ENUMERATION OF WEIGHTED $p$-LINE ARRAYS

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Let $F_p(n,k; q_1, q_2, \cdots, q_p) = F_p(n,k)$ be defined by

$$F_p(n,k) = \sum_{i=1}^{p} q_i^{r_i},$$

where the summation is over all $p$-line arrays of positive integers

\[
\begin{array}{cccc}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{p1} & a_{p2} & \cdots & a_{pn}
\end{array}
\]

subject to the following conditions:

$$\max\{a_{ij} : 1 \leq i \leq p\} \leq \min\{a_{ij+1} : 1 \leq i \leq p\}, \quad 1 \leq j \leq n - 1,$$

$$\max\{a_{ij} : 1 \leq i \leq p\} \leq j, \quad 1 \leq j \leq n,$$

and

$$a_{in} = k, \quad 1 \leq i \leq p.$$ 

Assuming $\Pi_{i=1}^{p} q_i = 1$, formulas for $F_p(n,k)$ and two other enumerants, which are closely related to $F_p(n,k)$, are obtained in this paper. These three functions generalize enumerants which Carlitz has determined.

1. **Introduction.** We consider the enumeration of $p$-line arrays of positive integers

\[
\begin{array}{cccc}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{p1} & a_{p2} & \cdots & a_{pn}
\end{array}
\]

(1.1)

satisfying certain conditions. We first require that

$$\max\{a_{ij} : 1 \leq i \leq p\} \leq \min\{a_{ij+1} : 1 \leq i \leq p\}, \quad 1 \leq j \leq n - 1,$$
and

\[(1.3) \quad \max\{a_{ij} : 1 \leq i \leq p\} \leq j, \quad 1 \leq j \leq n.\]

We indicate further requirements by defining the enumerants we seek. Let \(f_p(n, k; s_1, s_2, \cdots, s_p)\) represent the number of arrays (1.1) subject to the restrictions (1.2), (1.3),

\[(1.4) \quad \sum_{j=1}^{n} a_{ij} = s_i, \quad 1 \leq i \leq p,\]

and

\[(1.5) \quad a_{in} = k, \quad 1 \leq i \leq p,\]

and let

\[F_p(n, k; q_1, q_2, \cdots, q_p) = F_p(n, k) = F_p = \sum_{*} f_p(n, k; s_1, s_2, \cdots, s_p)q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p},\]

where \(\sum^{*}\) is the sum over the \(p\)-tuples \((s_1, s_2, \cdots, s_p)\). (We may view \((s_1, s_2, \cdots, s_p)\) as the weight of the array (1.1).) Let \(g_p(n, k; s_1, s_2, \cdots, s_p)\) denote the number of arrays (1.1) satisfying (1.2), (1.3), (1.4) and

\[(1.6) \quad \max\{a_{in} : 1 \leq i \leq p\} = k,\]

and let

\[G_p(n, k; q_1, q_2, \cdots, q_p) = G_p(n, k) = G_p = \sum_{*} g_p(n, k; s_1, s_2, \cdots, s_p)q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p}.\]

Finally, we use \(h_p(n, k; s_1, s_2, \cdots, s_p)\) to represent the number of arrays (1.1) subject to conditions (1.2), (1.3), (1.4), (1.6) and

\[(1.7) \quad a_{i+1,j} \leq a_{ij}, \quad 1 \leq i \leq p - 1, \quad 1 \leq j \leq n,\]

and we let

\[H_p(n, k; q_1, q_2, \cdots, q_p) = H_p(n, k) = H_p = \sum_{*} h_p(n, k; s_1, s_2, \cdots, s_p)q_1^{s_1} q_2^{s_2} \cdots q_p^{s_p}.\]

The functions \(F_1, G_1\) and \(H_1\) coincide, and if \(q_1 = 1\), they enumerate what MacMahon [7, p. 167] called two-element lattice
permutations. Carlitz and Riordan [6] have studied these functions and a $q$-generalization. A related $q$-generalization, in fact $F_1(n, k; q)$, has also been investigated by Carlitz [5]. If $p = 2$ and $q_1 = q_2 = 1$, $F_2, G_2$ and $H_2$ are the enumerants $f, g$ and $h$ which Carlitz [4] has explicitly determined.

In this paper we first generalize some identities which Carlitz stated for $f, g$ and $h$. Then, by assuming

$$\prod_{i=1}^{p} q_i = 1,$$

we are able to use these results and Carlitz's technique for finding $f, g$ and $h$ to obtain formulas for $F_p, G_p$ and $H_p$. In general these formulas are in terms of functions $t(n, k)$ which are defined by

$$\Phi^p(x) = \sum_{k=0}^{\infty} t(n, k)x^k,$$

where $\Phi(x)$ is a known function. In some special cases the enumerants can be expressed in terms of binomial or $q$-binomial coefficients. For example, if $q_1 = q_2 = q$ and $q^2 = 1$,

$$F_2(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n - j)b_2(n, j; q)$$

and

$$G_2(n, k + 1) = \frac{n - k}{n} b_2(n, k; q)$$

where

$$b_2(n, k; q) = \sum_{m=0}^{k} \binom{n}{k - m} \binom{2n + m - 1}{m} q^m$$

$$= \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k - 2j} \binom{2n + 2j - 1}{2j} + \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{n}{k - 2j - 1} \binom{2n + 2j}{2j + 1} q.$$

We also find that

$$H_2(n, k + 1) = \frac{n - k}{n} \sum_{m=0}^{k} \binom{n + k - m - 1}{k - m} \binom{n + m - 1}{m} q^m$$
if \( q_1q_2 = 1 \), and

\[
H_p(n, k + 1) = \frac{n - k}{n} \left[ \frac{np + k - 1}{k} \right]_q
\]

if \( q_1 = q_2 = \cdots = q_p = q \) and \( q^p = 1 \). In §6 we interpret the formulas for \( F_p \), \( G_p \) and \( H_p \) as partition theorems. It would be of interest to determine these enumerants without the restriction \((1.8)\).

In a subsequent paper we shall consider

\[
I_p(n, k; q_1, q_2, \cdots, q_p) = \sum_i i_p(n, k; s_1, s_2, \cdots, s_p)q_1^s_1 q_2^s_2 \cdots q_p^s_p,
\]

where \( i_p(n, k; s_1, s_2, \cdots, s_p) \) represents the number of arrays \((1.1)\) satisfying \((1.2)\), \((1.3)\), \((1.4)\) and \((1.7)\), and having \( k \) columns in which

\[ a_{ij} = a_{2j} = \cdots = a_{pj}. \]

Carlitz [3] called such columns coincidences and has proved that

\[
I_p(n, k; 1, 1, \cdots, 1) = \frac{1}{k} \left( \frac{n - 1}{k - 1} \right) \sum_{i=0}^{n-k-1} (-1)^{n-k-t} \binom{n-k}{t} \binom{2n + (p-1)t}{n-1}
\]

for \( q_1 = q_2 = \cdots = q_p = 1 \).

2. Preliminary results. Generalizing \((2.1)-(2.4), (2.7)\) and \((2.9)\) of [4], we have

\[(2.1) \quad F_p(n + 1, k) = \left[ \prod_{i=1}^{p} q_i \right]^k \sum_{m=1}^{k} \left[ \prod_{i=1}^{p} [k - m + 1]_q \right] \prod_{i=1}^{p} q_i [k - m]_q \right] F_p(n, m), \quad k \leq n + 1, \]

where

\[ [k]_q = \sum_{j=0}^{k-1} q_j, \]

\[(2.2) \quad F_p(n + 1, k) = \left[ \prod_{i=1}^{p} q_i \right]^k \sum_{m=1}^{k} G_p(n, m), \quad k \leq n + 1, \]

\[(2.3) \quad G_p(n + 1, k) = \sum_{m=1}^{k} \left[ \prod_{i=1}^{p} q_i \right]^m \left[ \prod_{i=1}^{p} [k - m + 1]_q \right] - \prod_{i=1}^{p} [k - m]_q \right] G_p(n, m), \quad k \leq n + 1, \]
(2.4) \[ G_p(n, k) = \sum_{m=1}^{k} \left[ \prod_{i=1}^{p} [k - m + 1]_{q_i} - \left(1 + \prod_{i=1}^{p} q_i \right) \prod_{i=1}^{p} [k - m]_{q_i} + \prod_{i=1}^{p} q_i \prod_{i=1}^{p} [k - m - 1]_{q_i} \right] F_p(n, m), \quad k \leq n, \]

(2.5) \[ G_p(n + k, k) = \sum_{m=1}^{k} \left\{ \prod_{i=1}^{p} q_i \right\}^{(n+k-m)(m-1)} G_p(m, m) G_p(n + k - m, k - m + 1), \quad n \geq 1, \]

and

(2.6) \[ F_p(n + k, k) = \sum_{m=1}^{k} \left\{ \prod_{i=1}^{p} q_i \right\}^{(n+k-m)(m-1)} G_p(m, m) F_p(n + k - m, k - m + 1), \quad n \geq 1. \]

Let

(2.7) \[ \theta_p(k; q_2, q_3, \ldots, q_p) = \sum q_i^{a_{in}}, \quad p \geq 2, \]

where the summation is over all \((p - 1)\)-tuples \((a_{2n}, a_{3n}, \ldots, a_{pn})\) with the \(a_{in}\) satisfying \(0 \leq a_{2n} \leq \cdots \leq a_{3n} \leq a_{2n} \leq k\). Then corresponding to (6.3) and (6.8) of [4] we have

(2.8) \[ H_p(n, k) = \sum_{m=1}^{k} \left\{ \prod_{i=1}^{p} q_i \right\}^{m} q_i^{k-m} \theta_p(k-m; q_2, q_3, \cdots, q_p) H_p(n-1, m), \]

where it is understood that \(H_p(n - 1, n) = 0\), and

(2.9) \[ H_p(n + k, k) = \sum_{j=1}^{k} \left\{ \prod_{i=1}^{p} q_i \right\}^{(n+k-m)(m-1)} H_p(m, m) H_p(n + k - m, k - m + 1), \quad n \geq 1. \]

The proofs of (2.1)–(2.6), (2.8) and (2.9) are simply generalizations of the proofs of their analogues in [4]. To prove (2.1) it suffices to assume \(k \leq n\) since \(F_p(n + 1, n + 1) = F_p(n + 1, n)\). For \(k \leq n\) we consider the array
satisfying (1.2), (1.3) and

\[
\text{(2.10)} \quad \max\{a_{in} : 1 \leq i \leq p\} \leq k.
\]

Let

\[
\text{(2.11)} \quad \min\{a_{in} : 1 \leq i \leq p\} = m.
\]

Using (2.10), (2.11) and the definition of \( F_p(n, k) \), we have

\[
\text{(2.12)} \quad F_p(n + 1, k) = \left[ \prod_{i=1}^{p} q_i \right]^k \sum_{m=1}^{k} \sum_{l=1}^{p} q_i^{q_{il} - q_{m}} F_p(n, m),
\]

where the inner summation is over all arrays (1.1) satisfying (1.2), (1.3), (2.10) and (2.11). From (2.12) we get

\[
\text{(2.13)} \quad F_p(n + 1, k) = \left[ \prod_{i=1}^{p} q_i \right]^k \sum' \prod_{i=1}^{p} q_i^{q_{il} - m} F_p(n, m),
\]

where \( \sum' \) is the sum over all \( p \)-tuples \((a_{1n}, a_{2n}, \ldots, a_{pn})\) subject to conditions (2.10) and (2.11). Since

\[
\sum' \prod_{i=1}^{p} q_i^{q_{il} - m} = \sum_{m \leq a_{1n} \leq k} \prod_{i=1}^{p} q_i^{q_{il} - m} - \sum_{m+1 \leq a_{1n} \leq k} \prod_{i=1}^{p} q_i^{q_{il} - m}
\]

\[
= \sum_{0 \leq a_{1n} \leq m} \prod_{i=1}^{p} q_i^{q_{il}} - \sum_{0 \leq a_{1n} \leq m-1} \prod_{i=1}^{p} q_i^{q_{il} + 1}
\]

\[
= \prod_{i=1}^{p} [k - m + 1] q_i - \prod_{i=1}^{p} q_i [k - m] q_i,
\]

(2.1) follows from (2.13).

Equation (2.2) follows immediately from the definitions. The proof of (2.3) is similar to the proof of (2.1), but to obtain (2.3) we consider the array
where conditions (1.2) and (1.3) (with \( n \) replaced by \( n + 1 \)) are satisfied and where

\[
\max\{a_i : 1 \leq i \leq p\} = m
\]

and

\[
\max\{a_{i,n+1} : 1 \leq i \leq p\} = k.
\]

As (2.14) and (1.2) imply that

\[
\min\{a_{i,n+1} : 1 \leq i \leq p\} \geq m,
\]

we find that

\[
G_p(n + 1, k) = \sum_{m=1}^{k} \sum_{i=1}^{p} q_{i,n}^{a_i-m} G_p(n, m),
\]

where the inner sum is over all \( p \)-tuples \((a_{1,n+1}; a_{2,n+1}; \ldots; a_{p,n+1})\) satisfying (2.15) and (2.16). From (2.17) we get (2.3).

To prove (2.4) consider the array (1.1) subject to conditions (1.2), (1.3), (1.6) and (2.11). Corresponding to (2.13) and (2.17) in the previous proofs, we have

\[
G_p(n, k) = \sum_{m=1}^{k} \sum_{i=1}^{p} q_{i,n}^{a_i-m} F_p(n, m),
\]

where the inner summation is over the \( p \)-tuples \((a_{1n}, a_{2n}, \ldots, a_{pn})\) satisfying (1.6) and (2.11). From (2.18) we get

\[
G_p(n, k) = \sum_{m=1}^{k} \left[ \sum_{m \leq a_n} \prod_{i=1}^{p} q_{i,n}^{a_i-m} - \sum_{m \leq a_n} \prod_{i=1}^{p} q_{i,n}^{a_i-m} \right. \\
- \left. \sum_{m \leq a_n} \prod_{i=1}^{p} q + \sum_{m + 1 \leq a_n} \prod_{i=1}^{p} q \right] F_p(n, m),
\]

and (2.4) follows.
Since the proofs of (2.5), (2.6) and (2.9) are similar, we shall only establish (2.5). To this end we observe that

$$\max\{a_{i1}: 1 \leq i \leq p\} = 1$$

implies that there exists a greatest \( m \) such that

$$\max\{a_{im}: 1 \leq i \leq p\} = m.$$  

Therefore

$$\max\{a_{i,m+1}: 1 \leq i \leq p\} = m,$$

$$a_{i,m+1} = m,$$  

and we can divide our original array into two sub-arrays as follows:

\[
\begin{array}{cccc}
1 & \ldots & a_{im} & m & \ldots & a_{1,n+k} \\
1 & \ldots & a_{2m} & m & \ldots & a_{2,n+k} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \ldots & a_{pm} & m & \ldots & a_{p,n-k} \\
\end{array}
\]

(2.19)

By subtracting \( m - 1 \) from each entry in the right sub-array of (2.19), we get

$$g_p(n + k, k; s_1, s_2, \ldots, s_p)$$

$$= \sum_{m=1}^{k} g_p(m, m; u_1, u_2, \ldots, u_p)g_p(n + k - m, k - m + 1; v_1, v_2, \ldots, v_p),$$

where \( u_i + v_i = s_i - (n + k - m)(m - 1), 1 \leq i \leq p \). Now (2.5) follows immediately.

We obtain (2.8) by considering the array

\[
\begin{array}{cccc}
a_{11} & a_{12} & \ldots & m & k \\
a_{21} & a_{22} & \ldots & a_{2,n-1} & a_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{p1} & a_{p2} & \ldots & a_{p,n-1} & a_{pn} \\
\end{array}
\]

where (1.2)–(1.4), (1.6), (1.7) and the condition

$$a_{1,n-1} = m$$
are satisfied. Clearly
\[
\begin{align*}
  h_p(n, k; s_1, s_2, \ldots, s_p) &= \sum_{m=1}^{k} \sum^{n-1} h_p(n - 1, m; s_1 - k, s_2 - a_{2n}, \ldots, s_p - a_{pn}),
\end{align*}
\]
where \( \sum^{n} \) is the sum over all \((p-1)\)-tuples such that
\[
  m \leq a_{pn} \leq a_{p-1,n} \leq \cdots \leq a_{2n} \leq k.
\]
Thus
\[
  H_p(n, k) = \sum_{m=1}^{k} q^m \sum^{p} \prod_{i=2}^{p} q^m H_p(n - 1, m)
\]
and we have (2.8).

3. Techniques for determining the enumerants. To obtain the following results we use Carlitz’s method [4] for finding \(f, g\) and \(h\) in a somewhat more general setting. Assuming (1.8), Theorem 1 provides formulas for \(G_p\) and \(H_p\) while Corollary 1 yields an expression for \(F_p\).

Before stating the theorem it is convenient to define some functions. Using \(\mathbb{N}\) to denote the nonnegative integers and \(\mathbb{N}^*\) to represent the positive integers, let \(r(n, k)\) be a function from \(\mathbb{N}^* \times \mathbb{N}^*\) into a field \(F\) and \(\phi(n)\) be a function from \(\mathbb{N}\) into \(F\). Let
\[
  R_n(x) = \sum_{k=1}^{n} r(n + k - 1, k)x^{n+k-1}, \quad n \geq 1,
\]
and
\[
  \Phi(x) = \sum_{n=0}^{\infty} \phi(n)x^n.
\]
Furthermore we define \(t(n, k)\), a function from \(\mathbb{N} \times \mathbb{N}\) into \(F\), by
\[
  \Phi^e(x) = \sum_{k=0}^{\infty} t(n, k)x^k.
\]

**Theorem 1.** If \(r(n, k)\) and \(\phi(n)\) satisfy
\[
  (3.1) \quad r(1, 1) = 1,
\]
(3.2) \[ r(n, k) = 0, \quad n < k, \]

(3.3) \[ \phi(0) = 1, \]

(3.4) \[ r(n + 1, k) = \sum_{m=1}^{k} \phi(k - m)r(n, m), \quad 1 \leq k \leq n + 1, \]

and

(3.5) \[ r(n + k, k) = \sum_{m=1}^{k} r(m, m)r(n + k - m, k - m + 1), \quad n \geq 1, \]

and if \( \Phi(z) \) is analytic about \( z = 0 \), then

(3.6) \[ r(n, k + 1) = \frac{n - k}{n} t(n, k). \]

**Proof.** By (3.5)

\[
R_{n+1}(x) = \sum_{k=1}^{n} x^{n+k} \sum_{m=1}^{k} r(m, m)r(n + k - m, k - m + 1)
\]

\[ = \sum_{m=1}^{n} r(m, m)x^{m} \sum_{k=1}^{n} r(n + k - 1, k)x^{n+k-1} \]

\[ = R_1(x) R_n(x). \]

Thus

\[ R_n(x) = R_1^n(x) \quad n \geq 1. \]

Using (3.1)–(3.4), we find that

\[
\Phi(R_n(x)) = 1 + \sum_{n=1}^{\infty} \phi(n) R_n(x)
\]

\[ = 1 + \sum_{n=1}^{\infty} \phi(n) \sum_{k=1}^{n} r(n + k - 1, k)x^{n+k-1} \]

\[ = 1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \phi(n - k) r(n - 1, k)x^{n-1} \]

\[ = 1 + \sum_{n=2}^{\infty} r(n, n)x^{n-1} \]

\[ = x^{-1} R_1(x). \]
It follows that

\[ x = \frac{z}{\Phi(z)}, \]

where \( z = R_1(x) \).

By the Lagrange expansion formula [9, p. 125] the equation

\[ x = \frac{z}{\Phi(z)} \quad (\Phi(0) = 1), \]

where \( \Phi(z) \) is analytic in a neighborhood of \( z = 0 \), implies

\[ f(z) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n} \left[ \frac{d^{n-1}}{dz^{n-1}} f'(z) \Phi^n(z) \right]_{z=0}. \]

Since

\[ R_m(x) = R_m^1(x) = z^m, \]

we can determine \( R_m(x) \) by letting \( f(z) = z^m \) in (3.7). Thus we have

\[
R_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \left[ \frac{d^{n-1}}{dz^{n-1}} (mz^{n-1}) \sum_{k=0}^{\infty} t(n, k)z^k \right]_{z=0} = \sum_{n=1}^{\infty} \frac{x^n m}{n} t(n, n - m),
\]

and

\[ \sum_{n=m+1}^{\infty} r(n, n - m + 1)x^n = \sum_{n=1}^{\infty} x^n m \frac{t(n, n - m)}{n}. \]

We obtain (3.6) by equating coefficients of \( x^n \) in (3.8).

To state the corollary we must introduce two more functions. Let \( s(n, k) \) be a function from \( \mathbb{N}^* \times \mathbb{N}^* \) into \( F \) and define \( S_n(x) \) by

\[ S_n(x) = \sum_{k=1}^{\infty} s(n + k - 1, k)x^{n+k-1}, \quad n \geq 1. \]

**Corollary 1.** If \( r(n, k), s(n, k) \) and \( \phi(n) \) satisfy (3.1)–(3.5),

\begin{align*}
(3.9) & \quad s(1, 1) = 1, \\
(3.10) & \quad s(n, n) = s(n, n - 1), \quad n \geq 2,
\end{align*}

and

\( s(n + k, k) = \sum_{m=1}^{k} r(m, m)s(n + k - m, k - m + 1), \quad n \geq 1, \)  \( (3.11) \)

then

\( s(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n - j) t(n, j). \)  \( (3.12) \)

**Proof.** Using (3.9)–(3.11), we find that

\[
S_1(x) = x + \sum_{k=1}^{\infty} x^{k+1} \sum_{m=1}^{k} r(m, m) s(k - m + 1, k - m + 1)
= x + R_1(x) S_1(x),
\]

and from (3.11) we get

\[
S_{n+1}(x) = \sum_{k=1}^{\infty} x^{n+k} \sum_{m=1}^{k} r(m, m) s(n + k - m, k - m + 1)
= S_n(x) R_1(x), \quad n \geq 1.
\]

Thus

\[
S_{n+1}(x) = S_1(x) R_1^n(x)
= \frac{x R_1^n(x)}{1 - R_1(x)}
\]
or

\[
\frac{S_{m+1}(x)}{x} = \frac{z^m}{1 - z},
\]

where \( z = R_1(x) \). Again making use of (3.7), this time with

\[
f(z) = \frac{z^m}{1 - z},
\]

we see that

\[
x^{-1} S_{m+1}(x) = \delta_{m,0} + \sum_{n=1}^{\infty} \frac{x^n}{n!} \left[ \frac{d^{n-1}}{dz^{n-1}} \left[ \frac{mz^{m-1}}{1 - z} + \frac{z^m}{(1 - z)^2} \right] \sum_{k=0}^{\infty} t(n, k)z^k \right]_{z=0}.
\]
Hence we have
\[ \sum_{n=m-1}^{\infty} s(n+1, n-m+1)x^n = \delta_{m,0} + \sum_{n=1}^{\infty} x^n \frac{1}{n} \sum_{j=0}^{n-m} (m+j) t(n, n-m-j) \]
and (3.12) follows.

4. The functions $F_p$ and $G_p$. Throughout the rest of this paper we assume condition (1.8) holds. With this assumption we can use the results of the two previous sections to determine $F_p$, $G_p$ and $H_p$. In this section we consider $F_p$ and $G_p$ and in §5 we find $H_p$.

**Theorem 2.** If $\prod_{i=1}^{q_i} q_i = 1$, then

\[ G_p(n, k + 1) = \frac{n-k}{n} b_p(n, k) \]

and

\[ F_p(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n-j) b_p(n, j), \]

where $b_p(n, k ; q_1, \cdots, q_p) = b_p(n, k)$ is defined by
\[
\left[ \sum_{k=0}^{n} \left[ \prod_{i=1}^{p} [k+1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i} \right] x^k \right]^n = \sum_{k=0}^{\infty} b_p(n, k)x^k.
\]

**Proof.** This theorem is an immediate consequence of Theorem 1 and Corollary 1, for (1.8), (2.3), (2.5), (2.6) and the definitions of $F_p$ and $G_p$ imply that the hypotheses of these results are satisfied if $r(n, k) = G_p(n, k)$, $s(n, k) = F_p(n, k)$ and

\[ \phi(k) = \prod_{i=1}^{p} [k+1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i}. \]

Instead of Corollary 1 we can use (2.2) to obtain (4.2). We remark that by virtue of (1.8) it is possible to reduce
\[ \prod_{i=1}^{p} [k+1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i}, \]
and thus $b_p(n, k)$, to a function of $p - 1$ $q_i$’s.
Now
\[ \sum_{k=0}^{\infty} \left[ \prod_{i=1}^{p} [k + 1]_{q_i} - \prod_{i=1}^{p} [k]_{q_i} \right] x^k = x^{-1} H_p(x^{-1} | q_1, q_2, \ldots, q_p), \]

where $H_p(x | q_1, q_2, \ldots, q_p) = H_p(x)$ is the generalized Eulerian function defined and studied by Roselle [12]. Because $H_p(x)$ is quite complicated for $p \geq 3$, in general it is not feasible to find a simple formula for $b_p(n, k)$ in terms of more familiar coefficients. However, we can find $b_2(n, k)$ without difficulty.

**Corollary 2.** Let $p = 2$. If $q_1 q_2 = 1$, then
\[ G_2(n, k + 1) = \frac{n - k}{n} b_2(n, k) \]

and
\[ F_2(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n - j) b_2(n, j), \]

where

\[ b_2(n, k) = \sum_{m=0}^{k} \binom{n}{k - m} \sum_{j=0}^{m} \binom{n + j - 1}{j} \binom{n + m - j - 1}{n - 1} q_1^{n - 2j}. \]  

**Proof.** Since
\[ H_2(x) = \frac{x + q_1 q_2}{(x - q_1)(x - q_2)} \]

and $q_1 q_2 = 1$, we find that
\[ x^{-1} H_2(x^{-1}) = \frac{1 + x}{(1 - q_1 x)(1 - q_1^{-1} x)}. \]

Thus
\[ (x^{-1} H_2(x^{-1}))^n = \sum_{m=0}^{n} \binom{n}{m} x^m \sum_{k=0}^{\infty} \binom{n + k - 1}{k} q_1^k x^k \sum_{j=0}^{\infty} \binom{n + j - 1}{j} q_1^{-j} x^j \]
\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{n}{k - m} \sum_{j=0}^{m} \binom{n + j - 1}{j} \binom{n + m - j - 1}{m - j} q_1^{-2j} x^k \]

and (4.3) follows.
Another special case of interest is that in which \( q_1 = q_2 = \cdots = q_p = q \). In this case condition (1.8) becomes \( q^p = 1 \) and the possible values for \( q \) are the \( p \)-th roots of unity. Let \( H_p(x \mid q_1, q_2, \cdots, q_p) = H_p(x \mid q) \) and \( b_p(n, k ; q_1, q_2, \cdots, q_p) = b_p(n, k ; q) \) when \( q_1 = \cdots = q_p = q \). We also require the notation

\[
[x] = [x]_q = \frac{q^x - 1}{q - 1}
\]

and

\[
\left[ \begin{array}{c} x \\ m \end{array} \right] = \left[ \begin{array}{c} x \\ m \end{array} \right]_q = \prod_{j=1}^{m} \frac{q^{x-j+1} - 1}{q^j - 1}.
\]

Carlitz [2] has proved that

\[
(4.4) \quad H_p(x \mid q) = \sum_{j=1}^{p} A_{p,j}(q) x^{j-1}/\prod_{n=1}^{p} (x - q^n),
\]

where \( A_{p,j}(q) \) are the \( q \)-Eulerian numbers defined by

\[
[x]^m = \sum_{s=1}^{m} A_{m,s}(q) \left[ \begin{array}{c} x + s - 1 \\ m \end{array} \right], \quad m \geq 1.
\]

Now (4.4) implies

\[
x^{-1} H_p(x^{-1} \mid q) = \sum_{j=1}^{p} A_{p,j}(q) x^{j-1}/\prod_{n=1}^{p} (1 - q^n x).
\]

If we define \( a_p(n, k ; q) \) by

\[
(4.5) \quad \left[ \sum_{j=1}^{p} A_{p,j}(q) x^{j-1} \right]^n = \sum_{k=0}^{n} a_p(n, k ; q) x^k,
\]

then

\[
(x^{-1} H_p(x^{-1} \mid q))^n = \sum_{k=0}^{n} a_p(n, k ; q) x^k \sum_{m=0}^{n} \left[ \begin{array}{c} np + m - 1 \\ m \end{array} \right] (qx)^m
\]

\[
= \sum_{k=0}^{n} \sum_{m=0}^{k} a_p(n, k - m ; q) \left[ \begin{array}{c} np + m - 1 \\ m \end{array} \right] q^m x^k.
\]
Thus we have proved

**Corollary 3.** If \( q_1 = q_2 = \cdots = q_p = q \) and \( q^p = 1 \), then

\[
G_p(n + 1, k + 1) = \frac{n - k}{n} b_p(n, k; q)
\]

and

\[
F_p(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n - j) b_p(n, k; q),
\]

where

\[
b_p(n, k; q) = \sum_{m=0}^{k} a_p(n, k - m; q) \binom{np + m - 1}{m} q^m
\]

and \( a_p(n, k; q) \) is defined by (4.5).

The following result follows immediately from Corollaries 2 and 3.

**Corollary 4.** Let \( p = 2 \). If \( q_1 = q_2 = q \) and \( q^2 = 1 \), then

\[
G_2(n, k + 1) = \frac{n - k}{n} b_2(n, k; q)
\]

and

\[
F_2(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n - j) b_2(n, j; q),
\]

where

\[
b_2(n, k; q) = \sum_{m=0}^{k} \binom{n}{k - m} \binom{2n + m - 1}{m} q^m
\]

or

\[
b_2(n, k; q) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{n}{k - 2j} \binom{2n + 2j - 1}{2j}
\]

\[
+ \sum_{j=0}^{\lfloor (k-1)/2 \rfloor} \binom{n}{k - 2j - 1} \binom{2n + 2j}{2j + 1} q
\]
or

\[ b_2(n, k; q) = \sum_{m=0}^{k} \binom{n}{k-m} \binom{2n+m-1}{m} q^{n-k}. \]

**Proof.** We deduce (4.7) from Corollary 2 by using the hypothesis \( q^2 = 1 \) and a binomial identity found in Riordan [10, p. 9]. Because \( q^2 = 1 \), (4.8) follows from (4.7). We get (4.9) from Corollary 3 by observing that

\[ a_2(n, k; q) = \binom{n}{k} q^{n-k}. \]

If \( q = 1 \) in Corollary 4 we have

**Corollary 5.** Let \( p = 2 \). If \( q_1 = q_2 = 1 \), then

\[ G_2(n, k + 1) = \frac{n-k}{n} \sum_{m=0}^{k} \binom{n}{k-m} \binom{2n+m-1}{m}. \]

and

\[ F_2(n + 1, k + 1) = \frac{1}{n} \sum_{j=0}^{k} (n-j) \sum_{m=0}^{j} \binom{n}{j-m} \binom{2n+m-1}{m}. \]

We note that (4.10) is precisely Carlitz's formula for \( g(n, k + 1) \), while (4.11) is equivalent to his formula for \( f(n + 1, k + 1) \).

If \( p = 1 \), condition (1.8) implies \( q_1 = 1 \). In this case we get

**Corollary 6.** Let \( p = 1 \). If \( q_1 = 1 \), then

\[ F_1(n, k + 1) = G_1(n, k + 1) = \frac{n-k}{n} \binom{n+k-1}{k}. \]

**Proof.** Letting \( p = 1 \) and \( q = 1 \) in Corollary 3 we obtain

\[ G_1(n, k + 1) = \frac{n-k}{n} \sum_{m=0}^{k} a_1(n, k-m; 1) \binom{n+m-1}{m}. \]

Since

\[ a_1(n, k; 1) = \delta_{k,0}, \]
(4.13) reduces to

\[ G_i(n, k + 1) = \frac{n - k}{n} \binom{n + k - 1}{k}. \]

Then (4.12) follows because \( F_i(n, k + 1) = G_i(n, k + 1) \) by definition. Formula (4.12) is the result given by Bertrand [1] as the number of two-element lattice permutations.

5. The function \( H_p \). By virtue of (1.8), (2.8) and (2.9), the hypotheses of Theorem 1 are satisfied if \( r(n, k) = H_p(n, k) \) and

\[ \phi(n) = q_1^p \theta_p(n; q_2, q_3, \ldots, q_p), \]

where \( \theta_p(n; q_2, q_3, \ldots, q_p) \), or more briefly \( \theta_p(n) \), is defined by (2.7) for \( p \geq 2 \) and by \( \theta_1(n) = 1 \) for \( p = 1 \). Thus we can express \( H_p(n, k) \) in terms of the coefficients \( c_p(n, k) = c_p(n, k; q_1, q_2, \ldots, q_p) \) defined by

\[ \Theta_p^*(x) = \sum_{k=0}^{\infty} c_p(n, k)x^k, \]

where

\[ \Theta_p(x) = \Theta_p(x | q_1, q_2, \ldots, q_p) = \sum_{k=0}^{\infty} \theta_p(k)(q_1x)^k. \]

In fact we have

**Theorem 3.** If \( \Pi_{i=1}^{p} q_i = 1 \), then

\[ H_p(n, k + 1) = \frac{n - k}{n} c_p(n, k), \]

where \( c_p(n, k) \) is defined by (5.1).

(In view of condition (1.8) it is possible to express \( \Theta_p(x) \) and \( c_p(n, k) \) as a function of \( p - 1 \) \( q_i \)'s.)

Since the coefficients \( c_p(n, k) \) are so closely related to \( \theta_p(k) \) and \( \Theta_p(x) \), we reduce \( \theta_p(k) \) from a \((p - 1)\)-tuple summation to a single sum. Using this simplification of \( \theta_p(k) \), we can write \( \Theta_p(x) \) as a single, finite sum.
Theorem 4. If \( p \geq 2 \), then

\[
\theta_p(k) = \frac{1}{\gamma_i(q_2, \ldots, q_p)} \sum_{j=2}^{p} \frac{(-1)^{j+1} q_2 q_3^2 \cdots q_j^{-1}(q_3 q_4 \cdots q_i)^k}{\gamma_j(q_2, \ldots, q_p)},
\]

where

\[
\gamma_i(q_2, \ldots, q_p) = (1 - q_2)(1 - q_2 q_3) \cdots (1 - q_2 \ldots q_p),
\]

and

\[
\gamma_j(q_2, \ldots, q_p) = (1 - q_2 \cdots q_i)(1 - q_3 \cdots q_i) \cdots (1 - q_j)(1 - q_{j+1}) \cdots (1 - q_{j+1} \cdots q_p), \quad j \geq 2,
\]

with the understanding that

\[
(1 - q_i \cdots q_i) = 1 \quad \text{if} \quad i > j.
\]

Proof. We use induction on \( p \). From definition (2.7) we get

\[
\theta_2(k) = \sum_{m=0}^{k} q_2^m.
\]

Thus

\[
\theta_2(k) = \frac{1 - q_2^{k+1}}{1 - q_2}
\]

and we have verified (5.3) for \( p = 2 \). Using first the definition of \( \theta_p+1(k) \) and then (5.3) (as the induction hypothesis), we find that

\[
\begin{align*}
\theta_{p+1}(k; q_2, \ldots, q_{p+1}) &= \sum_{a_2=0}^{k} q_2^{a_2} \sum_{a_3=0}^{a_2} q_3^{a_3} \cdots \sum_{a_{p+1}=0}^{a_p} q_{p+1}^{a_{p+1}} \\
&= \sum_{a_2=0}^{k} q_2^{a_2} \theta_p(a_2; q_3, \ldots, q_{p+1}) \\
&= \sum_{m=0}^{k} q_2^m \left[ \frac{1}{\gamma_2(q_3, \ldots, q_{p+1})} \sum_{j=2}^{p} \frac{(-1)^{j+1} q_3 q_4^2 \cdots q_j^{-1}(q_3 q_4 \cdots q_i)^k}{\gamma_j(q_3, \ldots, q_{p+1})} \right]
\end{align*}
\]
To complete the induction it suffices to prove

**Lemma 1.** We have

\[
\frac{1}{(1-q_1)(1-q_2)(1-q_3)\cdots(1-q_p)} + \sum_{j=2}^{p+1} \frac{(-1)^{j-1}q_2^j \cdots q_{j-2}^j(1-(q_2 \cdots q_j)^{k+1})}{\gamma_j(q_2, \cdots, q_{p+1})} = \frac{1}{(1-q_1)(1-q_1q_2)\cdots(1-q_1 \cdots q_p)},
\]

(5.6)

where \( \gamma_i \) is defined by (5.4) and (5.5).

**Proof.** This proof was suggested by Carlitz. In the expression

\[
\frac{(1-q_1q_2)\cdots(1-q_1 \cdots q_p)}{(1-q_2)(1-q_3)\cdots(1-q_2 \cdots q_p)} + \sum_{j=2}^{p} \frac{(-1)^{j+1}q_2^j \cdots q_{j-1}^j(1-q_j) \cdots (1-q_1 \cdots q_p)}{\gamma_j(q_1, \cdots, q_p)}
\]

(5.7)

fix \( q_2, \cdots, q_p \). Then (5.7) is a polynomial in \( q_1 \) of degree \( p - 1 \). Since this polynomial is 1 for \( p \) values of \( q_1 \), namely

\[ q_1 = 1, \frac{1}{q_2}, \cdots, \frac{1}{q_2q_3 \cdots q_p}, \]

it is identically 1 and (5.6) follows. Using the definition of \( \Theta_p(x) \) and (5.3), we find that

\[
\Theta_p(x) = \frac{1}{(1-q_1x)\gamma_1(q_2, \cdots, q_p)} + \sum_{j=2}^{p} \frac{(-1)^{j+1}q_2^j \cdots q_{j-1}^j}{(1-q_1 \cdots q_x)\gamma_j(q_2, \cdots, q_p)}.
\]

In general \( \Theta_p(x) \) is quite complicated and it is not feasible to determine the coefficients \( c_p(n, k) \) explicitly. However, for \( p = 2 \) and \( q_1q_2 = 1 \), we have
$$\Theta_2(x) = \frac{1}{(1-q_1 x)(1-q_2)} - \frac{q_2}{(1-x)(1-q_2)}$$
$$= \frac{1}{(1-q_1 x)(1-x)} .$$

It follows that

$$\theta_2^p(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q^n x^k$$

and

$$c_2(n, k) = \sum_{m=0}^{k} \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q^n .$$

Hence, as a corollary of Theorem 3, we have

**Corollary 7.** Let \( p = 2 \). If \( q_1 q_2 = 1 \), then

$$H_2(n, k+1) = \frac{n-k}{n} \sum_{m=0}^{k} \binom{n+k-m-1}{k-m} \binom{n+m-1}{m} q^n .$$

If, in addition to (1.8), we assume \( q_1 = q_2 = \cdots = q_p = q \), \( \Theta_p(x) \) is considerably simpler. In this case let \( \Theta_p(x) = \Theta_p(x \mid q) \), \( \theta_p(k) = \theta_p(k \mid q) \) and \( c_p(n, k) = c_p(n, k \mid q) \). From the definition of \( \theta_p(k \mid q) \) it follows that

$$\theta_p(k \mid q) = \sum q^n ,$$

where the summation is over all \((p-1)\)-tuples \((a_2, \cdots, a_p)\) such that \( 1 \leq a_p \leq \cdots \leq a_2 \leq k+1 \), and

$$m + p - 1 = \sum_{i=2}^{p} a_i .$$

Hence it is evident that \( \theta_p(k \mid q) \) generates the number of partitions of \( m + p - 1 \) into \( p-1 \) parts with each part at most \( k+1 \). It is well-known (see, for example, [8, p. 5]) that such a function is

$$\left[ \begin{array}{c} k + p - 1 \\ k \end{array} \right] .$$

Thus
\[ \Theta_p(x | q) = \sum_{k=0}^{\infty} \left[ \frac{k + p - 1}{k} \right] (qx)^k \]

Since \( q^p = 1 \),

\[ \Theta_p(x | q) = \prod_{k=0}^{p-1} (1 - q^k x)^{-1}, \]

and

\[ \Theta_p^n(x | q) = \prod_{k=0}^{np-1} (1 - q^k x)^{-n} \]

\[ = \prod_{k=0}^{np-1} (1 - q^k x)^{-1} \]

\[ = \sum_{k=0}^{\infty} \left[ \frac{np + k - 1}{k} \right] x^k. \]

Therefore

\[ c_p(n, k ; q) = \left[ \frac{np + k - 1}{k} \right], \]

and we have proved

**Corollary 8.** If \( q_1 = q_2 = \cdots = q_p = q \) and \( q^p = 1 \), then

\[ H_p(n, k + 1) = \frac{n - k}{n} \left[ np + k - 1 \right]. \]

We observe that if \( q \) is a primitive \( p \)th root of unity, say \( \xi \), (5.9) reduces to

\[ \Theta_p(x | \xi) = (1 - x^p)^{-1}. \]

Then

\[ \Theta_p^n(x | \xi) = \sum_{k=0}^{\infty} \left( \frac{n + k - 1}{k} \right) x^{pk}. \]
From (5.11) it follows that

\[
(5.12) \quad c_p(n, k; \xi) = \begin{cases} 
\binom{n+k/p - 1}{n-1} & \text{if } p/k \\
0 & \text{otherwise}
\end{cases}
\]

As an immediate consequence of Corollary 8 we have

**Corollary 9.** If \( q_i = 1, 1 \leq i \leq p \), then

\[
(5.13) \quad H_p(n, k + 1) = \frac{n-k}{n} \binom{np + k - 1}{k}.
\]

We can also obtain (5.13) by viewing \( \theta_p(k; 1) \) as the number of \((p-1)\)-combinations with repetition of \( k+1 \) distinct objects. Then we know that (see Riordan [11, p. 7])

\[
(5.14) \quad \theta_p(k; 1) = \binom{k+p-1}{p-1}.
\]

From (5.14) we can deduce (5.13).

If \( p = 2, q_1 = q_2 = q \) and \( q^2 = 1 \), the formulas for \( H_2(n, k + 1) \), given in the following corollary, are particularly simple.

**Corollary 10.** Let \( p = 2 \). If \( q_1 = q_2 = 1 \), then

\[
(5.15) \quad H_2(n, k + 1) = \frac{n-k}{n} \binom{2n + k - 1}{k};
\]

if \( q_1 = q_2 = -1 \),

\[
(5.16) \quad H_2(n, k + 1) = \begin{cases} 
\frac{n-k}{n} \binom{n+k/2 - 1}{n-1} & \text{if } k \text{ is even} \\
0 & \text{if } k \text{ is odd}
\end{cases}
\]

**Proof.** From either (5.8) or (5.13) we get (5.15), while (5.16) follows from (5.12). Carlitz’s result for \( h(n, k + 1) \) coincides with (5.15).

Since (5.13) is valid for \( p = 1 \), we have the expected result

**Corollary 11.** Let \( p = 1 \). If \( q_1 = 1 \), then

\[
(5.17) \quad H_1(n, k + 1) = \frac{n-k}{n} \binom{n+k - 1}{k}.
\]
6. **Partitions.** From the definition of $g_p(n, k; s_1, s_2, \ldots, s_p)$ it is clear that this enumerant is the number of partitions of the $p$-partite $(s_1, s_2, \ldots, s_p)$ of the form

$$\sum_{j=1}^{n} a_{ij} = s_i, \quad 1 \leq i \leq p,$$

where the $a_{ij}$ are positive integers subject to conditions (1.2), (1.4) and (1.6). Thus $G_p(n, k)$ generates these partitions. If we replace (1.6) by (1.5), these statements are true for $f_p(n, k)$ and $F_p(n, k)$. Adding (1.7) to the conditions for $g_p(n, k)$, we have a partition interpretation for $h_p(n, k)$ and $H_p(n, k)$.

Since we have only obtained $F_p, G_p$ and $H_p$ under the assumption (1.8), we describe how this restriction affects the partitions. If $\prod_{i=1}^{p} q_i = 1$ and $p \geq 2$,

$$G_p(n, k) = \sum_{*} g_p(n, k; s_1, s_2, \ldots, s_p) q_1^{-s_1} q_2^{-s_2} \cdots q_p^{-s_p},$$

where $\sum_{*}$ is as defined in §1. Thus we have

**Theorem 5.** If $\prod_{i=1}^{p} q_i = 1$, the function $G_p(n, k)$ generates the number of partitions of the $(p - 1)$-partite $(m_1, m_2, \ldots, m_{p-1})$ of the form

$$m_i = \sum_{j=1}^{n} a_{ij} - \sum_{j=1}^{n} a_{pj},$$

where the $a_{ij}$ are positive integers satisfying (1.2), (1.4) and (1.6).

We have corresponding interpretations for $F_p$ and $H_p$.

For example, by Corollaries 2 and 7 we have

$$G_{2}(n, k + 1) = \frac{n - k}{n} \sum_{m = -k}^{k} \sum_{j = \max(0, -m)}^{[(k - m)/2]} \binom{n}{k - m - 2j} \binom{n + j - 1}{j} \binom{n + m + j - 1}{m + j} q^m,$$

$$F_{2}(n + 1, k + 1) = \frac{1}{n} \sum_{m = -k}^{k} \sum_{j = \max(m, -m)}^{[(j - m)/2]} (n - j) \sum_{s = \max(0, -m)}^{[(j - m)/2]} \binom{n}{j - m - 2s} \binom{n + s - 1}{s} \binom{n + m + s - 1}{m + s} q^m.$$
and

\[ H_2(n, k + 1) = \frac{n - k}{n} \sum_{m=0}^{k} \binom{n + k - m - 1}{m} q^{m}. \]

Thus

(6.1) \( G_2(3, 2) = 2q_1^{-1} + 2 + 2q_1 \),

(6.2) \( F_2(3, 2) = q_1^{-1} + 2 + q_1 \)

and

(6.3) \( H_2(3, 2) = 2 + 2q_1 \).

The following arrays are enumerated by \( G_2(3, 2) \):

1

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<tbody>
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<td>1 1 2</td>
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(2)

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<td>1 1 1</td>
</tr>
</tbody>
</table>

Now arrays (1) and (2) satisfy

\[-1 = \sum_{j=1}^{3} a_{1j} - \sum_{j=1}^{3} a_{2j};\]

for (3) and (4) we have

\[0 = \sum_{j=1}^{3} a_{1j} - \sum_{j=1}^{3} a_{2j},\]

while (5) and (6) are subject to the condition

\[1 = \sum_{j=1}^{3} a_{1j} - \sum_{j=1}^{3} a_{2j}.\]

The arrays counted by \( F_2(3, 2) \) are (2) – (5). Array (2) accounts for the coefficient of \( q_1^{-1} \), (3) and (4) are enumerated by the constant term, and (5) is counted by the coefficient of \( q_1 \). Finally, \( H_2(3, 2) \) counts arrays (3) – (6). The first two of these are enumerated by the constant term; (5) and (6) account for the coefficient of \( q_1 \).
If we assume \( q_1 = \cdots = q_p = q \),

\[
G_p(n, k) = \sum_{s_1, \ldots, s_p} g_p(n, k; s_1, \ldots, s_p)q^{\sum_{i=1}^{p} s_i}.
\]

Then condition (1.8) implies \( q^p = 1 \) and we have

\[
G_p(n, k) = \sum_{m=0}^{p-1} \sum_{s_1, \ldots, s_p} g_p(n, k; s_1, \ldots, s_p)q^m
\]

where the inner sum is over all \( p \)-tuples satisfying

\[
\sum_{i=1}^{p} s_i \equiv m \pmod{p}.
\]

Therefore, we have

\[
\text{THEOREM 6. If } q_1 = q_2 = \cdots = q_p = q, \text{ and } G_{p,n,k}(r) \text{ is the number of partitions of the form}
\]

\[
r = \sum_{i=1}^{p} s_i = \sum_{i=1}^{p} \sum_{j=1}^{n} a_{ij},
\]

where the \( a_{ij} \) are positive integers subject to conditions (1.2), (1.3), (1.4) and (1.6), then \( G_p(n, k) \) generates

\[
\sum_{t} \Pi_{p,n,k+1}(m + tp), \quad 0 \leq m \leq p - 1.
\]

For example, from Corollary 4 we see that

\[
\sum_{t} \Pi_{2,n,k+1}(2t) = \frac{n-k}{n} \sum_{j=0}^{[k/2]} \binom{n}{k-2j} \binom{2n+2j-1}{2j}
\]

and

\[
\sum_{t} \Pi_{2,n,k+1}(2t + 1) = \frac{n-k}{n} \sum_{j=0}^{[(k-1)/2]} \binom{n}{k-2j-1} \binom{2n+2j}{2j+1}.
\]

We have corresponding results for \( F_p \) and \( H_p \).

Returning to the illustration used above, if \( q_1 = q_2 = q \), we may write (6.1), (6.2) and (6.3) as

\[
G_2(3, 2) = 2 + 4q,
\]

\[
G_2(3, 2) = 2 + 4q,
\]

\[
G_2(3, 2) = 2 + 4q,
\]
(6.5) \[ F_2(3, 2) = 2 + 2q, \]
and
(6.6) \[ H_2(3, 2) = 2 + 2q. \]

Since the sums of arrays (3) and (4) are even, these arrays are counted by the constant term in (6.4)–(6.6). In each case the coefficient of \( q \) enumerates the arrays having an odd sum.

REFERENCES


Received December 11, 1973.

DUKE UNIVERSITY
D. E. Bennett, *Strongly unicoherent continua* .............................................. 1

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