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THE $\bar{\beta}$ TOPOLOGY FOR W^*-ALGEBRAS

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Let A be a W^*-algebra and A_* its unique predual. A new locally convex topology $\bar{\beta}$ is developed for the study of the algebra A. It is shown that if A is a type I W^*-algebra, that is either countably decomposable, commutative, or a factor, then $\bar{\beta}$ is the Mackey topology for the dual pair $\langle A, A_*\rangle$. Consequently, when $A = L^\infty(X,\mu)$, where X is completely regular and μ is a compact regular Borel measure on X, $A^*_{\bar{\beta}} = L^1(X,\mu)$ and $\bar{\beta}$ convergence on uniformly bounded sets is equivalent to convergence in measure.

Let X be a locally compact Hausdorff space, βX the Stone-Čech compactification of X, and $C(\beta X)$ the collection of all complex-valued continuous functions on βX. In 1958, R. C. Buck **[2]** introduced a new locally convex topology for $C(\beta X)$ that gave new insight into the intricate structure of $C(\beta X)$. This locally convex topology for $C(\beta X)$, which Buck called the strict topology, is the topology generated by the seminorms $\{\lambda_f\}_{f\in C_0(X)}$, where $\lambda_f(g) = \|fg\|_\infty$. Here, $C_0(X)$ denotes those functions in $C(\beta X)$ that vanish on $\beta X\backslash X$. Although Buck's approach is very useful in the study of $C(\beta X)$, X locally compact, it does not lend itself to the study of $C(\beta X)$, X completely regular, since $C_0(X)$ may be the $\{0\}$ subspace in this setting. In **[18]**, F. D. Sentilles was able to overcome this possibility by introducing a new topology which, in the locally compact setting, reduces to the strict topology. Sentilles' topology, β, is defined as follows: for each $Q \subseteq \beta X\backslash X$, let β_Q be the strict topology on $C(\beta X)$ determined by $C_0(\beta X\backslash Q)$. Then β is defined as the inductive limit of the topologies β_Q as Q ranges over all compact subsets of $\beta X\backslash X$ **[18]**. Note that β is determined by the collection of open sets V, $\beta X \supseteq V \supseteq X$, whose Stone-Čech compactification is βX and is therefore not a unique topology, since it depends on the underlying subspace X. Using this topology, substantial progress has been made in the study of $C(\beta X)$, X completely regular, by Sentilles, Wheeler and others (see **[8]**, **[18]**, **[24]**, **[25]**).

The purpose of this paper is to define and study a noncommutative analogue of the topology introduced by Sentilles. Noncommutative versions of Buck's topology already exist in a Banach module setting **[19]** and in the C^*-algebra of double centralizers $M(B)$ of the C^*-algebra B **[3]**, **[20]**, **[22]**. In the double centralizer setting, B is viewed as a closed two-sided ideal in $M(B)$, and the strict topology for $M(B)$ is generated by the seminorms $\{\lambda_b, \rho_b\}_{b \in B}$, where $\lambda_b(x) = \| bx \|$ and $\rho_b(x) = \|xb\|$ for $x \in M(B)$. This topology has been very useful in the study of the C^*-algebra $M(B)$. In general it would be desirable to use this approach to study C^*-algebras A with identity, that is, develop a locally convex topology for A with the essential properties of the strict topology. It would be natural to try to find a closed two-sided ideal $J \subseteq A$ such that $M(J) = A$, but this in general is difficult to do. Consequently, we find it necessary to place additional restrictions on our C^*-algebra. Namely, we will require A to be a W^*-algebra. Here we view a W^*-algebra as a C^*-algebra which is the dual of a unique Banach space A_* **[14]**. In a W^*-algebra A, it is known that a closed two-sided ideal $J \subseteq A$ has the property that $M(J) = A$ if and only if J is essential, that is, $J^0 \equiv \{x \in A : xJ = \{0\}\} = \{0\}$ (see **[22]**). Since it is probable that more than one ideal with this property exists, it seems natural to apply Sentilles' method to our setting. Consequently, we define the $\bar{\beta}$ topology for a W^*-algebra A as follows: for each essential closed two-sided ideal $J \subseteq A$, we define the strict topology β_J for A to be the locally convex topology generated by the seminorms $\{\lambda_a, \rho_a\}_{a \in J}$ as in the double centralizer setting above. We then define the $\bar{\beta}$ topology to be the inductive limit of the β_J topologies **[13]**. The algebra A under the $\bar{\beta}$ topology will be denoted by $A_{\bar{\beta}}$. If A is topologically simple, then the $\bar{\beta}$ topology is the norm topology, since A is the only ideal $J \subseteq A$ such that $M(J) = A$. Note that our $\bar{\beta}$ topology is space free and unique while Sentilles' topology is generated by a subclass of these ideals and, consequently, in Sentilles' setting our topology is a weaker topology than his β topology. The main question that we consider in this paper is the following: for a countably decomposable W^*-algebra (for example, A_* separable), what are necessary and sufficient conditions for the dual of $A_{\bar{\beta}}$, denoted $A_{\bar{\beta}}{}^*$, to be A_*? We show that a sufficient condition is for A to be a type I W^*-algebra and we have evidence to suggest it is a necessary condition as well. When $A_{\bar{\beta}}{}^*$ is A_*, then $\bar{\beta}$ is the Mackey topology $\tau(A, A_*)$ as studied by Sakai **[14]**, Akemann **[1]** and others. In the special case when A is $L^\infty(\Omega, \mu)$, $\bar{\beta}$ is the mixed topology of Dazord and Jourlin **[4]**.

In §2 we discuss hyper-Stonean spaces as related to a W^*-algebra and §3 is devoted to the study of essential ideals. The general study of the $\bar{\beta}$ topology is presented in §4 with our main results appearing in §5. The reader is referred to **[5]**, **[6]**, and **[14]** for definitions and basic concepts of C^*-algebras and W^*-algebras.

2. Hyper-Stonean topological spaces. Let Ω be a compact Hausdorff space and $C(\Omega)$ the space of all complex-valued continuous functions on Ω. The space Ω is called *Stonean* if the closure of every open set is open, or equivalently, $C(\Omega)$ is a conditionally complete lattice [9, 3N. 6, p. 52]. Now suppose Ω is Stonean. A finite positive regular Borel measure μ on Ω is said to be *normal* if it satisfies the following property: if $\{f_\alpha\}$ is a uniformly bounded increasing directed set of positive functions in $C(\Omega)$, then l.u.b. $\int_\Omega f_\alpha d\mu = \int_\Omega$ l.u.b. $f_\alpha d\mu$. A finite complex regular Borel measure is called normal if it is a linear combination of positive normal measures. We denote by $M(\Omega)$ the finite complex regular Borel measures on Ω and by $N(\Omega)$ the closed subspace of normal measures. The Stonean space Ω is said to be *hyper-Stonean* if the union of the supports of the positive normal measures is dense in Ω, or equivalently, $C(\Omega)$ is a W^*-algebra [14, p. 46].

Throughout this section we shall assume that Ω is a hyper-Stonean space. The results in this section are due to Dixmier [7] and we include them here for completeness.

2.1. PROPOSITION. *Let $\{f_\alpha\}$ be an increasing net of continuous functions in $C(\Omega)$ which is bounded above. If f is the lattice supremum and f' the upper envelope ($f'(x) = \sup_\alpha f_\alpha(x)$, $x \in \Omega$), then f and f' differ on a set of first category.*

Proof. For the proof, see [7, p. 154].

2.2. PROPOSITION. *In order that the measure μ in $M(\Omega)$ be normal it is necessary and sufficient that $\mu(\Delta) = 0$ for all nowhere dense Borel subsets Δ of Ω.*

Proof. For a proof, see [7, Proposition 1, p. 157].

2.3. PROPOSITION. *Let μ be a positive normal measure on Ω and f a μ-measurable complex-valued function. Then there exists a continuous function f' on Ω such that $f = f'$ almost everywhere.*

Proof. For a proof, see [7, Proposition 2, p. 157].

2.4. COROLLARY. *If the support of μ is Ω, then $C(\Omega)$ is $*$-isomorphic to $L^\infty(\Omega, \mu)$.*

We note that by [14, 1.2.6, p. 5] every $*$-isomorphism of C^*-algebras is an isometry.

2.5. PROPOSITION. *Let μ be a positive normal measure on Ω and Δ a μ-measurable subset of Ω. Then Δ coincides, except on a set of μ-measure zero, with the closure $\bar{\Delta}$, with the interior Δ^i, with the closure of Δ^i, and with the interior of $\bar{\Delta}$.*

Proof. For a proof, see [7, Corollary, p. 158].

2.6. COROLLARY. *The support of μ is both open and closed.*

2.7. COROLLARY. *If the support of μ is Ω and Δ is a μ-measurable set such that $\mu(\Delta) = 0$, then Δ is nowhere dense.*

A measure space (Γ, ν) is said to be *localizable* if there exists a family $\{(\Gamma_\alpha, \nu_\alpha)\}$ of finite measure spaces such that $\Gamma = \cup \Gamma_\alpha$, $\nu = \Sigma \oplus \nu_\alpha$, and the family $\{\Gamma_\alpha\}$ is pairwise disjoint. Note that $L^\infty(\Gamma, \nu) = \Sigma \oplus L^\infty(\Gamma_\alpha, \nu_\alpha)$. The measure space (Γ, ν) is called *W^*-localizable* if each Γ_α is a hyper-Stonean space and ν_α is a positive normal measure on Γ_α with support Γ_α.

2.8. PROPOSITION. *Let Z be a commutative W^*-algebra. Then Z is $*$-isomorphic to some $L^\infty(\Gamma, \nu)$, where (Γ, ν) is a W^*-localizable measure space. Moreover, the Stone-Čech compactification of Γ is the spectrum of Z.*

Proof. Since Z is $*$-isomorphic to $C(\Omega)$, Ω hyper-Stonean, the result follows from the proof of [7, Theorem 1, p. 169].

3. Essential ideals in W^*-algebras.

Let A be a W^*-algebra and J a closed two-sided ideal of A. The ideal J is called essential if $J^0 \equiv \{x \in A : xJ = \{0\}\} = \{0\}$. The essential ideals of A will be denoted by $\mathscr{E}_A$, or $\mathscr{E}$ is A is understood. We do not assume J is proper.

A double centralizer of the ideal J is an ordered pair (S, T) of functions from J to J such that $xS(y) = T(x)y$ for all x, y in J. In [3] Busby shows S and T are bounded linear maps with $\|S\| = \|T\|$ and the space of all double centralizers of J, denoted by $M(J)$, is a C^*-algebra under the natural algebraic operations and norm $\|(S, T)\| = \|S\|$. There is a natural embedding of A into $M(J)$, namely, the map $x \to (L_x, R_x)$ where $L_x(y) = xy$ and $R_x(y) = yx$ for all $y \in J$. Our next result connects double centralizer algebras and essential ideals. For basic concepts and definitions of double centralizers, we refer the reader to [3], [20] and [22].

3.1. LEMMA. *Let J be a closed two-sided ideal of the W^*-algebra A. Then the map $x \to (L_x, R_x)$ is a $*$-homomorphism of A onto $M(J)$. Moreover, the map is a $*$-isomorphism if and only if J is essential.*

Proof. Let A_0 be the W^*-subalgebra of A generated by J. It is easy to show that J is essential in A. The conclusion follows from [**22**, Theorem 2.1 and Corollary 2.2, p. 478].

3.2. PROPOSITION. *Let A be a W^*-algebra and I, J and K closed two-sided ideals of A. The following statements are true*:

(1) *If $J \subseteq K$ and $J \in \mathscr{E}$, then $K \in \mathscr{E}$.*
(2) *If $I, J \in \mathscr{E}$, then $I + J \in \mathscr{E}$.*
(3) *If $I, J \in \mathscr{E}$, then $I \cap J \in \mathscr{E}$.*

Proof. The proof of (1) is trivial. It is well-known that $I + J$ is a closed two-sided ideal, so (2) follows immediately from (1). It is straightforward to show, by utilizing 3.1, that $\|x\| = \sup\{\|xy\|: y \in I \cap J, \|y\| \leqq 1\}$, since I and J are essential. Thus the map of 3.1 is an isometry and (3) follows.

The next result shows that W^*-algebras in general have an ample supply of essential ideals.

3.3. PROPOSITION. *Let A be a W^*-algebra. Then A can be written as follows*: *$A = \Sigma_{\alpha \in \pi} \bigoplus A_\alpha$, where each W^*-algebra A_α is either topologically simple or each maximal two-sided ideal of A_α is essential with respect to A_α.*

Proof. Let F be the family of all sets $\{P_\alpha\}$ of central projections with the following properties: (1) $P_\alpha P_\beta = 0$ for $\alpha \neq \beta$; (2) $P_\alpha A$ is topologically simple. It is easy to see, by using Zorn's lemma, that there is a maximal such family $\{P_\alpha\}$. Let $A_\alpha = P_\alpha A$ and $P = \Sigma P_\alpha$. It is straightforward to verify that $A = (\Sigma \bigoplus A_\alpha) \bigoplus (1 - P)A$. Now suppose J is a maximal ideal of $(1 - P)A$ that is not essential. It follows that J^0 is a nonzero topologically simple two-sided ideal of $(1 - P)A$ which is closed in the $\sigma(A, A_*)$ topology. Therefore, there is a central projection Q such that $QA = J^0$[**14**, 1.10.5, p. 25]. But this contradicts the fact that $\{P_\alpha\}$ was maximal. Hence our proof is complete.

It is well known that a factor contains a smallest nonzero, not necessarily proper, closed two-sided ideal [**26**, Remark 3, p. 61]. We will use this fact in the following proposition.

3.4. PROPOSITION. *Suppose that the W^*-algebra A is a factor. Then every nonzero closed two-sided ideal of A is essential.*

Proof. Let J be the smallest nonzero closed two-sided ideal of A. By virtue of 3.2, we need only show J is essential. If $J^0 \neq \{0\}$, then

$J \subseteq J^0$. But this is clearly a contradiction. Hence $J^0 = \{0\}$ and our proof is complete.

Let (Ω, μ) be a localizable measure space and A a W^*-algebra with separable predual A_*. We let $L^\infty(\Omega, \mu, A)$ denote the Banach space of all A-valued essentially bounded weakly* μ-locally measurable functions on Ω (see [**11**, 3.5, p. 72]). In [**14**, 1.22.13, p. 68], Sakai shows $L^\infty(\Omega, \mu, A)$ is a W^*-algebra under pointwise multiplication and its predual is $L^1(\Omega, \mu, A_*)$, where $L^1(\Omega, \mu, A_*)$ is the Banach space of all A_*-valued Bochner μ-integrable functions on Ω. The next lemma connects W^*-tensor products with the space $L^\infty(\Omega, \mu, A)$. For basic definitions and concepts of tensor products of C^*-algebras, we refer the reader to [**14**, 1.22, pp. 58–70]. For the definition of the $s(A, A_*)$ and $s^*(A, A_*)$ topologies see, [**14**, p. 20].

3.5. LEMMA. *Let Z be a commutative W^*-algebra and A a W^*-algebra with separable predual. Then $Z \bar{\otimes} A$ is $*$-isomorphic to $\sum_{\alpha \in \pi} \oplus L^\infty(\Omega_\alpha, \mu_\alpha, A)$, where each Ω_α is hyper-Stonean and μ_α is a positive normal measure with support Ω_α.*

Proof. The proof follows immediately from 2.8 and [**14**, 1.22.13, p. 68].

3.6. LEMMA. *Let Z be a commutative W^*-algebra and A a factor with A_* separable. If J is a closed two-sided ideal of $Z \bar{\otimes} A$ such that $J \cap (Z \otimes_{\alpha_0} A) = \{0\}$, then $J = \{0\}$.*

Proof. By virtue of 3.5 we may assume $Z \bar{\otimes} A = L^\infty(\Omega, \mu, A)$, where Ω is hyper-Stonean and μ is a positive normal measure with support Ω. Moreover, by virtue of 2.4 and [**14**, 1.22.3, p. 61], we may assume $Z \otimes_{\alpha_0} A = C(\Omega, A)$, where $C(\Omega, A)$ is viewed as a subalgebra of $L^\infty(\Omega, \mu, A)$ in the natural way. Note that it follows from 2.4 that the center of $L^\infty(\Omega, \mu, A)$ is $C(\Omega) \cdot 1$, where 1 denotes the identity of A.

First, suppose A is finite. Then, by [**14**, 2.6.1, p. 98], $L^\infty(\Omega, \mu, A)$ is finite. The conclusion follows directly from Corollary 1 of Proposition 2 in [**5**, p. 256].

Next, suppose A is semi-finite. By [**14**, p. 157] there exists an increasing net of projections $\{e_\alpha\}$ which are finite and such that sup $e_\alpha = 1$. Set $A_\alpha = L^\infty(\Omega, \mu, e_\alpha A e_\alpha)$. Then A_α is a W^*-subalgebra of $L^\infty(\Omega, \mu, A)$. Suppose J is a closed two-sided ideal of $L^\infty(\Omega, \mu, A)$ such that $J \cap C(\Omega, A) = \{0\}$. Then $J \cap C(\Omega, e_\alpha A e_\alpha) = \{0\}$ and therefore $J \cap A_\alpha = \{0\}$, since $e_\alpha A e_\alpha$ is a finite factor and the above applies. Now let $x \in J^+$ and set $E_\alpha(t) = e_\alpha$ for all $t \in \Omega$. It follows that $E_\alpha x E_\alpha \in J^+ \cap A_\alpha$ and consequently $E_\alpha x E_\alpha = 0$. Since $\{E_\alpha\}$ converges to the identity of $L^\infty(\Omega, \mu, A)$ in the $s(L^\infty(\Omega, \mu, A), L^1(\Omega, \mu, A_*))$ topology [**14**, 1.13.4, p. 30] and multiplication is jointly $s(L^\infty(\Omega, \mu, A), L^1(\Omega, \mu, A_*))$ continuous on uniformly bounded spheres [**14**, 1.8.12, p. 21], it follows that $E_\alpha x E_\alpha$, converges to x. Hence $x = 0$. Since x was chosen arbitrarily, $J = \{0\}$.

Finally, suppose A is purely infinite. Since A_* is separable, A is countably decomposable [**14**, 2.1.9, p. 80]. Moreover, since the support of μ is Ω, it follows from [7, Proposition 7, p. 161] that $C(\Omega)$ is countably decomposable. Hence $L^\infty(\Omega, \mu, A)$ is a countably decomposable type III (purely infinite) W^*-algebra [**14**, 2.6.6, p. 101]. Now, if J is a closed two-sided ideal of $L^\infty(\Omega, \mu, A)$ such that $J \cap C(\Omega, A) = \{0\}$, then it follows directly from [**14**, 4.1.5, p. 155] that $J = \{0\}$.

Since A must be either finite, semi-finite or purely infinite, our proof is complete.

3.7. COROLLARY. *Let Ω be a hyper-Stonean space, μ a positive normal measure with support Ω, and A a factor with separable predual A_*. If J is a closed two-sided ideal of $L^\infty(\Omega, \mu, A)$ such that $J \cap C(\Omega, A) = \{0\}$, then $J = \{0\}$.*

3.8. THEOREM. *Let Z be a commutative W^*-algebra and A a factor with separable predual A_*. If J is an essential ideal of $Z \bar{\otimes} A$, then $J \cap (Z \otimes_{\alpha_0} A)$ is an essential ideal of $Z \otimes_{\alpha_0} A$.*

Proof. Just as in 3.6, we may assume $Z \bar{\otimes} A = L^\infty(\Omega, \mu, A)$, where Ω is hyper-Stonean and μ is a positive normal measure with support Ω. Moreover, we may assume $Z \otimes_{\alpha_0} A = C(\Omega, A)$. Now suppose J is an essential ideal of $L^\infty(\Omega, \mu, A)$ such that $J_1 \equiv C(\Omega, A) \cap J$ is not essential in $C(\Omega, A)$.

First, we will show that there exists an open and closed subset G of Ω such that $x(t) = 0$ for each $x \in J_1$ and $t \in G$. Since $J_1^0 \neq \{0\}$, we may choose a nonzero $y \in J_1^0$. Because $t \to \|y(t)\|$ is a continuous map, it is clear that there is an open and closed set G for which $\|y(t)\| > 0$ for each $t \in G$. Now suppose there is a t_0 in G and an x in J_1 such that $x(t_0) \neq 0$. Then $K_{t_0} = \{x(t_0): x \in J_1\}$ is a nonzero closed two-sided ideal of A and moreover, by 3.4, K_{t_0} is essential in A. Since $y \in J_1^0$, $y(t_0)x(t_0) = 0$ for each $x \in J_1$. Thus, $y(t_0) = 0$ since K_{t_0} is essential in A. But this is a contradiction because $y(t_0) \neq 0$. So, $x(t) = 0$ for each $x \in J_1$ and $t \in G$.

Due to the fact that J is essential in $L^\infty(\Omega, \mu, A)$, it is straightforward to show that $\chi_G J$ is a nonzero ideal of $L^\infty(\Omega, \mu, A)$. Thus, by 3.7,

$\chi_G J \cap C(\Omega, A) \neq \{0\}$. It follows that there must be an $x \in J_1$ for which $x(t) \neq 0$ for some $t \in G$. But this contradicts the defining properties of G. Consequently, J_1 must be essential in $C(\Omega, A)$ and our proof is complete.

3.9. COROLLARY. *Let Ω, μ, and A be defined as in* 3.7. *If J is an essential ideal of $L^\infty(\Omega, \mu, A)$, then $J \cap C(\Omega, A)$ is an essential ideal of $C(\Omega, A)$.*

3.10. PROPOSITION. *Let Ω, μ, and A be defined as in* 3.7. *If K is an essential closed two-sided ideal in A, then $L^\infty(\Omega, \mu, K)$ is an essential closed two-sided ideal of $L^\infty(\Omega, \mu, A)$.*

Proof. Let $J = L^\infty(\Omega, \mu, K)$ and suppose $J^0 \neq \{0\}$. Since J^0 is a closed two-sided ideal, there exists by 3.6 a nonzero x in $J^0 \cap C(\Omega, A)$. In particular, we have $xy = 0$ for all $y \in C(\Omega, K)$. Since $C(\Omega, K)$ is essential in $C(\Omega, A)$ it follows that $x = 0$, contradicting that $x \neq 0$. Thus J is essential and our proof is complete.

Let H be a separable Hilbert space, $B(H)$ the bounded linear operators on H, $B_0(H)$ the compact operators, and $T(H)$ the trace class operators. It is well known that the dual of $B_0(H)$ is $T(H)$ [**14**, 1.19.1, p. 47] and that the predual of $B(H)$ is $T(H)$ [**14**, 1.15.3, p. 39]. Furthermore, $T(H)$ is separable whenever H is separable [**14**, 2.1.10, p. 81]. These facts will be used in the following examples.

3.11. EXAMPLE. Let Ω and μ be defined as in 3.7 and H as above. Then $L^\infty(\Omega, \mu, B_0(H))$ is an essential ideal of $L^\infty(\Omega, \mu, B(H))$.

3.12. EXAMPLE. Let J be a closed two-sided ideal of $L^\infty(\Omega, \mu, B(H))$. Then J is essential if and only if there exists a closed nowhere dense, possibly empty, subset E of Ω with the property that for each $x \in J$ and $\epsilon > 0$, there is an open neighborhood V of E such that $\|x \mid V\| \leqq \epsilon$. We will denote by J_E the essential ideal consisting of all those elements of $L^\infty(\Omega, \mu, B_0(H))$ which satisfy this property. If $B(H)$ is the complex number system, this essential ideal of $L^\infty(\Omega, \mu)$ will be denoted by I_E.

4. The $\bar{\beta}$ topology. In this section, A will always denote a W^*-algebra and A_* its predual. Let J be an essential ideal of A. Recall that the β_J topology for A is the locally convex topology generated by the family of seminorms $\{\lambda_a, \rho_a\}_{a \in J}$, where $\lambda_a(x) = \|ax\|$ and $\rho_a(x) = \|xa\|$ for all $x \in A$, and the $\bar{\beta}$ topology for A is the inductive limit [**13**, p. 79] of all the β_J topologies. As before, let $\mathscr{E}_A$, or $\mathscr{E}$ if A is understood, denote the family of all essential ideals of A. In this section we study the algebra A under the $\bar{\beta}$ topology.

The proofs of 4.1 through 4.5 are by virtue of [**20**, Corollary 2.7, p. 638], simple adaptations of arguments given by Sentilles. Consequently, we do not include them, but rather refer the reader to [**18**, pp. 317–318] and [**20**, pp. 636–638].

4.1. THEOREM. *Let W be a convex, balanced and absorbing subset of A. Then W is a $\bar{\beta}$ neighborhood of zero if and only if, for each $r > 0$ and $J \in \mathscr{E}$, there is a β_J neighborhood of zero V_J such that $V_J \cap \{x : \|x\| \leqq r\} \subseteq W$. Consequently, the strongest locally convex topology for A that agrees with the $\bar{\beta}$ topology on uniformly bounded subsets of A is the $\bar{\beta}$ topology.*

4.2. COROLLARY. *The continuity of linear maps on $A_{\bar{\beta}}$ is determined on the uniformly bounded subsets of A.*

4.3. COROLLARY. *Let B be a locally convex space and $T: A \to B$ a linear or conjugate linear map. Then T is $\bar{\beta}$ continuous if and only if T is β_J continuous for each $J \in \mathscr{E}$.*

4.4. COROLLARY. *The mappings $x \to ax$, $x \to xa$ and $x \to x^*$ are $\bar{\beta}$ continuous for x, $a \in A$.*

4.5. PROPOSITION. *The following statements are true*: (1) *as subsets of A^*, $A^*_{\bar{\beta}} = \cap_{J \in \mathscr{E}} A^*_{\beta_J}$*; (2) *if, for each $J \in \mathscr{E}$, β_J is the Mackey topology of the dual pair $\langle A, A^*_{\beta_J} \rangle$, then $\bar{\beta}$ is the Mackey topology of the dual pair $\langle A, A^*_{\bar{\beta}} \rangle$.*

Note that A_* is a uniformly closed subspace of A^*.

4.6. THEOREM. *For the dual pair $\langle A, A_* \rangle$, we have $\tau(A, A_*) \leqq \bar{\beta}$, where $\tau(A, A_*)$ denotes the Mackey topology of the dual pair $\langle A, A_* \rangle$.*

Proof. By virtue of [**14**, 1.16.7, p. 41], A can be viewed as a weakly closed self-adjoint subalgebra of $B(H)$, where H is some Hilbert space with the property $H = \{T(h): T \in A, h \in H\}$. Let J be an essential ideal in A. By the Cohen-Hewitt factorization theorem [**10**, Theorem 2.5, p. 151], $H_0 \equiv \{T(h): T \in J, h \in H\}$ is a closed subspace of H. Furthermore, since J is essential, $H = H_0$. It follows that the β_J topology is stronger than the strong operator topology and therefore stronger than the $s(A, A_*)$ topology on uniformly bounded spheres [**14**, 1.15.2, p. 35]. Moreover, due to the fact that the map

$x \to x^*$ is β_J continuous and multiplication is jointly β_J continuous on uniformly bounded spheres, the β_J topology is stronger than the $s^*(A, A_*)$ topology on uniformly bounded spheres. But, Akemann has shown that on uniformly bounded spheres the $\tau(A, A_*)$ and $s^*(A, A_*)$ topologies agree [**1**, Theorem II.7, p. 292]. The conclusion now follows from 4.1.

4.7. COROLLARY. *The Banach space A_* is equal to $A_{\bar\beta}^*$ if and only if $\bar\beta = \tau(A, A_*)$.*

4.8. COROLLARY. *A set $V \subseteq A$ is $\bar\beta$ bounded if and only if V is uniformly bounded.*

4.9. COROLLARY. *The unit ball of A is closed in the $\bar\beta$ topology.*

4.10. PROPOSITION. *The following statements are equivalent*
(1) $\mathscr{E} = \{A\}$
(2) *$\bar\beta$ is normable*
(3) *$\bar\beta$ is metrizable*
(4) *$\bar\beta$ is bornological*
(5) *$\bar\beta$ is barrelled.*

Proof. It is clear that (1) implies (2), (2) implies (3), and (3) implies (4). Assume (4) holds, that is, $\bar\beta$ is bornological. Then $\bar\beta$ is the strongest locally convex topology on A with the same class of bounded sets. Thus, by 4.8 $\bar\beta$ is the norm topology and therefore barrelled. Now assume $\bar\beta$ is barrelled. It follows that the unit ball B_1 of A is a $\bar\beta$ neighborhood of zero. Thus the norm and β_J topologies agree on A for all $J \in \mathscr{E}$. From [**8**, 3.2.4, p. 78] we have, for $J \in \mathscr{E}$, $A = M(J) = J$ and our proof is complete.

4.11. PROPOSITION. *Suppose $\{A_\alpha\}$ is a family of W^*-algebras such that $A = \Sigma_{\alpha\in\pi} \bigoplus A_\alpha$. Then $A_{\bar\beta}^* = (\Sigma_{\alpha\in\pi} \bigoplus (A_\alpha)_{\bar\beta}^*)_{l_1}$* [**14**, 1.1.5, p. 2]. *Consequently, $A_{\bar\beta}^* = A_*$ if and only if $(A_\alpha)_{\bar\beta}^* = (A_\alpha)_*$ for each $\alpha \in \pi$.*

Proof. Note that essential ideals of A of the form $(\Sigma_{\alpha\in\pi} \bigoplus J_\alpha)_0$, where J_α is essential in A_α, generate the $\bar\beta$ topology for A. By $(\Sigma_{\alpha\in\pi} \bigoplus J_\alpha)_0$ we mean those $\{x_\alpha\}$ in A such that $x_\alpha \in J_\alpha$ and $\alpha \to \|x_\alpha\|$ vanishes at infinity. By using this fact together with 4.3 and 4.13, the proof becomes straightforward.

4.12. PROPOSITION. *Let f be a hermitian $\bar\beta$ continuous linear functional on A. Then there exists a unique decomposition $f = f_1 - f_2$, where f_1 and f_2 are positive $\bar\beta$ continuous linear functionals such that $\|f\| = \|f^+\| + \|f^-\|$.*

Proof. The proof follows directly from [**6**, 12.3.4, p. 245], [**21**, Corollary 2.6, p. 164] and 4.3.

4.13. COROLLARY. *The space $A^*_{\bar{\beta}}$ is the linear span of its positive elements.*

For $f \in A^*$ and $x, y \in A$ we define the elements of A^* $x \cdot f$, $f \cdot x$ and $x \cdot f \cdot y$ by $(x \cdot f)(a) = f(ax)$, $(f \cdot x)(a) = f(xa)$ and $(x \cdot f \cdot y)(a) = f(yax)$ for all $a \in A$.

4.14. PROPOSITION. *Suppose J is an essential ideal of A. Then $A^*_{\bar{\beta}}$ is the linear span of all linear functionals in $A^*_{\bar{\beta}}$ of the form $x \cdot g \cdot x$, where $x \in J^+$ and g is a positive $\bar{\beta}$ continuous linear functional on A.*

Proof. Let $f \in A^*_{\bar{\beta}}$. Suppose $\{e_\lambda\}$ is a positive approximate identity for J. Since f is also β_J continuous, it follows from [**20**, Corollary 2.2, p. 635] that $\lim e_\lambda \cdot f = \lim f \cdot e_\lambda = f$. Due to the fact that $A^*_{\bar{\beta}}$ is both a left and right J-module, we see that $f = a \cdot h \cdot b$ by virtue of [**19**, Theorem 2.1, p. 142], where $h \in A^*_{\bar{\beta}}$. By a variant of [**19**, Theorem 2.1, p. 142] there exist elements x, y, z in J such that $x \geqq 0$ and $a = xy$ and $b = zx$. Thus $f = x \cdot g \cdot x$, where $g = y \cdot h \cdot z$. The remainder of the proof follows immediately from 4.13.

4.15. PROPOSITION. *Suppose A is a factor and J is the smallest closed two-sided ideal of A. Then $A^*_{\bar{\beta}} = A^*_{\beta_J}$.*

Proof. The proof is trivial.

4.16. PROPOSITION. *Suppose A is a factor. Then $A^*_{\bar{\beta}} = A_*$ if and only if A is of type I.*

Proof. If A is a type I factor, then $A = B(H)$ for some Hilbert space H. For this case $A_* = T(H)$, the trace class operators, and $A^*_{\bar{\beta}} = A^*_{\beta_J}$ where $J = B_0(H)$. But $T(H) = B_0(H)^* = A^*_{\beta_J}$ [**20**, Corollary 2.3, p. 635]. So, the first part of our proof is complete.

Now suppose $A^*_{\bar{\beta}} = A_*$. By 4.15 and [**20**, Corollary 2.3, p. 635], $A^*_{\bar{\beta}} = A^*_{\beta_J} = J^* = A_*$. So $A^{**}_{\beta_J} = A$ and by [**23**, Theorem 5.1, p. 533], A is of type I.

5. The main results. In this section, A will denote a W^*-algebra, A_* its unique predual, $\mathscr{E}_A$ the essential ideals of A, or $\mathscr{E}$ if A is understood, and $A^*_{\bar{\beta}}$ the dual of A under the $\bar{\beta}$ topology. We will now state one of the main results of the section.

THEOREM I. *If A is a countably decomposable type I W^*-algebra, then $A^*_{\bar\beta} = A_*$. Consequently, the $\bar\beta$ topology is the Mackey topology of the dual pair $\langle A, A_*\rangle$.*

Before we proceed with the proof of Theorem I, we will need the following three lemmas.

5.1. LEMMA. *If A is a countably decomposable type I_n W^*-algebra (n a cardinal number), then $n \leqq \aleph_0$. Consequently, A is $*$-isomorphic to $\Sigma_{\alpha\in\Gamma}\oplus L^\infty(\Omega_\alpha, \mu_\alpha, B(H))$, where Ω_α is hyper-Stonean, μ_α is a finite positive normal measure with support Ω_α, and the dimension of the Hilbert space H is n.*

Proof. The proof follows from [**14**, 2.3.3, p. 89] and 3.5.

In the next two lemmas we will assume Ω is hyper-Stonean, μ is a finite positive normal measure with support Ω, and H is a separable Hilbert space.

5.2. LEMMA. *Let B_1 be the set of all finitely-valued functions x in $C(\Omega, B_0(H))^+$ with $\|x\| \leqq 1$. Then $\bigcup_{J\in\mathscr{C}} Cl_{\beta_J}(B_1)$ is equal to D_1, the set of all x in $L^\infty(\Omega, \mu, B_0(H))^+$ with $\|x\| \leqq 1$. Here, $Cl_{\beta_J}(B_1)$ denotes the closure of B_1 in the β_J topology.*

Proof. By 4.9, $\bigcup_{J\in\mathscr{C}} Cl_{\beta_J}(B_1) \subseteq D_1$. Let $x \in D_1$. By [**11**, Corollary 1 and Corollary 2, p. 73], there exists a sequence $\{x_n\}_{n=1}^\infty$ in D_1 of countably-valued functions such that $\|x_n - x\| \to 0$. Thus, it suffices to assume that x is countably-valued. Let $\{T_i\}_{i=1}^\infty$ be the values in $B_0(H)^+$ assumed by x. Then set $E_i = \{t : x(t) = T_i\}$. By virtue of 2.2, 2.5 and 2.7, we may assume $x = \Sigma_{i=1}^\infty T_i\chi_{E_i}$, where $E_i \cap E_j = \emptyset$, $i \neq j$, and E_i is open and closed. Let $E = \overline{\bigcup_{i=1}^\infty E_i} \setminus \bigcup_{i=1}^\infty E_i$ and suppose $E \neq \emptyset$. Then E is closed and, by 2.5 and 2.7, nowhere dense. Let J_E be the ideal of $L^\infty(\Omega, \mu, B_0(H))$ defined in 3.12. We shall show that $x_n \to x$ in the β_{J_E} topology where $x_n = \Sigma_{i=1}^n T_i\chi_{E_i}$. Given $z \in J_E$ and $\epsilon > 0$, there exists an open set $V_\epsilon \supseteq E$ such that $\|z(t)\| < \epsilon/2\|x\|$ for almost all $t \in V_\epsilon$. Since x is continuous on $\tilde V_\epsilon$, the complement of V_ϵ, the functions $t \to \|x_n(t) - x(t)\|$ form a decreasing sequence of positive continuous functions that converge pointwise to 0 on the compact set $\tilde V_\epsilon$. Consequently, by Dini's Theorem there exists an integer N such that $\|x_n(t) - x(t)\| < \epsilon/\|z\|$ for all $t \in \tilde V_\epsilon$ and $n \geqq N$. It easily follows that $\|zx_n - zx\| < \epsilon$ and $\|x_n z - xz\| < \epsilon$ for $n \geqq N$. Thus, $x_n \to x$ in the β_{J_E} topology. If $E = \emptyset$, then by Dini's theorem $x_n \to x$ uniformly on Ω. Therefore $x \in \bigcup_{J\in\mathscr{C}} Cl_{\beta_J}(B_1)$ and our proof is complete.

In the following lemma let A denote the W^*-algebra $L^\infty(\Omega, \mu, B(H))$.

5.3. LEMMA. *If $F \in A^*_{\bar{\beta}}$ and x is a finitely valued function in $C(\Omega, B_0(H))$, then $x \cdot F \in A_*$.*

Proof. It is straightforward to verify, by utilizing the spectral theorem for compact operators, 4.13, and 2.5, that we can make the following assumptions: (1) $x = \chi_G P$, where G is an open and closed subset of Ω and P is a one-dimensional projection on H; (2) F is positive. First, we will show that $x \cdot F \cdot x \in A_*$, or equivalently, $x \cdot F \cdot x$ is normal **[14**, 1.13.2, p. 28**]**. Let $\{z_\alpha\}$ be an increasing net in A^+ with $z = \sup z_\alpha$. Since $x \cdot F \cdot x(z_\alpha) = F(xz_\alpha x)$ and F is $\bar{\beta}$ continuous, it will suffice to show $xz_\alpha x \to xzx$ in the $\bar{\beta}$ topology.

Let $E \subseteq \Omega$ be a closed nowhere dense set, I_E the ideal of $L^\infty(\Omega, \mu)$ as defined in 3.12 and J_E the corresponding ideal of $L^\infty(\Omega, \mu, B_0(H))$ (see 3.12). For $y \in J_E$ we have

$$
\begin{aligned}
&\| y(t)x(t)[z(t) - z_\alpha(t)]x(t)\| \\
&\quad \leqq \sup\{\| y(t)x(t)[z(t) - z_\alpha(t)]x(t)h \| : h \in H, \|h\| \leqq 1\} \\
&\quad \leqq \sup\{ |\langle h, h_0\rangle \langle (z(t) - z_\alpha(t))(h_0), h_0\rangle| \, \| y(t)(h_0)\| : h \in H, \|h\| \leqq 1\} \\
&\quad \leqq |\langle (z(t) - z_\alpha(t))(h_0), h_0\rangle| \, \| y(t)\|,
\end{aligned}
$$

where $P(h_0) = h_0$ and $\|h_0\| = 1$. By **[11**, Theorem 2.8.5, p. 34**]** and **[11**, Theorem 3.5.2, p. 72**]**, the map $t \to \| y(t)\|$ is measurable and thus equal to, almost everywhere, a continuous function that vanishes on E. Therefore, it sufficies to find a closed nowhere dense set E for which $\phi_\alpha(t) \equiv \langle z_\alpha(t)(h_0), h_0\rangle$ converges to $\phi(t) \equiv \langle z(t)(h_0), h_0\rangle$ in the β_{I_E} topology, for then $xz_\alpha x$ will converge to xzx in the β_{J_E} topology, and hence, in the $\bar{\beta}$ topology.

Since $xz_\alpha x \to xzx$ in the $\sigma(A, A_*)$ topology **[14**, 1.7.4, p. 15**]** and the predual of $L^\infty(\Omega, \mu, B(H))$ is $L^1(\Omega, \mu, T(H))$, it is easy to show that $\int_\Omega \phi_\alpha(t)d\mu \to \int_\Omega \phi(t)d\mu$. By virtue of 2.4 we can choose functions f_α, f in $C(\Omega)$ such that $f_\alpha = \phi_\alpha$ and $f = \phi$ almost everywhere. Note that f_α is an increasing net with $f \geqq f_\alpha$, so $f \geqq \sup f_\alpha \equiv f'$. Since μ is a positive normal measure, $\int_\Omega f_\alpha d\mu \to \int_\Omega f' d\mu$. Hence, $\int_\Omega (f - f')d\mu = 0$, or equivalently, $f = f'$. Now, let E be the closure of $\{t : f(t) > \sup f_\alpha(t)\}$. By 2.1, 2.2, 2.5 and 2.7, E is nowhere dense. By virtue of Dini's theorem, it is straightforward to show that $f_\alpha \to f$ in the β_{I_E} topology for $C(\Omega)$ and consequently $xz_\alpha x \to xzx$ in the β_{J_E} topology for $L^\infty(\Omega, \mu, B(H))$. Thus $x \cdot F \cdot x \in A_*$.

Finally, we must show $x \cdot F \in A_*$. Suppose $x_\alpha \to 0$ for the

$s^*(A, A_*)$ topology where $\|x_\alpha\| \leqq 1$. Then $x_\alpha^* \to 0$ for the $s^*(A, A_*)$ topology and by [14, 1.8.9, p. 20] and [14, 1.8.12, p. 21] it follows that $x_\alpha^* x_\alpha \to 0$ for the $s(A, A_*)$ topology. By utilizing the Schwartz inequality [14, p. 9], [14, 1.8.10, p. 21] and the above result for $x \cdot F \cdot x$ it is easy to show that $(x \cdot F)(x_\alpha) \to 0$. Thus $x \cdot F \in A_*$ and our proof is complete.

Proof of Theorem I. By [14, 2.3.2, p. 89], $A = \Sigma_{n \in \Gamma} \oplus A_n$, where each A_n is a type I_n W^*-algebra. By virtue of 4.11 we may assume A is of type I_n and consequently by 5.1 and 4.11 we may assume $A = L^\infty(\Omega, \mu, B(H))$ where Ω is hyper-Stonean, μ is a positive normal measure with support Ω, and H is a separable Hilbert space.

Let $F \in L^\infty(\Omega, \mu, B(H))^*_{\bar{\beta}}$ such that $F \geqq 0$. By 4.14 and 3.11 we may assume $F = x \cdot G \cdot x$ for some $x \in L^\infty(\Omega, \mu, B_0(H))^+$ and G a positive $\bar{\beta}$ continuous linear functional on $L^\infty(\Omega, \mu, B(H))$. Clearly, we may assume $\|x\| \leqq 1$. By 5.2 there exists an essential ideal J in $L^\infty(\Omega, \mu, B(H))$ such that $x \in Cl_{\beta_J}(B_1)$. Consequently, there exists a net $\{x_\alpha\}$ in B_1 that converges to x in the β_J topology. By 4.14 we may assume $G = y \cdot G_1 \cdot y$ for some $y \in J^+$ and G_1 a positive $\bar{\beta}$ continuous linear functional on $L^\infty(\Omega, \mu, B(H))$. Therefore $x_\alpha \cdot G \cdot x_\alpha \to F$ uniformly. Hence, by virtue of 5.3, $F \in A_*$. Since $A^*_{\bar{\beta}}$ is the linear span of its positive linear functionals, $A^*_{\bar{\beta}} = A_*$. That $\bar{\beta}$ is the Mackey topology of the dual pair $\langle A, A_* \rangle$, is an immediate consequence of 4.7 and our proof is now complete.

For a type I W^*-algebra, the condition of being countably decomposable is not necessary for Theorem I to hold. In fact, Theorem I holds for $A = B(H)$, H not separable (see 4.16), and for any commutative W^*-algebra Z. Moreover, it is easy to see from our proof, that Theorem I holds for $Z \bar{\otimes} B(H)$ where H is a separable Hilbert space. It is of interest to note that for the W^*-algebra $L^\infty = L^\infty(X, \nu)$, where X completely regular and ν is a compact regular Borel measure on X, we have $\bar{\beta} = \tau(L^\infty, L^1)$ where $L^1 = L^1(X, \nu)$. Thus, $\bar{\beta}$ is the mixed topology considered by Dazord and Jourlin [4]. These results lead us to the following two related questions.

5.4. *Question.* Suppose A is a type I W^*-algebra that is not countably decomposable. Must $A^*_{\bar{\beta}}$ be equal to A_*?

5.5. *Question.* Let A be a countably decomposable W^*-algebra such that $A^*_{\bar{\beta}} = A_*$. Must A be a type I W^*-algebra? In other words, does the converse of Theorem I hold?

Our next result suggests that the converse of Theorem I may indeed be true. We will now view A as a W^*-algebra on a separable Hilbert space.

THEOREM II. *Let $A = \int_\Gamma A(t)\mu(dt)$ be the direct integral decomposition of A into factors* [**17**, Corollary 10, p. 53]. *Let B be a factor and define Λ to be the set of all $t \in \Gamma$ for which $A(t)$ is spatially isomorphic to B. If $A^*_{\bar{\beta}} = A_*$ and $\mu(\Lambda) > 0$, then B must be a type I factor.*

Proof. By virtue of [**17**, Theorem 2, p. 228], 4.11, and 3.5, we need only consider the case $A = L^\infty(\Omega, \mu, B)$, where Ω is hyper-Stonean, μ is a positive normal measure with support Ω, and $\mu(\Omega) = 1$. Now, let D be the set of all elements in $C(\Omega, B)$ of the form $\Sigma_{k=1}^n x_k \chi_{E_k}$, where $x_k \in B$ and $\{E_k\}_{k=1}^n$ are pairwise disjoint sets that are both open and closed. Since D is a *-subalgebra of $C(\Omega, B)$ that separates points and contains the identity, we have by [**6**, 11.5.3, p. 234] that D is uniformly dense in $C(\Omega, B)$. Consequently, it follows from [**14**, 1.22.3, p. 61] and [**14**, p. 67] that D is $\sigma(A, A_*)$ dense in $L^\infty(\Omega, \mu, B)$. Now, for $f \in (B^*_{\bar{\beta}_B})^+$ define $\tilde{f}$ on D as follows: for $x = \Sigma_{k=1}^n x_k \chi_{E_k}$, set $\tilde{f}(x) = \Sigma_{k=1}^n f(x_k)\mu(E_k)$. We will now show that $\tilde{f}$ is continuous on the unit ball of D for the relative $\bar{\beta}_A$ topology. By 4.1 and 4.3, it will suffice to show that $\tilde{f}$ is continuous on the unit ball of D in the relative β_J topology for each $J \in \mathscr{E}_A$.

Let J be a closed two-sided essential ideal of A, I_0 the smallest closed two-sided ideal of B, and $\epsilon > 0$. Since f is $\bar{\beta}_B$ continuous on B, there exists a $b \in I_0$ such that $|f(x)| \leqq \epsilon/2$ whenever $\|bx\| + \|xb\| \leqq 1$. Now set $J_1 = J \cap C(\Omega, B)$. By 3.9, J_1 is essential in $C(\Omega, B)$, so the set $E = \{t \in \Omega\colon x(t) = 0 \text{ for all } x \in J_1\}$ is a closed nowhere dense subset of Ω. Since μ is normal, there exists an open and closed set $G \supseteq E$ such that $\mu(G) < \epsilon/2\|f\|$. For each $t_0 \in \tilde{G}$, the complement of G, the set $\{x(t_0)\colon x \in J_1\}$ contains I_0. Therefore, it is straightforward to show that there exists a subset V_{t_0} of $\tilde{G}$ and an element a_0 in J_1 that satisfy the following:

(1) $t_0 \in V_{t_0}$ and V_{t_0} is both open and closed;

(2) $\|a_0(t) - b\| < 1/8$ for $t \in V_{t_0}$ and $a_0(t) = 0$ otherwise. Since $\tilde{G}$ is compact and $\{V_t\}_{t\in\tilde{G}}$ is an open cover, there exists a finite collection $\{V_{t_i}\}_{i=1}^m$ that covers $\tilde{G}$. Due to the fact that each set V_{t_i} is both open and closed, we can construct an element a in J_1 such that $\|a(t) - b\| < 1/4$ for $t \in \tilde{G}$ and $a(t) = 0$, $t \in G$. Now let x be an element of the unit ball of D, where $x = \Sigma_{k=1}^n x_i \chi_{E_i}$, and suppose $\|xa\| + \|ax\| \leqq 1/2$. We may assume that, for some positive integer k, $E_1, E_2, \cdots, E_k$ are subsets of G

and $E_{k+1}, \cdots, E_n$ are subsets of $\tilde{G}$. Since $\|x\| \leqq 1$ and $\|ax\| + \|xa\| \leqq 1/2$, it is easy to show that $\|x_i b\| + \|bx_i\| \leqq 1$ for $i = k+1, \cdots, n$. Therefore

$$|\tilde{f}(x)| \leqq \sum_{i=1}^{k} |f(x_i)|\mu(E_i) + \sum_{i=k+1}^{n} |f(x_i)|\mu(E_i)$$
$$\leqq \|f\|\mu(G) + (\epsilon/2)\mu(\tilde{G}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, $\tilde{f}$ is $\bar{\beta}_A$ continuous on the unit ball of D. Since $A^*_{\bar{\beta}_A} = A_*$, the $\bar{\beta}_A$ topology is the Mackey topology of the dual pair $\langle A, A_* \rangle$. Consequently, it follows from **[14**, 1.9.1, p. 22] and **[14**, 1.8.10, p. 21] that $\tilde{f}$ can be extended uniquely to a $\bar{\beta}_A$ continuous positive linear functional on A. Hence $\tilde{f} \in A_*$ and this implies $f \in B_*$. But, by 4.16 this can only happen when B is a type I factor and our proof is complete.

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