

# Pacific Journal of Mathematics

**A REPRESENTATION THEOREM FOR ISOMETRIES OF  
 $C(X, E)$**

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## A REPRESENTATION THEOREM FOR ISOMETRIES OF $C(X, E)$

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Let  $X, Y$  be compact Hausdorff spaces and let  $E, F$  be Banach spaces such that their duals are strictly convex. We show that a linear map  $T: C(X, E) \rightarrow C(Y, F)$  is an isometric isomorphism if and only if there exists a homeomorphism  $\phi: Y \rightarrow X$  and a continuous map  $\lambda$  from  $Y$  to the set of isometric isomorphisms from  $E$  onto  $F$  (with the strong topology) such that  $Tf(y) = \lambda(y) \cdot f(\phi(y))$  for all  $y \in Y$ ,  $f \in C(X, E)$ .

1. Suppose  $E$  is a Banach space and  $X$  is a compact Hausdorff space, we use  $C(X, E)$  to denote the Banach space of continuous functions from  $X$  into  $E$ . In [3], Jerison gave a generalization of the Banach-Stone theorem, he showed that if  $X, Y$  are compact Hausdorff spaces,  $E$  is a strictly convex space and  $T: C(X, E) \rightarrow C(Y, E)$  is an isometric isomorphism, then there exists a homeomorphism  $\phi: Y \rightarrow X$ , a continuous map  $\lambda$  from  $Y$  into the set of rotations of  $E$  (i.e. the set of isometric isomorphisms from  $E$  onto  $E$ ) under the strong topology such that for each  $f \in C(X, E)$ ,  $y \in Y$ , we have

$$Tf(y) = \lambda(y) \cdot f(\phi(y)).$$

Makai [5] and Sundaresan [6] made some improvements of the result. In this paper, we will consider the isometric isomorphisms between  $C(X, E)$  and  $C(Y, F)$  where  $E^*, F^*$  are strictly convex spaces. Let  $E, F$  be Banach spaces, we use  $S(E)$  to denote the unit ball of  $E$ ,  $\partial S(E)$  the set of extreme points of  $S(E)$ ,  $L(E, F)$  the set of bounded linear operators from  $E$  into  $F$  and  $I(E, F)$  the set of isometric isomorphisms from  $E$  into  $F$ . We will show

**THEOREM.** *Suppose  $X, Y$  are compact Hausdorff spaces and  $E, F$  are Banach spaces with  $E^*, F^*$  strictly convex. Let*

$$T: C(X, E) \rightarrow C(Y, F)$$

*be an isometric isomorphism; then there exist a homeomorphism  $\phi: Y \rightarrow X$  and a continuous map  $\lambda: Y \rightarrow I(E, F)$  (with the strong topology) such that*

$$(*) \quad Tf(y) = \lambda(y) \cdot f(\phi(y)) \quad \text{for all } y \in Y, f \in C(X, E).$$

Conversely, if we are given  $\phi$  and  $\lambda$  as above, then there exists an isometric isomorphism  $T$  from  $C(X, E)$  onto  $C(Y, F)$  satisfies (\*).

We remark that the theorem will not be true for arbitrary Banach spaces (c.f. §3).

2. We will begin by showing the converse part of the theorem. The map  $T$  defined by (\*) is obviously linear and continuous. For  $g \in C(Y, F)$ , define  $\tau: X \rightarrow I(F, E)$  by  $\tau(x) = (\lambda(\phi^{-1}(x)))^{-1}$  and let  $f \in C(X, E)$  be defined by  $f(x) = \tau(x) \cdot g(\phi^{-1}(x))$  for all  $x \in X$ . Then  $Tf = g$  and  $T$  is onto. To show that  $T$  is an isometry, take any  $f \in C(X, E)$ , then

$$\begin{aligned} \|Tf\| &= \sup\{\|Tf(y)\|: y \in Y\} \\ &= \sup\{\|\lambda(y) \cdot f(\phi(y))\|: y \in Y\} \\ &= \sup\{\|f(\phi(y))\|: y \in Y\} \\ &= \sup\{\|f(x)\|: x \in X\} \\ &= \|f\|. \end{aligned}$$

The proof of the first part is divided into the subsequent lemmas.

LEMMA 1. *Let  $X$  be a compact Hausdorff space and let  $E$  be a Banach space; then the set of extreme points of  $S(C(X, E)^*)$  is of the form  $\delta_{x,u}$  where  $x \in X$ ,  $u \in \partial S(E^*)$ , and*

$$\delta_{x,u}(f) = u(f(x)), f \in C(X, E)$$

*Proof.* C.f. [4], Theorem 3.2.

Under the assumption of the Theorem, the adjoint map  $T^*: C(Y, F)^* \rightarrow C(X, E)^*$  is also an isometric isomorphism. It sends the extreme points of  $S(C(Y, F)^*)$  onto the set of extreme points of  $S(C(X, E)^*)$ , i.e., for  $y \in Y$ ,  $v \in \partial S(F^*)$ ,  $T^*(\delta_{y,v})$  is of the form  $\delta_{x,u}$ , where  $x \in X$  and  $u \in \partial S(E^*)$ .

LEMMA 2. (i) *For any  $y \in Y$ ,  $v \in F^*$ ,  $T^*(\delta_{y,v})$  is of the form  $\delta_{x,u}$  where  $x \in X$ ,  $u \in E^*$ .*

(ii) *Let  $y \in Y$ ,  $v, \bar{v} \in F^*$  and let  $T^*(\delta_{y,v}) = \delta_{x,u}$ ,  $T^*(\delta_{y,\bar{v}}) = \delta_{\bar{x},\bar{u}}$ ; then  $x = \bar{x}$ .*

(iii) For each fixed  $y \in Y$ , the map  $v \rightarrow u$ ,  $F^* \rightarrow E^*$  where  $T^*(\delta_{y,v}) = \delta_{x,u}$  is an isometric isomorphism. Moreover, this map is weak\* continuous.

*Proof.* Since  $F^*$  is strictly convex, every point of norm 1 in  $F^*$  is an extreme point of  $S(F^*)$ . By the preceding remark, (i) holds for all points of norm 1. Note also that  $\alpha\delta_{y,v} = \delta_{y,\alpha v}$  for all  $\alpha \in \mathbb{R}$ , so (i) is true for all  $v \in F^*$ . To prove (ii), suppose  $x \neq \bar{x}$  and consider  $T^*(\delta_{y,v+\bar{v}})$ ; by (i), it is of the form  $\delta_{x',u'}$  for some  $u' \in E^*$ ,  $x' \in X$  and

$$\delta_{x',u'} = \delta_{x,u} + \delta_{\bar{x},\bar{u}}.$$

Note that  $x' \neq x, \bar{x}$ . Indeed, if  $x' = x$  (or  $\bar{x}$ ), then we can choose  $f \in C(X, E)$ ,  $z \in E$  such that  $f(\bar{x}) = z$ ,  $\bar{u}(z) \neq 0$ , but  $f(x) = 0$ , then

$$\delta_{x',u'}(f) \neq \delta_{x,u}(f) + \delta_{\bar{x},\bar{u}}(f).$$

Since  $x' \neq x, \bar{x}$ , by a similar kind of argument, it is easily shown that there exists a  $g \in C(X, E)$  such that

$$\delta_{x',u'}(g) \neq \delta_{x,u}(g) + \delta_{\bar{x},\bar{u}}(g).$$

a contradiction. In (iii), it follows from (i), (ii) that the map is well defined and linear. To show that it is onto, we note that if  $T^*(\delta_{y_1,v_1}) = \delta_{x,u_1}$ ,  $T^*(\delta_{y_2,v_2}) = \delta_{x,u_2}$ , then  $y_1 = y_2$  (for we need only consider  $(T^*)^{-1}$  as in (ii)). For  $u_1 \in E^*$ , consider  $\delta_{x,u_1}$  where  $x \in X$  is such that  $T^*(\delta_{y,v}) = \delta_{x,u}$ ,  $v \in F^*$  (by (ii), the point  $x$  is well defined). Since  $T^*$  is onto, there exists  $\delta_{y_1,v_1} \in C(Y, F)^*$  such that  $T^*(\delta_{y_1,v_1}) = \delta_{x,u_1}$ . By the above remark,  $y_1 = y$  and hence  $T^*(\delta_{y,v_1}) = \delta_{x,u_1}$  and  $v_1$  is the preimage of  $u_1$ . To show that the map is an isometry, we need only observe that for any  $v \in F^*$  such that  $\|v\| = 1$ , the point  $\delta_{y,v}$  is an extreme point of  $S(C(Y, F)^*)$ , hence  $\delta_{x,u} = T^*(\delta_{y,v})$  is an extreme point of  $S(C(X, E)^*)$  and  $\|u\| = 1$ . The last assertion of (iii) follows from the weak\* continuity of  $T^*$ .

From Lemma 2 (ii), we can define a map  $\phi: Y \rightarrow X$  such that  $\phi(y) = x$ . For each  $y \in Y$ , we let  $\lambda(y)^*: F^* \rightarrow E^*$  be the map in Lemma 2 (iii). Since  $\lambda(y)^*$  is weak\* continuous, it induces a map  $\lambda(y): E \rightarrow F$  which is also an isometric isomorphism. Hence we can define the map  $\lambda: Y \rightarrow I(E, F)$  with  $y \rightarrow \lambda(y)$ . For any  $v \in F^*$ ,  $y \in Y$  and  $f \in C(X, E)$ , we have

$$\begin{aligned} &v(Tf(y)) \\ &= \delta_{y,v}(Tf) = T^*(\delta_{y,v})f \\ &= (\delta_{\phi(y), \lambda(y)^*v})(f) = (\lambda(y)^*v)(f(\phi(y))) \\ &= v(\lambda(y) \cdot f(\phi(y))). \end{aligned}$$

Thus

$$Tf(y) = \lambda(y) \cdot f(\phi(y)).$$

It remains to show

LEMMA 3. *The map  $\phi$  is a homeomorphism.*

*Proof.* That  $\phi$  is onto follows from the fact  $T^*$  sends the set of elements of the form  $\delta_{y,v}$ ,  $y \in Y$ ,  $v \in F^*$  onto the set of elements of the form  $\delta_{x,u}$ ,  $x \in X$ ,  $u \in E^*$ . That  $\phi$  is one-to-one follows from the remark in the proof of the onto part in Lemma 2 (iii). It remains to show that  $\phi$  is continuous. ( $\phi^{-1}$  will then be continuous since  $X, Y$  are compact Hausdorff spaces). Let  $\{y_\alpha\}$  be a net in  $Y$  converging to  $y$ . Fix  $v \in F^*$  and let  $T^*(\delta_{y_\alpha, v}) = \delta_{x_\alpha, u_\alpha}$ ; then  $\{\delta_{x_\alpha, u_\alpha}\}$  converges weak\* to  $T^*(\delta_{y, v}) = \delta_{x, u}$ . We want to show that  $\{x_\alpha\}$  converges to  $x$ . Let  $\{x_\beta\}, \{u_\beta\}$  be subsets of  $\{x_\alpha\}, \{u_\alpha\}$  which converge weak\* to  $\bar{x}, \bar{u}$  respectively. For  $f$  in  $C(X, E)$ ,

$$\begin{aligned} & |\delta_{x,u}(f) - \delta_{\bar{x},\bar{u}}(f)| \\ & \leq |\delta_{x,u}(f) - \delta_{x_\beta, u_\beta}(f)| + |\delta_{x_\beta, u_\beta}(f) - \delta_{\bar{x}, \bar{u}}(f)| \\ & \quad + |\delta_{\bar{x}, \bar{u}}(f) - \delta_{\bar{x}, \bar{u}}(f)| \\ & \leq |\delta_{x,u}(f) - \delta_{x_\beta, u_\beta}(f)| + |u_\beta(f(x_\beta)) - u_\beta(f(\bar{x}))| \\ & \quad + |u_\beta(f(\bar{x})) - \bar{u}(f(\bar{x}))| \\ & \leq |\delta_{x,u}(f) - \delta_{x_\beta, u_\beta}(f)| + \|f(x_\beta) - f(\bar{x})\| \|v\| \\ & \quad + |u_\beta(f(\bar{x})) - \bar{u}(f(\bar{x}))|. \end{aligned}$$

The right side converges to zero as  $\{x_\beta\}$  and  $\{u_\beta\}$  converge to  $\bar{x}$  and  $\bar{u}$  respectively. This shows that  $x = \bar{x}$ . The net  $\{x_\alpha\}$  is in the compact set  $X$  and has only one limit point  $x$ , thus  $\{x_\alpha\}$  converges to  $x$ .

LEMMA 4. *The map  $\lambda: Y \rightarrow I(E, F)$  is continuous with respect to the strong topology on  $I(E, F)$ .*

*Proof.* Let  $\{y_\alpha\}$  be a net in  $Y$  converging to  $y_0$ . For each  $z$  in  $E$ , we can find an  $f$  such that  $f(x) = z$  for all  $x$  in  $X$ , thus

$$\|\lambda(y_\alpha)z - \lambda(y_0)z\| = \|Tf(y_\alpha) - Tf(y_0)\|.$$

Since  $Tf$  is in  $C(Y, F)$ , the right side converges to 0 as  $\{y_\alpha\}$  converges to  $y_0$ . This shows that  $\lambda$  is continuous.

3. We give an example which shows that the theorem is not true if we do not assume that  $E^*$ ,  $F^*$  are strictly convex. Let  $X$  be a compact Hausdorff space and let  $R^2$  be the two dimensional linear space with the maximum norm ( $\|(r, s)\| = \max\{|r|, |s|\}$ ,  $r, s \in R$ ). It is clear that  $C(X, R^2)$  is a Banach lattice with an order unit  $f_e$  where  $f_e(x) = (1, 1)$  for all  $x$  in  $X$ . Also the norm satisfies  $\|f \vee g\| = \|f\| \vee \|g\|$  for all  $f, g$  in the positive cone of  $C(X, R^2)$ . By Kakutani's representation theorem of abstract  $M$  spaces [2],  $C(X, R^2)$  is isometrically isomorphic to  $C(Y, R)$  for some compact Hausdorff space  $Y$ . Thus, the theorem does not hold.

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