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In this note we give a discrete analogue, the so called q-analogue, of the well known fractional version of Leibniz formula, i.e., the formula which expresses the fractional integral of the product of two functions in terms of the derivatives and fractional integrals of each. Our discrete analogue is naturally suited to be applied to basic or Heine series. We give three such applications.

By the Leibniz formula we mean

$$D^n\{f(x)g(x)\} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x) \qquad (D = d/dx). \tag{1.1}$$

This formula has been generalized [7] to arbitrary complex values of n to

$$I^\alpha\{f(x)g(x)\} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} D^k f(x)\, I^{\alpha+k}\{g(x)\} \tag{1.2}$$

where

$$I^\alpha\{f(x)\} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \tag{1.3}$$

is the familiar Riemann-Liouville fractional integral. For other extensions based on (1.3) see [8, 9].

The q-difference operator is defined by means of

$$D_q\{f(x)\} = \frac{f(qx)-f(x)}{x(q-1)} \tag{1.4}$$

(note that if f is differentiable then $\lim_{q\to 1} D_q f(x) = f'(x)$).

For the q-difference operator we can define two inverse operations, the so called q-integrals,

$$I_q f(x) = \int_0^x f(t)d(t;q) = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k) \tag{1.5}$$

and

$$K_q f(x) = \int_x^\infty f(t)d(t; q) = x(1-q) \sum_{k=1}^\infty q^{-k} f(xq^{-k}) \tag{1.6}$$

both of which reduce, for certain classes of functions, to the corresponding Riemann integrals $\int_0^x f(t)dt$ and $\int_x^\infty f(t)dt$ when $q \to 1$.

Many algebraic as well as function theoretic q-analogues have been considered (see e.g. **[3, 4, 5]**). We shall require in this work the q-binomial coefficient

$$\begin{bmatrix} x \\ 0 \end{bmatrix}_q = 1, \qquad \begin{bmatrix} x \\ n \end{bmatrix}_q = \frac{(1-q^x)(1-q^{x-1})\cdots(1-q^{x-n+1})}{(1-q)(1-q^2)\cdots(1-q^n)}, \qquad (n \geqq 1)$$

and q-factorial notation

$$\begin{aligned} &[a]_0 = [a]_{0,q} = 1 \\ &[a]_n = [a]_{n,q} = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \, . \\ &[a]_1 = [a], \quad [n]! = [1][2]\cdots[n], \quad [0]! = 1. \end{aligned}$$

If there is no danger of confusion we shall write $[1-a]_n$ or equivalently $[1-a]_{n,q}$ to mean the quantity defined above, i.e., $[a]_n$.

Two q-analogues of the exponential function are in use.

$$e_q(x) = \sum_{n=0}^\infty \frac{x^n}{[q]_{n,q}} = \prod_{n=0}^\infty (1-xq^n)^{-1} \qquad |q| < 1.$$

The infinite product converges for all x provided that $|q| < 1$ and

$$E_q(x) = \sum_{n=0}^\infty (-1)^n \frac{x^n}{[q]_{n,q}} q^{\frac{1}{2}n(n-1)} = \prod_{n=0}^\infty (1-xq^n) \qquad |q| < 1$$

which is an entire function of x.

It is easy to see that $\lim_{q\to 1} e_q(x(1-q)) = \lim_{q\to 1} E_q(x(q-1)) = e^x$.

We shall also make use of the function

$$\Gamma_q(a) = \frac{e_q(q^\alpha)}{e_q(q)} (1-q)^{1-\alpha} \quad \text{defined for} \quad \alpha \neq 0, -1, -2, \cdots.$$

This is a q-analogue of the gamma function and satisfies the functional equation $\Gamma_q(\alpha+1) = ((1-q^\alpha)/(1-q))\Gamma_q(\alpha)$.

Furthermore we write

(1.7) $$[x-y]_\beta = x^\beta \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \beta \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} \left(\frac{y}{x}\right)^k$$

as a generalization for the finite product $[x-y]_n = (x-y)\cdot(x-ay)\cdots(x-q^{n-1}y)$. It is easy to see that when $\beta = n$ formula (1.7) reduces to a well known formula of Euler. On the other hand if β is not a positive integer and if $|q|<1$ then the series in (1.7) converges absolutely to the value

$$x^\beta \frac{e_q\left(q^\beta \frac{y}{x}\right)}{e_q\left(\frac{y}{x}\right)} = .x^\beta \prod_{n=0}^{\infty} \frac{1-\frac{y}{x}q^n}{1-\frac{y}{x}q^{\beta+n}}.$$

The Heine series referred to above are series of the form

$$ {}_r\Phi_s \begin{bmatrix} \alpha_1, \alpha_2, \cdots, \alpha_r; & \\ & x \\ \beta_1, \beta_2, \cdots, \beta_s; & \end{bmatrix} = \sum_{n=0}^{\infty} \frac{[\alpha_1]_n [\alpha_2]_n \cdots [\alpha_r]_n}{[q]_n [\beta_1]_n \cdots [\beta_s]_n} x^n.$$

Now corresponding to (1.1) we have **[4]** the q-Leibniz formula

(1.8) $$D_q^n\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k f(xq^{n-k})g(x)$$

valid for $n = 0, 1, 2, \cdots$.

Hence our goal here is to extend (1.8) to "fractional" values of n. We need therefore a concept of fractional q-integral. This has been done in **[1, 2]**, by means of

(1.9) $$\begin{aligned} I_q^\alpha\{f(t); x\} &= \frac{1}{\Gamma_q(x)} \int_0^x [x-qt]_{\alpha-1} f(t) d(t; q) \\ &= x^\alpha (1-q)^\alpha \sum_{k=0}^{\infty} \frac{[q^\alpha]_k}{[q]_k} q^k f(xq^k) \end{aligned}$$

and

$$K_q^{-\alpha}\{f(t); x\} = \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_q(\alpha)} \int_x^\infty [t-x]_{\alpha-1} f(tq^{1-\alpha}) d(t; q)$$

(1.10) $$= x^\alpha (1-q)^\alpha q^{-\frac{1}{2}\alpha(\alpha+1)} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} \cdot f(xq^{-\alpha-k}).$$

When there is no danger of confusion we shall simply write $I_q^\alpha f(x)$ and $K_q^{-\alpha} f(x)$ for (1.9) and (1.10).

We also remark that $I_q^0 f(x) = K_q^0 f(x) = f(x)$.

The operators I_q^α and $K_q^{-\alpha}$ are closely related. In fact one can see from (1.9) and (1.10) that if we put $pq = 1$ then

$$I_p^\alpha\{f(t); x\} = q^{\frac{1}{2}\alpha(\alpha+1)} \frac{(1-p)^\alpha}{(1-q)^\alpha} K_q^{-\alpha}\{f(tq^\alpha_-); x\}. \tag{1.11}$$

In view of this we shall confine our discussion to only one of the two operators, say, to I_q.

Note that the operators (1.9) and (1.10) reduce, for integral values of α, to

$$I_q^{-N}\{f(x)\} = (-1)^N K_q^N\{f(x)\} = D_q^N\{f(x)\},$$

whereas $I_q^N\{f(x)\}$ and $K_q^{-N}\{f(x)\}$ are the N repeated operators (1.5) and (1.6) respectively.

If $U(x) = \Sigma\, c_n x^n$ is a power series whose radius of convergence is R then we have from (1.9)

$$I_q^\alpha\{U(x)\} = \frac{x^\alpha}{\Gamma_q(\alpha+1)} \sum_{n=0}^{\infty} c_n \frac{[q]_n}{[q^{\alpha+1}]_n} x^n \tag{1.12}$$

which for $|q| < 1$ has the same radius of convergence as that of U.

It is clear that (1.9) is absolutely convergent if $U(x) = O(x^{\lambda-1})$ as $x \to 0$ for $\mathrm{Re}(\lambda) > 0$ so that (1.9) is absolutely convergent for the cases $U(x) = x^{\lambda-1} E_q(x)$. Similar remark holds when we shall take $U(x) = [1-x]_N$ or $U(x) = x^{\lambda+n+1}$.

2. q-Newton Series. Such a series were given by Jackson **[6]** in the form

$$f(x) = \sum_{n=0}^{\infty} \frac{D_q^n f(a)}{[q]_n} (1-q)^n [x-a]_n. \tag{2.1}$$

However we shall require such a formula in a slightly different form which we state as

$$f(x) = \sum_{n=0}^{\infty} (-1)^n q^{-n(n-1)/2} \frac{D_q^n f(aq^{-n})}{[q]_n} (1-q)^n [a-x]_n \tag{2.2}$$

To verify the validity (at least formally) of (2.2) we put

$$f(x)=\sum_{n=0}^{\infty} C_n[a-x]_n \tag{2.3}$$

But

$$D_q^m[a-x]_n=\frac{(-1)^m[q]_n}{[q]_{n-m}}\, q^{\frac{1}{2}m(m-1)}[a-xq^m]_{n-m},$$

so that if we q-difference (2.3) m times and put $x=aq^{-m}$ we get the right value of c_m.

We shall require (2.2) when x is replaced by xq^n and a by $xq^{-\alpha}$. It becomes after some simplification

$$U(xq^n)=\sum_{k=0}^{\infty}(-1)^k q^{-\frac{1}{2}k(k-1)-\alpha k}\frac{[q^{\alpha+n}]_k}{[q]_k}\cdot x^k\{D_q^kU(xq^{-\alpha-k})\}. \tag{2.4}$$

If U is a polynomial then the right hand side of (2.4) is a finite sum and no question of convergence arises. The formula can also be seen to be valid if $U(x)$ has a convergent power series expansion and at the same time $U(xq^{-\alpha-k})$ has a power series expansion for all k. In case $|q|<1$ it is then sufficient to assume that $U(x)$ is entire. In all these cases (2.4) is absolutely convergent.

3. Fractional q-Leibniz Formula. We now have from (1.9) that

$$I_q^\alpha\{U(x)V(x)\}=x^\alpha(1-q)^\alpha\sum_{n=0}^{\infty}\frac{[q^\alpha]_n}{[q]_n}\, q^nU(xq^n)V(xq^n). \tag{3.1}$$

Replacing in (3.1) for $U(xq^n)$ its value obtained in (2.4) we get

$$\begin{aligned} I_q^\alpha\{U(x)V(x)\}&=x^\alpha(1-q)^\alpha\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{[q^\alpha]_n[q^{\alpha+n}]_m}{[q]_n[q]_m}\\ &\quad\cdot q^nx^mq^{-\frac{1}{2}m(m-1)-\alpha m}V(xq^n)D_q^mU(xq^{-\alpha-m})\\ &=x^\alpha(1-q)^\alpha\sum_{m=0}^{\infty}(-1)^mq^{-\frac{1}{2}m(m-1)-m\alpha}\frac{[q^\alpha]_m}{[q]_m}\\ &\quad\cdot x^mD_q^mU(xq^{-\alpha-m})\sum_{n=0}^{\infty}q^n\frac{[q^{\alpha+m}]_n}{[q]_n}V(xq^n). \end{aligned} \tag{3.2}$$

Here we have used the fact that

$$[q^\alpha]_n[q^{\alpha+n}]_m=[q^\alpha]_{n+m}=[q^\alpha]_m[q^{\alpha+m}]_n.$$

If we now evaluate the inside sum by means of (1.9) we get

$$I_q^\alpha\{U(x)V(x)\} = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q D_q^m U(xq^{-\alpha-m}) I_q^{\alpha+m} V(x). \tag{3.3}$$

In case $\alpha = -N$, a negative integer, we obtain the well-known formula (1.5).

In case $V(x) = 1$ we have

$$I_q^{\alpha+k}\{1\} = \frac{x^{\alpha+k}}{\Gamma_q(\alpha+k+1)}$$

and hence (3.3) yields

$$I_q^\alpha U(x) = \sum_{n=0}^{\infty} \begin{bmatrix} -\alpha \\ n \end{bmatrix}_q \frac{x^{\alpha+n}}{\Gamma_q(\alpha+n+1)} D_q^n U(xq^{-\alpha-n}) \tag{3.4}$$

which can also be written as

$$I_q^\alpha U(x) = \frac{1}{\Gamma_q(x)} \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+\alpha}}{[n]!} \frac{q^{-\frac{1}{2}n(n-1)-\alpha n}}{[\alpha+n]} \cdot D_q^n U(xq^{-\alpha-n}). \tag{3.5}$$

If $q \to 1$ formulas (3.3) and (3.5) reduce to the following formulas (Davis [3]).

$$I^\nu[U(x)V(x)] = \sum_{k=0}^{\infty} \binom{-\nu}{k} U^{(k)}(x) V^{(-\nu-k)}(x)$$

and

$$I^\nu u(x) = \sum_{m=0}^{\infty} (-1)^m \frac{\nu}{\nu+m} \frac{(x-c)^{\nu+m}}{\Gamma(\nu+1)} u^{(m)}(x)$$

where I^ν is the νth fractional integral of Liouville (1.3).

Although the derivation of formula (3.3) given above was only formal, it is easy to see that (3.3) is valid whenever the functions $U(x)$ and $V(x)$ are such that the series in (1.9), (2.4), and (3.1) are absolutely convergent. For example if $U(x)$ is a polynomial then (2.4) is only a finite sum and the interchange of summation in (3.2) is justified. In all the applications that we give in the next section all the functions U, V are chosen so that (3.3) is valid.

4. Applications. Our first application is to take $U(x) = [1-x]_N$ and $V(x) = x^{\lambda-1}$ where N is a positive integer and $\mathrm{Re}(\lambda) > 0$. By easy calculation we have

$$D_q^k[1-x]_N = q^{Nk} \frac{[q^{-N}]_k}{[q]_k} [1-xq^k]_{N-k}$$

and by virtue of (1.9)

$$\begin{aligned} I_q^{\alpha+\lambda}\{x^{\lambda-1}\} &= x^{\alpha+k+j}(1-q)^{\alpha+k} \sum_{j=0}^{\infty} \frac{[q^{\alpha+k}]_j}{[q]_j} q^{j\lambda} \\ &= x^{\alpha+k+\lambda-1}(1-q)^{\alpha+k} \prod_{s=0}^{\infty} \left\{ \frac{1-q^{\alpha+k+\lambda+s}}{1-q^{-\lambda+s}} \right\} \\ &= x^{\alpha+k+\lambda-1} \frac{(1-q)^{\alpha+k}}{[q^{\alpha+\lambda}]_k} \frac{e_q(q^\lambda)}{e_q(q^{\alpha+\lambda})} . \end{aligned}$$

Replacing these values in (3.3) we get, for $x \neq q^{\alpha-j}$ ($j = 0, 1, 2, \cdots, N-1$)

$$\begin{aligned} (4.1) \quad I_q^\alpha\{x^{\lambda-1}[1-x]_N\} &= \sum_{k=0}^{N} \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{Nk} \frac{[q^{-N}]_k}{[q]_k} \\ &\quad \cdot [1-xq^{-\alpha}]_{N-k} \cdot x^{\alpha+k+\lambda+1} \frac{(1-q)^\alpha}{[q^{\alpha+\lambda}]_k} \frac{e_q(q^\lambda)}{e_q(q^{\alpha+\lambda})} \\ &= x^{\alpha+\lambda+1}(1-q)^\alpha \frac{e_q(q^\lambda)}{e_q(q^{\alpha+\lambda})} \\ &\quad \cdot [1-xq^{-\alpha}]_N \cdot {}_2\Phi_2 \left[\begin{matrix} q^\alpha, q^{-N} & ; \\ q^{\alpha+\lambda}, (1/x)q^{1+\alpha-N} & ; \end{matrix} \, q \right] . \end{aligned}$$

On the other hand we can calculate the left-hand side of (4.1) directly by means of (1.9). We get

$$\begin{aligned} (4.2) \quad I_q^\alpha\{x^{\lambda-1}[1-x]_N\} &= x^{\alpha+\lambda-1}(1-q)^\alpha \sum_{k=0}^{\infty} q^{k\lambda} \frac{[q^\alpha]_k}{[q]_k} [1-xq^k]_N \\ &= x^{\alpha+\lambda-1}(1-q)^\alpha [1-x]_N \cdot {}_2\Phi_1 \left[\begin{matrix} q^\alpha, xq^N ; \\ x \;\; ; \end{matrix} \, q^\lambda \right] . \end{aligned}$$

Comparing (4.1) and (4.2) we get (putting $x = q^c$) the transformation formula

$$\begin{aligned} (4.3) \qquad {}_2\Phi_1 \left[\begin{matrix} q^\alpha, q^{N+c} ; \\ q^c ; \end{matrix} \, q^\lambda \right] &= \prod_{j=0}^{\infty} \left\{ \frac{1-q^{\alpha+\lambda+j}}{1-q^{\lambda+j}} \right\} \cdot \frac{[q^{c-\alpha}]_N}{[q^c]_N} \\ &\quad \cdot {}_2\Phi_2 \left[\begin{matrix} q^{-N}, q^\alpha & ; \\ q^{\alpha+\lambda}, q^{1+\alpha-N-c} & ; \end{matrix} \, q \right] . \end{aligned}$$

provided $|q| < 1$, $\mathrm{Re}(\lambda) > 0$ and N is a positive integer.

For our next application let us consider the fractional q-integral $I_q^{\alpha}\{x^{\lambda-1}E_q(x)\}$ and evaluate it in two different ways. By using the definition (1.9) we get, for $|q|<1$, and $\mathrm{Re}(\lambda)>0$,

$$I_q^{\alpha}\{x^{\lambda-1}E_q(x)\}=E_q(x)x^{\alpha+\lambda-1}(1-q)^{\alpha}{}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ x\,;\end{matrix}\ q\right]. \tag{4.4}$$

On the other hand if we apply our Leibniz formula (3.3) with $U(x)=E_q(x)$, $V(x)=x^{\lambda-1}$ we get, for $|xq^{-\alpha}|<1$,

$$\begin{aligned}I_q^{\alpha}\{x^{\lambda-1}E_q(x)\}&=\prod_{s=0}^{\infty}\left\{\frac{(1-q^{\alpha+\lambda+s})(1-xq^{-\alpha+s})}{(1-q^{\lambda+s})}\right\}\\&\quad\cdot(1-q)^{\alpha}x^{\alpha+\lambda-1}{}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ q^{\lambda+\alpha}\,;\end{matrix}\ xq^{-\alpha}\right].\end{aligned} \tag{4.5}$$

Comparing (4.4) and (4.5) we get the transformation formula (putting $x=q^c$)

$$ {}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ q^{c}\,;\end{matrix}\ q^{\lambda}\right]=\prod_{s=0}^{\infty}\left\{\frac{(1-q^{\alpha+\lambda+s})(1-q^{c-\alpha+s})}{(1-q^{c+s})(1-q^{\lambda+s})}\right\}\cdot{}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ q^{\lambda+\alpha}\,;\end{matrix}\ q^{c-\alpha}\right] \tag{4.6}$$

provided $|q|<1$, $\mathrm{Re}(\lambda)>0$, $\mathrm{Re}(c-\alpha)>0$. Note that c and λ in the left hand side interchanged positions in the right hand side.

For our third and final application we consider $I_q^{\alpha}\{x^{n+\lambda-1}E_q(x)\}$ where n is a positive integer. We apply our Leibniz formula in two different ways. Once we let $U(x)=x^n$, $V(x)x^{\lambda-1}E_q(x)$. We then put $U(x)=E_q(x)$ and $V(x)=x^{n+\lambda-1}$. Equating the results of these two calculations we obtain

$$\begin{aligned}&\frac{[q^{\lambda}]_n}{[q^{\alpha+\lambda}]_n}\,q^{n\alpha}\,{}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ q^{\alpha+\lambda+n}\,;\end{matrix}\ xq^{n}\right]\\&=\sum_{k=0}^{n}\frac{[q^{-n}]_k[q^{\alpha}]_k[xq^{n-k}]_k}{[q]_k[q^{\alpha+\lambda}]_k}\,q^{k}\,{}_1\Phi_1\left[\begin{matrix}q^{\alpha+k}\,;\\ q^{\alpha+\lambda+k}\,;\end{matrix}\ xq^{n-k}\right]\end{aligned} \tag{4.7}$$

provided that $|q|<1$, $|x|<1$, $\mathrm{Re}(\lambda-\alpha)>0$.

As a corollary of this we obtain, putting $n=1$, the contiguous relation

$$\begin{aligned}(q^{\alpha}-q^{\lambda})\,{}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ q^{\lambda+1}\,;\end{matrix}\ xq\right]&=(1-q)^{\lambda})\,{}_1\Phi_1\left[\begin{matrix}q^{\alpha}\,;\\ q^{\lambda}\,;\end{matrix}\ xq\right]\\&\quad-(1-x)(1-q^{\alpha})\,{}_1\Phi_1\left[\begin{matrix}q^{\alpha+1}\,;\\ q^{\lambda+1}\,;\end{matrix}\ x\right].\end{aligned}$$

REFERENCES

1. P. P. Agarwal, *Certain fractional g-integrals and g-derivatives*, Proc. Camb. Phil. Soc., **66** (1969), 365–370.

2. W. A. Al-Salam, *Some fractional q-integrals and g-derivatives*, Proc. Edinburgh Math. Soc., **15** (1966), 135–140.

3. H. T. Davis, *The application of fractional operators to functional equations*, American J. of Mathematics, **49** (1927), 123–142.

4. W. Hahn, *Über orthogonale Polynome, die q-Differenzengleichungen genügen*, Mathematische Nachrichten, **2** (1949), 4–34.

5. W. Hahn, *Über die höheren Heineschen Reihen und einheitliche Theorie der sogenannten speziellen Funktionen*, Math. Nach., **3** (1950), 257–294.

6. F. H. Jackson, *q-Form of Taylor's theorem*, Messenger of Mathematics, **39** (1909), 62–64.

7. J. Liouville, *Sur le calcul des differentielles à indices quelconques*, Journale de l'Ecole Polytechnique Ser. 1, **21** (1832), 71–161.

8. T. J. Osler, *Leibniz rule for fractional derivatives generalized and application to infinite series*, SIAM J. Appl. Math., **18** (1970), 658–674.

9. Y. Watanabe, *Notes on the generalized derivative of Riemann-Liouville and its application to Leibniz formula*, Tohoku Math. J., **34** (1931), 8–41.

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