leslie C. Glaser, *On tame Cantor sets in spheres having the same projection in each direction* ................................................................. 87
Zensiro Goseki, *On semigroups in which \( x = xzx = x x z \) if and only if \( x = x y z x \).* .......................................................... 103
Emil Grosswald, *Rational valued series of exponentials and divisor functions* ................. 111
David E. Handelman, *Strongly semiprime rings* ........................................... 115
Jackson Neal Henry and Donald Curtis Taylor, *The \( \beta \) topology for \( W^* \)-algebras* .......................................................... 123
Margaret Jones Hodel, *Enumeration of weighted p-line arrays* ......................... 141
Surender Kumar Jain and Surjeet Singh, *Rings with quasi-projective left ideals* ............... 169
S. Jeyaratnam, *The Diophantine equation \( Y(Y + m)(Y + 2m)(Y + 3m) = 2X(X + m)(X + 2m)(X + 3m) \) ......... 183
Richard Michael Kane, *On loop spaces without p torsion* ................................. 189
Alvin John Kay, *Nonlinear integral equations and product integrals* ................. 203
Alexander S. Kechris, *Countable ordinals and the analytical hierarchy. I* ............ 223
Ka-Sing Lau, *A representation theorem for isometries of \( C(X, E) \).................. 229
Ib Henning Madsen, *On the action of the Dyer-Lashof algebra in \( H_*(G) \) ... 235
Augusto Nobile, *Some properties of the Nash blowing-up* .............................. 297
Gerald E. Peterson and Grant Welland, *Plessner's theorem for Riesz conjugates* ........ 307