

Pacific Journal of Mathematics

A FRACTIONAL LEIBNIZ q -FORMULA

WALEED A. AL-SALAM AND A. VERMA

A FRACTIONAL LEIBNIZ q -FORMULA

W. A. AL-SALAM AND A. VERMA

In this note we give a discrete analogue, the so called q -analogue, of the well known fractional version of Leibniz formula, i.e., the formula which expresses the fractional integral of the product of two functions in terms of the derivatives and fractional integrals of each. Our discrete analogue is naturally suited to be applied to basic or Heine series. We give three such applications.

By the Leibniz formula we mean

$$(1.1) \quad D^n \{f(x)g(x)\} = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x) \quad (D = d/dx).$$

This formula has been generalized [7] to arbitrary complex values of n to

$$(1.2) \quad I^\alpha \{f(x)g(x)\} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} D^k f(x) I^{\alpha+k} \{g(x)\}$$

where

$$(1.3) \quad I^\alpha \{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

is the familiar Riemann-Liouville fractional integral. For other extensions based on (1.3) see [8, 9].

The q -difference operator is defined by means of

$$(1.4) \quad D_q \{f(x)\} = \frac{f(qx) - f(x)}{x(q-1)}$$

(note that if f is differentiable then $\lim_{q \rightarrow 1} D_q f(x) = f'(x)$).

For the q -difference operator we can define two inverse operations, the so called q -integrals,

$$(1.5) \quad I_q f(x) = \int_0^x f(t) d(t; q) = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k)$$

and

$$(1.6) \quad K_q f(x) = \int_x^\infty f(t) d(t; q) = x(1-q) \sum_{k=1}^\infty q^{-k} f(xq^{-k})$$

both of which reduce, for certain classes of functions, to the corresponding Riemann integrals $\int_0^x f(t) dt$ and $\int_x^\infty f(t) dt$ when $q \rightarrow 1$.

Many algebraic as well as function theoretic q -analogues have been considered (see e.g. [3, 4, 5]). We shall require in this work the q -binomial coefficient

$$\begin{bmatrix} x \\ 0 \end{bmatrix}_q = 1, \quad \begin{bmatrix} x \\ n \end{bmatrix}_q = \frac{(1-q^x)(1-q^{x-1}) \cdots (1-q^{x-n+1})}{(1-q)(1-q^2) \cdots (1-q^n)}, \quad (n \geq 1)$$

and q -factorial notation

$$\begin{aligned} [a]_0 &= [a]_{0,q} = 1 \\ [a]_n &= [a]_{n,q} = (1-a)(1-aq)(1-aq^2) \cdots (1-aq^{n-1}) . \\ [a]_1 &= [a], \quad [n]! = [1][2] \cdots [n], \quad [0]! = 1. \end{aligned}$$

If there is no danger of confusion we shall write $[1-a]_n$ or equivalently $[1-a]_{n,q}$ to mean the quantity defined above, i.e., $[a]_n$.

Two q -analogues of the exponential function are in use.

$$e_q(x) = \sum_{n=0}^\infty \frac{x^n}{[q]_{n,q}} = \prod_{n=0}^\infty (1-xq^n)^{-1} \quad |q| < 1.$$

The infinite product converges for all x provided that $|q| < 1$ and

$$E_q(x) = \sum_{n=0}^\infty (-1)^n \frac{x^n}{[q]_{n,q}} q^{\frac{1}{2}n(n-1)} = \prod_{n=0}^\infty (1-xq^n) \quad |q| < 1$$

which is an entire function of x .

It is easy to see that $\lim_{q \rightarrow 1} e_q(x(1-q)) = \lim_{q \rightarrow 1} E_q(x(q-1)) = e^x$.

We shall also make use of the function

$$\Gamma_q(a) = \frac{e_q(q^a)}{e_q(q)} (1-q)^{1-a} \quad \text{defined for } a \neq 0, -1, -2, \dots$$

This is a q -analogue of the gamma function and satisfies the functional equation $\Gamma_q(\alpha+1) = ((1-q^\alpha)/(1-q))\Gamma_q(\alpha)$.

Furthermore we write

$$(1.7) \quad [x - y]_\beta = x^\beta \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \beta \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} \left(\frac{y}{x}\right)^k$$

as a generalization for the finite product $[x - y]_n = (x - y) \cdot (x - ay) \cdots (x - q^{n-1}y)$. It is easy to see that when $\beta = n$ formula (1.7) reduces to a well known formula of Euler. On the other hand if β is not a positive integer and if $|q| < 1$ then the series in (1.7) converges absolutely to the value

$$x^\beta \frac{e_q\left(q^\beta \frac{y}{x}\right)}{e_q\left(\frac{y}{x}\right)} = x^\beta \prod_{n=0}^{\infty} \frac{1 - \frac{y}{x} q^n}{1 - \frac{y}{x} q^{\beta+n}}.$$

The Heine series referred to above are series of the form

$$r^{\Phi_S} \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_r; \\ \beta_1, \beta_2, \dots, \beta_s; \end{matrix} x \right] = \sum_{n=0}^{\infty} \frac{[\alpha_1]_n [\alpha_2]_n \cdots [\alpha_r]_n}{[q]_n [\beta_1]_n \cdots [\beta_s]_n} x^n.$$

Now corresponding to (1.1) we have [4] the q -Leibniz formula

$$(1.8) \quad D_q^n \{f(x)g(x)\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k f(xq^{n-k})g(x)$$

valid for $n = 0, 1, 2, \dots$.

Hence our goal here is to extend (1.8) to “fractional” values of n . We need therefore a concept of fractional q -integral. This has been done in [1, 2], by means of

$$(1.9) \quad \begin{aligned} I_q^\alpha \{f(t); x\} &= \frac{1}{\Gamma_q(\alpha)} \int_0^x [x - qt]_{\alpha-1} f(t) d(t; q) \\ &= x^\alpha (1 - q)^\alpha \sum_{k=0}^{\infty} \frac{[q^\alpha]_k}{[q]_k} q^k f(xq^k) \end{aligned}$$

and

$$(1.10) \quad \begin{aligned} K_q^{-\alpha} \{f(t); x\} &= \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_q(\alpha)} \int_x^\infty [t - x]_{\alpha-1} f(tq^{1-\alpha}) d(t; q) \\ &= x^\alpha (1 - q)^\alpha q^{-\frac{1}{2}\alpha(\alpha+1)} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} \cdot f(xq^{-\alpha-k}). \end{aligned}$$

When there is no danger of confusion we shall simply write $I_q^\alpha f(x)$ and $K_q^{-\alpha} f(x)$ for (1.9) and (1.10).

We also remark that $I_q^0 f(x) = K_q^0 f(x) = f(x)$.

The operators I_q^α and $K_q^{-\alpha}$ are closely related. In fact one can see from (1.9) and (1.10) that if we put $pq = 1$ then

$$(1.11) \quad I_p^\alpha \{f(t); x\} = q^{\frac{1}{2}\alpha(\alpha+1)} \frac{(1-p)^\alpha}{(1-q)^\alpha} K_q^{-\alpha} \{f(tq^\alpha); x\}.$$

In view of this we shall confine our discussion to only one of the two operators, say, to I_q .

Note that the operators (1.9) and (1.10) reduce, for integral values of α , to

$$I_q^{-N} \{f(x)\} = (-1)^N K_q^N \{f(x)\} = D_q^N \{f(x)\},$$

whereas $I_q^N \{f(x)\}$ and $K_q^{-N} \{f(x)\}$ are the N repeated operators (1.5) and (1.6) respectively.

If $U(x) = \sum c_n x^n$ is a power series whose radius of convergence is R then we have from (1.9)

$$(1.12) \quad I_q^\alpha \{U(x)\} = \frac{x^\alpha}{\Gamma_q(\alpha+1)} \sum_{n=0}^{\infty} c_n \frac{[q]_n}{[q^{\alpha+1}]_n} x^n$$

which for $|q| < 1$ has the same radius of convergence as that of U .

It is clear that (1.9) is absolutely convergent if $U(x) = O(x^{\lambda-1})$ as $x \rightarrow 0$ for $\text{Re}(\lambda) > 0$ so that (1.9) is absolutely convergent for the cases $U(x) = x^{\lambda-1} E_q(x)$. Similar remark holds when we shall take $U(x) = [1-x]_N$ or $U(x) = x^{\lambda+n+1}$.

2. q -Newton Series. Such a series were given by Jackson [6] in the form

$$(2.1) \quad f(x) = \sum_{n=0}^{\infty} \frac{D_q^n f(a)}{[q]_n} (1-q)^n [x-a]_n.$$

However we shall require such a formula in a slightly different form which we state as

$$(2.2) \quad f(x) = \sum_{n=0}^{\infty} (-1)^n q^{-n(n-1)/2} \frac{D_q^n f(aq^{-n})}{[q]_n} (1-q)^n [a-x]_n$$

To verify the validity (at least formally) of (2.2) we put

$$(2.3) \quad f(x) = \sum_{n=0}^{\infty} C_n [a-x]_n$$

But

$$D_q^m [a-x]_n = \frac{(-1)^m [q]_n}{[q]_{n-m}} q^{\frac{1}{2}m(m-1)} [a-xq^m]_{n-m},$$

so that if we q -difference (2.3) m times and put $x = aq^{-m}$ we get the right value of c_m .

We shall require (2.2) when x is replaced by xq^n and a by $xq^{-\alpha}$. It becomes after some simplification

$$(2.4) \quad U(xq^n) = \sum_{k=0}^{\infty} (-1)^k q^{-\frac{1}{2}k(k-1)-\alpha k} \frac{[q^{\alpha+n}]_k}{[q]_k} \cdot x^k \{D_q^k U(xq^{-\alpha-k})\}.$$

If U is a polynomial then the right hand side of (2.4) is a finite sum and no question of convergence arises. The formula can also be seen to be valid if $U(x)$ has a convergent power series expansion and at the same time $U(xq^{-\alpha-k})$ has a power series expansion for all k . In case $|q| < 1$ it is then sufficient to assume that $U(x)$ is entire. In all these cases (2.4) is absolutely convergent.

3. Fractional q -Leibniz Formula. We now have from (1.9) that

$$(3.1) \quad I_q^\alpha \{U(x)V(x)\} = x^\alpha (1-q)^\alpha \sum_{n=0}^{\infty} \frac{[q^\alpha]_n}{[q]_n} q^n U(xq^n) V(xq^n).$$

Replacing in (3.1) for $U(xq^n)$ its value obtained in (2.4) we get

$$(3.2) \quad \begin{aligned} I_q^\alpha \{U(x)V(x)\} &= x^\alpha (1-q)^\alpha \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{[q^\alpha]_n [q^{\alpha+n}]_m}{[q]_n [q]_m} \\ &\quad \cdot q^n x^m q^{-\frac{1}{2}m(m-1)-\alpha m} V(xq^n) D_q^m U(xq^{-\alpha-m}) \\ &= x^\alpha (1-q)^\alpha \sum_{m=0}^{\infty} (-1)^m q^{-\frac{1}{2}m(m-1)-\alpha m} \frac{[q^\alpha]_m}{[q]_m} \\ &\quad \cdot x^m D_q^m U(xq^{-\alpha-m}) \sum_{n=0}^{\infty} q^n \frac{[q^{\alpha+m}]_n}{[q]_n} V(xq^n). \end{aligned}$$

Here we have used the fact that

$$[q^\alpha]_n [q^{\alpha+n}]_m = [q^\alpha]_{n+m} = [q^\alpha]_m [q^{\alpha+m}]_n.$$

If we now evaluate the inside sum by means of (1.9) we get

$$(3.3) \quad I_q^\alpha \{U(x)V(x)\} = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q D_q^m U(xq^{-\alpha-m}) I_q^{\alpha+m} V(x).$$

In case $\alpha = -N$, a negative integer, we obtain the well-known formula (1.5).

In case $V(x) = 1$ we have

$$I_q^{\alpha+k} \{1\} = \frac{x^{\alpha+k}}{\Gamma_q(\alpha+k+1)}$$

and hence (3.3) yields

$$(3.4) \quad I_q^\alpha U(x) = \sum_{n=0}^{\infty} \begin{bmatrix} -\alpha \\ n \end{bmatrix}_q \frac{x^{\alpha+n}}{\Gamma_q(\alpha+n+1)} D_q^n U(xq^{-\alpha-n})$$

which can also be written as

$$(3.5) \quad I_q^\alpha U(x) = \frac{1}{\Gamma_q(x)} \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+\alpha}}{[n]!} \frac{q^{-\frac{1}{2}n(n-1)-\alpha n}}{[\alpha+n]} \cdot D_q^n U(xq^{-\alpha-n}).$$

If $q \rightarrow 1$ formulas (3.3) and (3.5) reduce to the following formulas (Davis [3]).

$$I^\nu [U(x)V(x)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} U^{(k)}(x) V^{(-\nu-k)}(x)$$

and

$$I^\nu u(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\nu}{\nu+m} \frac{(x-c)^{\nu+m}}{\Gamma(\nu+1)} u^{(m)}(x)$$

where I^ν is the ν th fractional integral of Liouville (1.3).

Although the derivation of formula (3.3) given above was only formal, it is easy to see that (3.3) is valid whenever the functions $U(x)$ and $V(x)$ are such that the series in (1.9), (2.4), and (3.1) are absolutely convergent. For example if $U(x)$ is a polynomial then (2.4) is only a finite sum and the interchange of summation in (3.2) is justified. In all the applications that we give in the next section all the functions U, V are chosen so that (3.3) is valid.

4. Applications. Our first application is to take $U(x) = [1-x]_N$ and $V(x) = x^{\lambda-1}$ where N is a positive integer and $\text{Re}(\lambda) > 0$. By easy calculation we have

$$D_q^k[1-x]_N = q^{Nk} \frac{[q^{-N}]_k}{[q]_k} [1-xq^k]_{N-k}$$

and by virtue of (1.9)

$$\begin{aligned} I_q^{\alpha+\lambda}\{x^{\lambda-1}\} &= x^{\alpha+k+i}(1-q)^{\alpha+k} \sum_{j=0}^{\infty} \frac{[q^{\alpha+k}]_i}{[q]_j} q^{j\lambda} \\ &= x^{\alpha+k+\lambda-1}(1-q)^{\alpha+k} \prod_{s=0}^{\infty} \left\{ \frac{1-q^{\alpha+k+\lambda+s}}{1-q^{-\lambda+s}} \right\} \\ &= x^{\alpha+k+\lambda-1} \frac{(1-q)^{\alpha+k}}{[q^{\alpha+\lambda}]_k} \frac{e_q(q^\lambda)}{e_q(q^{\alpha+\lambda})}. \end{aligned}$$

Replacing these values in (3.3) we get, for $x \neq q^{\alpha-i}$ ($j = 0, 1, 2, \dots, N-1$)

$$\begin{aligned} (4.1) \quad I_q^\alpha\{x^{\lambda-1}[1-x]_N\} &= \sum_{k=0}^N \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{Nk} \frac{[q^{-N}]_k}{[q]_k} \\ &\quad \cdot [1-xq^{-\alpha}]_{N-k} \cdot x^{\alpha+k+\lambda+1} \frac{(1-q)^\alpha}{[q^{\alpha+\lambda}]_k} \frac{e_q(q^\lambda)}{e_q(q^{\alpha+\lambda})} \\ &= x^{\alpha+\lambda+1}(1-q)^\alpha \frac{e_q(q^\lambda)}{e_q(q^{\alpha+\lambda})} \\ &\quad \cdot [1-xq^{-\alpha}]_N \cdot {}_2\Phi_2 \left[\begin{matrix} q^\alpha, q^{-N} \\ q^{\alpha+\lambda}, (1/x)q^{1+\alpha-N} \end{matrix}; q \right]. \end{aligned}$$

On the other hand we can calculate the left-hand side of (4.1) directly by means of (1.9). We get

$$\begin{aligned} (4.2) \quad I_q^\alpha\{x^{\lambda-1}[1-x]_N\} &= x^{\alpha+\lambda-1}(1-q)^\alpha \sum_{k=0}^{\infty} q^{k\lambda} \frac{[q^\alpha]_k}{[q]_k} [1-xq^k]_N \\ &= x^{\alpha+\lambda-1}(1-q)^\alpha [1-x]_N \cdot {}_2\Phi_1 \left[\begin{matrix} q^\alpha, xq^N \\ x \end{matrix}; q^\lambda \right]. \end{aligned}$$

Comparing (4.1) and (4.2) we get (putting $x = q^c$) the transformation formula

$$\begin{aligned} (4.3) \quad {}_2\Phi_1 \left[\begin{matrix} q^\alpha, q^{N+c} \\ q^c \end{matrix}; q^\lambda \right] &= \prod_{j=0}^{\infty} \left\{ \frac{1-q^{\alpha+\lambda+j}}{1-q^{\lambda+j}} \right\} \cdot \frac{[q^{c-\alpha}]_N}{[q^c]_N} \\ &\quad \cdot {}_2\Phi_2 \left[\begin{matrix} q^{-N}, q^\alpha \\ q^{\alpha+\lambda}, q^{1+\alpha-N-c} \end{matrix}; q \right]. \end{aligned}$$

provided $|q| < 1$, $\text{Re}(\lambda) > 0$ and N is a positive integer.

For our next application let us consider the fractional q -integral $I_q^\alpha\{x^{\lambda-1}E_q(x)\}$ and evaluate it in two different ways. By using the definition (1.9) we get, for $|q| < 1$, and $\text{Re}(\lambda) > 0$,

$$(4.4) \quad I_q^\alpha\{x^{\lambda-1}E_q(x)\} = E_q(x)x^{\alpha+\lambda-1}(1-q)^\alpha {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ x \end{matrix}; q\right].$$

On the other hand if we apply our Leibniz formula (3.3) with $U(x) = E_q(x)$, $V(x) = x^{\lambda-1}$ we get, for $|xq^{-\alpha}| < 1$,

$$(4.5) \quad I_q^\alpha\{x^{\lambda-1}E_q(x)\} = \prod_{s=0}^{\infty} \left\{ \frac{(1-q^{\alpha+\lambda+s})(1-xq^{-\alpha+s})}{(1-q^{\lambda+s})} \right\} \\ \cdot (1-q)^\alpha x^{\alpha+\lambda-1} {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ q^{\lambda+\alpha} \end{matrix}; xq^{-\alpha}\right].$$

Comparing (4.4) and (4.5) we get the transformation formula (putting $x = q^c$)

$$(4.6) \quad {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ q^c \end{matrix}; q^\lambda\right] = \prod_{s=0}^{\infty} \left\{ \frac{(1-q^{\alpha+\lambda+s})(1-q^{c-\alpha+s})}{(1-q^{c+s})(1-q^{\lambda+s})} \right\} \cdot {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ q^{\lambda+\alpha} \end{matrix}; q^{c-\alpha}\right]$$

provided $|q| < 1$, $\text{Re}(\lambda) > 0$, $\text{Re}(c - \alpha) > 0$. Note that c and λ in the left hand side interchanged positions in the right hand side.

For our third and final application we consider $I_q^\alpha\{x^{n+\lambda-1}E_q(x)\}$ where n is a positive integer. We apply our Leibniz formula in two different ways. Once we let $U(x) = x^n$, $V(x) = x^{\lambda-1}E_q(x)$. We then put $U(x) = E_q(x)$ and $V(x) = x^{n+\lambda-1}$. Equating the results of these two calculations we obtain

$$(4.7) \quad \frac{[q^\lambda]_n}{[q^{\alpha+\lambda}]_n} q^{n\alpha} {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ q^{\alpha+\lambda+n} \end{matrix}; xq^n\right] \\ = \sum_{k=0}^n \frac{[q^{-n}]_k [q^\alpha]_k [xq^{n-k}]_k}{[q]_k [q^{\alpha+\lambda}]_k} q^k {}_1\Phi_1\left[\begin{matrix} q^{\alpha+k}; \\ q^{\alpha+\lambda+k} \end{matrix}; xq^{n-k}\right]$$

provided that $|q| < 1$, $|x| < 1$, $\text{Re}(\lambda - \alpha) > 0$.

As a corollary of this we obtain, putting $n = 1$, the contiguous relation

$$(q^\alpha - q^\lambda) {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ q^{\lambda+1} \end{matrix}; xq\right] = (1-q)^\lambda {}_1\Phi_1\left[\begin{matrix} q^\alpha; \\ q^\lambda \end{matrix}; xq\right] \\ - (1-x)(1-q^\alpha) {}_1\Phi_1\left[\begin{matrix} q^{\alpha+1}; \\ q^{\lambda+1} \end{matrix}; x\right].$$

REFERENCES

1. P. P. Agarwal, *Certain fractional g -integrals and g -derivatives*, Proc. Camb. Phil. Soc., **66** (1969), 365–370.
2. W. A. Al-Salam, *Some fractional q -integrals and g -derivatives*, Proc. Edinburgh Math. Soc., **15** (1966), 135–140.
3. H. T. Davis, *The application of fractional operators to functional equations*, American J. of Mathematics, **49** (1927), 123–142.
4. W. Hahn, *Über orthogonale Polynome, die q -Differenzgleichungen genügen*, Mathematische Nachrichten, **2** (1949), 4–34.
5. W. Hahn, *Über die höheren Heineschen Reihen und einheitliche Theorie der sogenannten speziellen Funktionen*, Math. Nach., **3** (1950), 257–294.
6. F. H. Jackson, *q -Form of Taylor's theorem*, Messenger of Mathematics, **39** (1909), 62–64.
7. J. Liouville, *Sur le calcul des différentielles à indices quelconques*, Journal de l'Ecole Polytechnique Ser. 1, **21** (1832), 71–161.
8. T. J. Osler, *Leibniz rule for fractional derivatives generalized and application to infinite series*, SIAM J. Appl. Math., **18** (1970), 658–674.
9. Y. Watanabe, *Notes on the generalized derivative of Riemann-Liouville and its application to Leibniz formula*, Tohoku Math. J., **34** (1931), 8–41.

Received May 14, 1974 and in revised form March 21, 1975.

THE UNIVERSITY OF ALBERTA
EDMONTON, CANADA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics
All Rights Reserved

Pacific Journal of Mathematics

Vol. 60, No. 2

October, 1975

Waleed A. Al-Salam and A. Verma, <i>A fractional Leibniz q-formula</i>	1
Robert A. Bekes, <i>Algebraically irreducible representations of $L_1(G)$</i>	11
Thomas Theodore Bowman, <i>Construction functors for topological semigroups</i>	27
Stephen LaVern Campbell, <i>Operator-valued inner functions analytic on the closed disc. II</i>	37
Leonard Eliezer Dor and Edward Wilfred Odell, Jr., <i>Monotone bases in L_p</i>	51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, <i>On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 = f$</i>	63
Y. Gordon, <i>Unconditional Schauder decompositions of normed ideals of operators between some l_p-spaces</i>	71
Gary Grefsrud, <i>Oscillatory properties of solutions of certain nth order functional differential equations</i>	83
Irvin Roy Hentzel, <i>Generalized right alternative rings</i>	95
Zensiro Goseki and Thomas Benny Rushing, <i>Embeddings of shape classes of compacta in the trivial range</i>	103
Emil Grosswald, <i>Brownian motion and sets of multiplicity</i>	111
Donald LaTorre, <i>A construction of the idempotent-separating congruences on a bisimple orthodox semigroup</i>	115
Pjek-Hwee Lee, <i>On subrings of rings with involution</i>	131
Marvin David Marcus and H. Minc, <i>On two theorems of Frobenius</i>	149
Michael Douglas Miller, <i>On the lattice of normal subgroups of a direct product</i>	153
Grattan Patrick Murphy, <i>A metric basis characterization of Euclidean space</i>	159
Roy Martin Rakestraw, <i>A representation theorem for real convex functions</i>	165
Louis Jackson Ratliff, Jr., <i>On Rees localities and H_i-local rings</i>	169
Simeon Reich, <i>Fixed point iterations of nonexpansive mappings</i>	195
Domenico Rosa, <i>B-complete and B_r-complete topological algebras</i>	199
Walter Roth, <i>Uniform approximation by elements of a cone of real-valued functions</i>	209
Helmut R. Salzmann, <i>Homogene kompakte projektive Ebenen</i>	217
Jerrold Norman Siegel, <i>On a space between BH and B_∞</i>	235
Robert C. Sine, <i>On local uniform mean convergence for Markov operators</i>	247
James D. Stafney, <i>Set approximation by lemniscates and the spectrum of an operator on an interpolation space</i>	253
Árpád Szász, <i>Convolution multipliers and distributions</i>	267
Kalathoor Varadarajan, <i>Span and stably trivial bundles</i>	277
Robert Breckenridge Warfield, Jr., <i>Countably generated modules over commutative Artinian rings</i>	289
John Yuan, <i>On the groups of units in semigroups of probability measures</i>	303