MONOTONE BASES IN $L_p$

Leonard Eliezer Dor and Edward Wilfred Odell, Jr.
MONOTONE BASES IN $L_p$

L. E. Dor and E. Odell

We prove that every monotone basis (decomposition) for $L_p(\mu)$, $1 < p < \infty$, is unconditional. The structure of such bases is closely related to that of the usual Haar basis. This structure is described here, and it is shown that there is an uncountable number of mutually non-equivalent monotone bases for $L_p$. The structure of monotone bases in $L_1$ is also considered, and the equivalence question there is characterized in analytic terms.

Introduction. The Theorem (2.1), that every monotone decomposition, and in particular every monotone basis for $L_p(\mu)$, $1 < p < \infty$, is unconditional was discovered also by A. Pełczyński and H. P. Rosenthal [10]. The remainder of §2 deals with the structure of monotone bases in $L_p(\mu)$ ($1 < p < \infty$). In Theorem 2.2 we obtain a representation of a monotone basis for $L_p(0,1)$ as a direct $l_p$-sum of what we call generalized Haar bases (which are in turn a natural generalization of the classical Haar system). Finally we show that there is a continuum of non-equivalent generalized Haar bases in $L_p$.

In §3 we study monotone bases on $L_1(0,1)$. First we show how a general monotone basis in $L_1(0,1)$ is obtained from generalized Haar bases, and then we characterize analytically the equivalence of two generalized Haar bases in $L_1(0,1)$.

Section 1 contains notation and preliminaries. Several open questions are stated throughout the paper.

We wish to thank Professors T. Figiel, W. B. Johnson and H. P. Rosenthal for many helpful discussions regarding the material presented here.

1. Notation and preliminaries. We use standard Banach space notation. A sequence of closed subspaces $X_n$ of a Banach space $X$ is said to be a (Schauder) decomposition if every $f \in X$ can be uniquely expressed as $f = \sum_{i=1}^{\infty} f_i$, where $f_i \in X_i$ for all $i$. The decomposition is called unconditional if $\sum_{i=1}^{\infty} f_i$ converges unconditionally for all $f$. This is equivalent to the condition $K = \sup \{\|P_E\|; E \subseteq N \text{ finite}\} < \infty$ where $P_E$ is defined by: $P_E f = \sum_{i \in E} f_i$. $K$ is called the unconditional constant of the decomposition. A decomposition is called monotone if $P_n = P_{\{1,2,\ldots,n\}}$ is a contractive (i.e. norm 1) projection for all $n$. Thus a monotone decomposition corresponds to a sequence $(P_i)$ of contractive projections satisfying $P_i P_j = P_{\min(i,j)}$. 

51
If \((Ω, \mathcal{F}, μ)\) is a measure space (μ is assumed to be finite unless otherwise stated), we shall refer to its \(L^p\)-space as \(L^p(μ)\), \(L^p(\mathcal{F}, μ)\), or \(L^p(Ω)\) according to convenience. If \(Ω_0 \subseteq Ω\) we shall identify \(L^p(Ω_0)\) with functions in \(L^p(Ω)\) vanishing off \(Ω_0\).

If \(\mathcal{J}\) is a sub \(σ\)-ring of \(\mathcal{F}\), \(S(\mathcal{J})\) will denote the support of \(\mathcal{J}\); i.e. its greatest element, and the conditional expectation \(\mathbb{E}_\mathcal{J}f = \mathbb{E}_{\mathcal{J}, μ}f\) of \(f \in L_1(Ω, \mathcal{F}, μ)\) with respect to \(\mathcal{J}\) and \(μ\) is defined as the unique \(g \in L_1(Ω_0, \mathcal{F}, μ)\) satisfying \(\int_E gdμ = \int_E fdμ\) for all \(E \in \mathcal{J}\). \(\mathbb{E}_\mathcal{J}\) is a contractive projection of \(L^p(\mathcal{F})\) onto \(L^p(\mathcal{J})\), for any \(p \geq 1\). For a function \(f\), \(S(f)\) will denote the support of \(f\); for a set \(A\), \(\sim A\) will denote the complement of \(A\). \(m\) is Lebesgue measure on \([0,1]\).

The contractive projections in \(L^p(μ)\) were characterized by Douglas [4] (for \(p = 1\)) and Ando [1] (for \(1 < p < ∞, p \neq 2\)) as follows (cf. also [9]):

**Theorem A.** (i) Let \(1 < p < ∞, p \neq 2\). If \(P\) is a contractive projection in \(L^p(μ)\), then there is a measure \(ν\) on \(\mathcal{F}\), an isometry \(T\) of \(L^p(μ)\) onto \(L^p(ν)\), and a sub \(σ\)-ring \(\mathcal{J}\) of \(\mathcal{F}\), so that

\[
TPT^{-1} = \mathbb{E}_\mathcal{J, ν}.
\]

(ii) Let \(p = 1\). If \(P\) is a contractive projection in \(L_1(μ)\), there are \(ν, T\) and \(\mathcal{J}\) as in (i) and a norm 1(nilpotent) operator \(N: L_1(\sim S(\mathcal{J})) \to L_1(\mathcal{J})\) so that

\[
TPT^{-1}(f) = \mathbb{E}_\mathcal{J}f + N(f_{\mid S(\mathcal{J})}).
\]

We outline the proof, since a similar construction will be used later. The main part of the proof is to show the following special case:

**Fact 1.** If \(P\) is a projection in \(L_p(μ)\) \((1 \leq p < ∞, p \neq 2)\) and \(χ_0\) is in the range \(R(\mathcal{P})\) of \(P\) then there is a sub \(σ\)-algebra \(\mathcal{G}\) of \(\mathcal{F}\) so that \(P = \mathbb{E}_\mathcal{G}\).

Also needed is:

**Fact 2.** Every closed subspace \(X\) of \(L_p(μ)\) \((1 \leq p < ∞)\) contains a function \(k\) with greatest support \(S(k)\) (i.e. for all \(f \in X, S(f) \subseteq S(k)\) \(μ\)-a.e.).

Now, let \(k_0\) be an element with greatest support in \(R(\mathcal{P})\), (we shall then write, \(S(\mathcal{P}) = S(k_0)\)) and let \(k = k_0 + χ_{S(\mathcal{P})}\). Define \(ν\) by \(dν = \lvert k \rvert^p dμ\) and \(T: L_p(μ) \to L_p(ν)\) by \(Tf = f/k\), if \(f \in L_p(μ)\). \(Q = TPT^{-1}\) is a contractive projection in \(L_p(ν)\), and \(χ_{S(\mathcal{Q})} = χ_{S(\mathcal{P})} \in R(\mathcal{Q})\). Therefore by Fact 1, \(Q_{\mid \mathcal{J}_0 S(\mathcal{P})} = \mathbb{E}_\mathcal{J, ν}\) for some sub \(σ\)-ring \(\mathcal{J}\) of \(\mathcal{F}\) with \(S(\mathcal{J}) = S(\mathcal{P})\). Denoting \(Q_{\mid \mathcal{J}_0 S(\mathcal{P})} = N\) we have:

\[
Qf = \mathbb{E}_\mathcal{J}(f_{\mid S(\mathcal{P})}) + N(f_{\mid S(\mathcal{P})}).
\]
Now, if $1 < p < \infty$, then $L_p(\nu)$ is smooth and hence contractive projections in $L_p(\nu)$ are uniquely determined by their range, (cf. [3]), implying that $N = 0$. For $p = 1$, $N$ can be any contraction.

The proof of our first result essentially extends Theorem A to sequences of contractive projections $(P_i)$ satisfying $P_iP_j = P_{\min(i,j)}$. We then apply the following result of Burkholder and Gundy (cf. [2], Theorem 9).

**Theorem B.** Let $1 < p < \infty$. If $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ is an increasing sequence of sub $\sigma$-algebras of $\mathcal{F}$ which generate the $\sigma$-algebra $\mathcal{F}$, then the monotone Schauder decomposition $(R(\mathcal{E}_{\mathcal{F}_i} - \mathcal{E}_{\mathcal{F}_{i-1}}), i = 1, 2, \cdots)$ for $L_p(\mathcal{F})$ is unconditional. Moreover, there is a constant $K_p$, depending only on $p$ so that the unconditional constant of this decomposition is smaller then $K_p$.

**2. Monotone bases in $L_p$ ($1 < p < \infty$).**

**Theorem 2.1.** Let $(P_i)$ be a sequence of contractive projections in $L_p(\Omega, \mathcal{F}, \mu)$, $(1 < p < \infty, p \neq 2)$, with $P_iP_j = P_{\min(i,j)}$. Then there is a measure $\nu$ on $\Omega$, an isometry $T$ of $L_p(\mu)$ onto $L_p(\nu)$ and a sequence of sub $\sigma$-rings $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ so that $P_i = T^{-1}\mathcal{E}_{\mathcal{F}_i}T$.

**Proof.** We first note that Theorem A (and the definition of $T$ in its proof) implies:

\begin{equation}
(*) \quad \text{If } h, f \in R(P), \text{ then } h \cdot \chi_{\mathcal{S}(f)} \in R(P).
\end{equation}

Let $k_1 \in R(P_1)$ with $S(k_1) = S(P_1)$. If $S(P_2) \supseteq S(P_1)$ use (*) to choose $k_2 \in R(P_2)$ with $S(k_1) \cap S(k_2) = \emptyset$ and $S(k_1) \cup S(k_2) = S(P_2)$. If $S(P_2) = S(P_1)$ we proceed to $P_3$ and continue in this manner. We obtain a (possibly finite) sequence $(k_i)$ of disjointly supported functions and integers $n(1) \leq n(2) \leq \cdots$ with the property that for each each $i$,

\[ S(P_i) = \bigcup_{j=1}^{n(i)} S(k_j) \]

and $k_j \in R(P_j)$ for $j \leq n(i)$. We may assume $k = \sum k_i \in L_p(\mu)$ and proceed to define $\nu$ and $T$ as in the proof of Theorem A, i.e. $d\nu = |k|^p d\mu$ and $Tf = f/k$. Clearly $Q_i = TP_iT^{-1}$ satisfies $Q_i(\chi_{\mathcal{S}(Q_i)}) = \chi_{\mathcal{S}(Q_i)}$ and $Q_iQ_j = Q_{\min(i,j)}$. By Theorem A there are sub $\sigma$-rings $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}$ with $Q_i = \mathcal{E}_{\mathcal{F}_i}$ for $i$.

**Corollary 1.** A monotone decomposition in $L_p(\mu)$, $(1 < p < \infty)$ is unconditional with constant $\leq K_p$, where $K_p$ depends only on $p$. 

Proof. For \( p = 2 \) this is well known. If \( p \neq 2 \), we apply 2.1, and observe, that in the notation of its proof, we have:

\[
Qf = \sum_i \chi_{S(k_i)} \bar{e}_i (\chi_{S(k_i)} : f), \quad f \in L_p(\nu),
\]

(a) for fixed \( j \) the non-zero projections \( f \to \chi_{S(k_j)} \bar{e}_j f \) in \( L_p(S(k_j)) \) are conditional expectations with respect to \( \sigma \)-algebras on \( S(k_j) \), and

(b) the direct \( L_p \)-sum of projections of norm smaller than \( K_p \) has norm smaller than \( K_p \). The rest follows from Theorem B.

Remark. Corollary 1 holds for arbitrary measures \( \mu \). In fact, for any given \( f \in L_p(\mu) \) we can find a sub \( \sigma \)-ring \( \Sigma_0 \subseteq \Sigma \) so that \( L_p(\Sigma_0) \) is separable, contains \( f \) and is an invariant subspace for each projection \( P_n \) (cf. [11], Lemma 1 and its proof). Then by Corollary 1,

\[
\left\| \sum_n \epsilon_n (P_n - P_{n-1}) f \right\| \leq K_p \| f \|
\]

for all \( \epsilon_n = \pm 1 \).

Corollary 2. If \( (X_i) \) is a monotone decomposition for \( L_p(\mu) \), \((1 < p < \infty)\) with each \( X_i \) finite-dimensional, then there is a monotone basis \( (x_i) \) and integers \( 1 = n(0) < n(1) < \cdots \) so that \( X_i = \{ x_j ; n(i-1) \leq j < n(i) \} \).

We proceed to describe more precisely the monotone bases in \( L_p(0,1) \), \( 1 < p < \infty, p \neq 2 \). For clarity of exposition we shall state the results for separable \( L_p(\mu) \) where \( \mu \) is a purely nonatomic probability measure.

A system of sets \( (A_{n,i} ; i \leq 2^n, n = 0,1,2, \cdots) \) is called a dyadic tree if for all \( n \) and \( i \leq 2^n \)

\[
A_{n+1,2i-1} \cap A_{n+1,2i} = \emptyset
\]

and

\[
A_{n+1,2i-1} \cup A_{n+1,2i} = A_{n,i}.
\]

Definition. Let \( 1 < p < \infty \), and let \( (A_{n,i} ; i \leq 2^n, n = 0,1,2, \cdots) \) be a dyadic tree in \( \mathcal{F} \). The generalized Haar system \( (h_k, k = 1,2, \cdots) \) with respect to \( (A_{n,i}) \) is defined as follows:

\[
h_1 = h_{0,1} = \chi_{A_{0,1}} \| \chi_{A_{0,1}} \|_p
\]

and:

\[
h_{2^{n-1}i} = h_{n,i} = H_{n,i} \| H_{n,i} \|_p.
\]
where
\[ H_n = \{ \chi_{A_{n_2^{-1}}} / \mu(A_{n_2^{-1}}) - \chi_{A_{n_2}} / \mu(A_{n_2}) \}, \]
for \( i \leq 2^{n-1}, n \geq 1. \)

The system \((h_n)\) is determined by the conditions: \( h_n \) is a linear combination of \( \chi_{A_{n_2^{-1}}} \) and \( \chi_{A_{n_2}} \), which is positive on \( A_{n_2^{-1}} \) and satisfies:
\[
\|h_n\|_p = 1 \quad \text{and} \quad \int h_n d\mu = 0 \quad (n \geq 1).
\]
If \((A_n)\) are the dyadic intervals in \([0,1]\) and \( \mu \) is the Lebesgue measure on \([0,1]\), this gives the usual Haar system in \( L_p \). It is easily seen, that a generalized Haar system is a monotone basic sequence, which spans the space \( L_p(\beta) \), where \( \beta \) is the \( \sigma \)-algebra generated by the \( A_n \). If \( \beta = S_f \) we must have \( \mu(A_n) \rightarrow 0 \); on the other hand if the \( A_n \) are intervals in \([0,1]\) and \( m(A_n) \rightarrow 0 \), then \( \beta = S_f \).

**Theorem 2.2.** Let \((x_n)\) be a normalized monotone basis for \( L_p(\mu) \), \( \mu \) purely nonatomic, \( 1 < p < \infty, p \neq 2 \). Then there is a measure \( \nu \), an isometry \( T \) of \( L_p(\mu) \) onto \( L_p(\nu) \) which sends \((x_n)\) to a basis \((y_n)\), and a sequence (possibly finite) of disjoint sets \((E_n)\) in \( \beta \), covering \( \Omega \), so that \((y_n)\) is the union of disjoint subsequences \((y_n^\alpha, i = 1, 2, \cdots, n = 1, 2, \cdots)\) where for each \( n \), \((y_n^\alpha, i = 1, 2, \cdots)\) is a permutation of a generalized Haar basis for \( L_p(E_n) \).

**Proof.** By Theorem 2.1 we may assume that \( P_i = E_{i,j} \) for each \( i \), where \( \beta_i \subseteq \beta_{i+1} \subseteq \cdots \) are sub \( \sigma \)-rings of \( \beta \), and \( P_i: L_p(\mu) \rightarrow [x_1, \cdots, x_n] \) are the projections associated with the basis \((x_n)\). For each \( i \), we have: \( L_p(\beta_i) = R(P_i) = [x_1, \cdots, x_n] \) and so \( \beta_i \) is generated by \( i \) atoms. For each \( i \) there are two cases:

1°. \( S(P_i) = S(P_{i-1}) \)

2°. \( S(P_i) \supsetneq S(P_{i-1}) \).

In case 1°, \( \beta_i \) is obtained from \( \beta_{i-1} \) by splitting some set \( A \) in \( \beta_{i-1} \) into two sets. Clearly \( S(x_i) = A \) and \( \int_A x_i = 0 \). In case 2° \( \beta_i \) is obtained from \( \beta_{i-1} \) by adding an atom \( D \) disjoint from the \( i-1 \) atoms of \( \beta_{i-1} \). Then \( P_{i-1} x_D = 0 \) so that \( x_i = \pm x_D / \nu(D)^{1/p} \) (being norm 1). We enumerate all the \( x_i \) obtained in 2° as \( \{ x_i^\alpha: n = 1, 2, \cdots \} \) and for each \( n \) enumerate the functions \( \{ x_i: S(x_i) \subseteq S(x_i^\alpha) \} \) as \( (x_i^\alpha)_{i=1}^\infty \). This is clearly the required partition.

**Remark.** In the above Theorem we could have let \( \nu = \text{Lebesgue measure} \ m, \) on \([0,1]\). Indeed there exist disjoint intervals \( E_n \subseteq [0,1] \) with
$m(E_n) = \mu(S(x^n))$ and a map $\phi$ from $\bigcup \mathcal{F}_i$ into the intervals contained in $[0,1]$ which preserves inclusion, disjointness and measure, such that for any $x_i$ of type $2^c$, $\phi(\{t: x_i > 0\})$ is to the left of $\phi(\{t: x_i < 0\})$. This map extends to an isomorphism of the measure space $(\Omega, \mathcal{F}, \mu)$ onto the Lebesgue measure space on $[0,1]$. Thus to study monotone bases in $L_p(\mu)$, one need only study generalized Haar systems with respect to dyadic trees and one can assume that the interval where $x_i$ is positive is to the left of the interval where it is negative.

We turn now to the question of equivalence of Haar bases for $L_p$, $1 < p \neq 2$. A basis $(x_n)$ is said to be $K$-equivalent to a basis $(y_n)$, $(x_n) \sim (y_n)$, if for all $n$ and all scalars $\alpha_1, \cdots, \alpha_n$,

$$K^{-1} \left\| \sum_{i=1}^n \alpha_i x_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i y_i \right\| \leq K \left\| \sum_{i=1}^n \alpha_i x_i \right\|.$$ 

If $(h_{n,i})$ is a generalized Haar basis for $L_p$, we define its generalized Rademacher functions $r_n$ by:

$$r_n = 2^{(1-n)p}(h_{n,1} + h_{n,2} + \cdots + h_{n,2^{n-1}}).$$

**Theorem 2.3.** There exist two nonequivalent generalized Haar bases for $L_p(0,1)$, $(1 < p < \infty, p \neq 2)$.

**Proof.** Let $(h_{n,i})$, $(r_i)$ denote the classical Haar and Rademacher systems. By Khintchine’s inequality (cf. [12]), $(r_i)$ is equivalent to the usual basis of $l_2$. We shall construct a generalized system $(h_{n,i}')$, $(r_i')$ so that $(r_{2^n})$ is equivalent to the usual basis for $L_p$, and hence $(h_{n,i}') \not\sim (h_{n,i})$.

It is easy to check, that if $h = a\chi_{E_1} - b\chi_{E_2}$ is a generalized Haar function, then $\|h_{E_i}\|$ approaches 1 as $m(E_1)/m(E_2) \to 0$. (This does not happen of course for $p = 1$).

We shall have $(r_{2^n}) \sim$ usual basis of $l_p$ if there are disjoint sets $E_n$ so that:

1. \( \int_{E_n} \left| r_{2^k} \right|^p > 1 - 4^{-(k+1)p}, \quad k = 1, 2, \cdots \)

and

2. \( \int_{E_n} \sum_{j=1}^{k-1} \left| r_{2^j} \right|^p < 4^{-(k+1)p}, \quad k = 1, 2, \cdots \)

(see [7], proof of Theorem 2).

Let $h_{0,1} = 1$, $h_{1,1} = h_{1,1}$, and assume that $(h_{n,i})$ and $E_j$ are chosen for $1 \leq k \leq 2n - 1$, $i \leq 2^{k-1}$, $1 \leq j \leq n - 1$, so that (1) and (2) hold for
$k = 1, \ldots, n - 1$. Let $(A_{k,i}, \quad k \leq 2n - 1, \quad i \leq 2^k)$ be the underlying intervals. For each $i \leq 2^{2n-1}$, divide $A_{2n-1,i}$ into two disjoint intervals $A_{2n,2i-1}$ and $A_{2n,2i}$ with $m(A_{2n,2i})$ so small that $\|h_{2n,i}|_{A_{2n,2i}}\| > 1 - 4^{-(n+1)p}$, and $\sum_{i=2^{2n-1}}^{2^{2n-1}} m(A_{2n,2i}) \leq \epsilon_n$, $\epsilon_n > 0$ being chosen so that $M(E) \leq \epsilon_n$ implies

$$\int_E \sum_{j \geq n-1} |r_{2j}^p| < 4^{-(n+1)p}.$$ 

Let $E = \bigcup_{i \geq 2^{2n-1}} A_{2n,2i}$. Then we have:

$$\int_{E_n} \left| r_{2n}^p \right| = \frac{1}{2^{n-1}} \sum_{i=1}^{2^{n-1}} \|h_{2n,i}|_{A_{2n,2i}}\|^p > 1 - 4^{-(n+1)p}.$$ 

Thus (1) and (2) hold for $k = n$. Define now the functions $h'_{2n+1,i} \quad i \leq 2^n$ by splitting each $A_{2n,i}$ into two intervals of equal measure. (This ensures that $m(A_{n,i}) \to 0$ and so $(h_{n,i})$ is a basis for all of $L_p$).

Using an idea of J. Hennefeld [5], we can now prove:

**COROLLARY.** There is an uncountable family of mutually nonequivalent generalized Haar bases for $L_p$.

**Proof.** Let $(E_n)$ be a partition of $[0,1]$ into infinitely many disjoint adjacent intervals, ordered from left to right. Define part of the tree $(A_{n,i})$ as follows: for any $n \geq 1$ let $A_{n,2^{n-1}} = E_n$, and $A_{n,2^n} = \bigcup_{j > n} E_j$. Now, given a sequence $(\epsilon_n)$, $\epsilon_n = \pm 1$, complete the tree $(A_{n,i})$ so that the system $(h_{n,i}) = \mathcal{H}$ satisfies the condition that for $\epsilon_m = 1$ the sequence $\{h \in \mathcal{H}; S(h) \subseteq E_m\}$ in its natural ordering is equivalent to the usual Haar basis (without constant term), while for $\epsilon_m = -1$ it is equivalent to the basis $h'_{n,i}$ of 2.3. Different sequences $(\epsilon_n)$ yield non-equivalent systems $(h_{n,i})$.

**Questions.** (1) Does every generalized Rademacher system span a complemented subspace of $L_p$? If so could this be used to construct an $L_p$ space not isomorphic to any of those already known?

(2) Do there exist two non-permutatively equivalent generalized Haar bases? (We can show that (for $p > 2$) some permutation of the generalized Haar basis constructed in 2.3 has its generalized Rademacher system equivalent to the unit vector basis of $l_2$.)

3. **Monotone bases in $L_1$.** Monotone bases in $L_1$ are also built from generalized Haar bases, however the "interlace" is somewhat more involved, due to the larger variety of contractive projections in $L_1$ (cf. Theorem A(ii).):
THEOREM 3.1. Let \((x_k)\) be a normalized monotone basis for \(L_1(\mu)\), \(\mu\) purely non-atomic. Then there is an isometry \(T\) of \(L_1(\mu)\) onto some \(L_1(\nu)\), which sends \((x_k)\) to a basis \((y_k)\), and a sequence (possibly finite) of disjoint sets \(E_n\) in \(\mathcal{F}\), covering \(\Omega\), so that \((y_k)\) is the union of disjoint subsequences \((y^*_n, i = 1, 2, \cdots)\), \(n = 1, 2, \cdots\), where for each \(n\), the sequence: \(\chi_{E_n} / \| \chi_{E_n} \|\), \(y^*_n, \cdots\) is a generalized Haar basis for \(L_1(E_n)\). Moreover, \(y^*_n = c_n \chi_{E_n} + f_n\), where \(\| f_n \| \leq \| c_n \chi_{E_n} \|\) and \(f_n\) is a combination of the elements \((y_k)\) preceding \(y^*_n\) in the original sequence \((y_k)\).

Proof. Let \((P_n)\) be the projections associated with the basis \((x_n)\). Using Theorem A(ii) and the proof of Theorem 2.1, we get an isometry \(T\) of \(L_1(\mu)\) onto some \(L_1(\nu)\) and a sequence of sub \(\sigma\)-rings \(\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}\) so that the projections \(Q_i = TP_iT^{-1}\) have the form \(Qf = E_{\mathcal{F}_i}f + N_i(f - S(\mathcal{F}_i)), N_i\) being some norm 1 operator from \(L_1(\Omega, \mathcal{F}_i, \nu)\) to \(L_1(\Omega, \mathcal{F}_i, \nu)\). Let \(y_i = Tx_i\). We have two cases: 1° \(S(Q_i) = S(Q_{i+1})\) and 2° \(S(Q_i) \supseteq S(Q_{i+1})\). In case 1°, \(\mathcal{F}_i\) is obtained by splitting an atom \(A\) in \(\mathcal{F}_{i-1}\), and \(y_i\) is a Haar function supported on \(A\), while in case 2°, \(\mathcal{F}_i\) is obtained by adding an atom \(A\) disjoint from \(\mathcal{F}_{i-1}\). In the latter case, \(Q_{i-1}(\chi_A - N_{i-1}\chi_A) = 0\), so \(y_i = \varphi(\chi_A - N_{i-1}\chi_A)\), where \(\| N_{i-1}\chi_A \| \leq \| \chi_A \|\) and \(N_{i-1}\chi_A\) is \(\mathcal{F}_{i-1}\)-measurable.

In the rest of this section, we examine the question when two generalized Haar bases in \(L_1(\mu)\) are equivalent. If \((h_n)\) is such a basis, then a sequence \((h_{ni}(n), n = 0, 1, 2, \cdots)\) will be called a chain if \(S(h_{ni}(n)) \subseteq \mathcal{S}(h_{ni-1}(n-1))\) for all \(n\). Now, two generalized Haar bases \((h_n)\) and \((g_n)\) are equivalent if (and only if) there is \(K\) so that every chain of \((h_n)\) is \(K\)-equivalent to the corresponding chain in \((g_n)\). In fact, suppose that \((h_n)\) is built on the dyadic tree of sets \((A_n)\). Then \([h_{ki}, 1 \leq i \leq 2^{k-1}, 0 \leq k \leq n] = [\chi_{A_{ni}}, i \leq 2^n]\) and any operator on this space attains its norm at one of the \(\chi_{A_{ni}},\) since the convex hull of \{\(\pm \chi_{A_{ni}}/m(A_{ni}), i \leq 2^n\}\) is the unit ball of \([\chi_{A_{ni}}, i \leq 2^n]\). But each \(\chi_{A_{ni}}\) is contained in the span of a chain.

Thus it is enough to consider the equivalence of chains. For simplicity, we shall consider only the chain \((h_n)\), however the results evidently apply to any chain.

THEOREM 3.2. Let \((h_n)\) be the generalized Haar system based on the dyadic tree of sets \((A_n)\), and let \((g_n)\) be the generalized Haar system based on \((B_n)\). Let \(T h_n = g_{n}, n,\) and define

\[
P_n = \int_{B_{n+1}} T(\chi_{A_{n+1}}/m(A_{n+1})); \quad q_n = \int_{A_{n+1}} T^{-1}(\chi_{B_{n+1}}/m(B_{n+1})).
\]

Then \((h_n) \sim (g_n)\) iff \(M = \max\{\var{p_n}, \var{q_n}\} < \infty\), and the equivalence constant \(K\) satisfies: \(M \leq K \leq 2M + 3\)

\[
\left(\text{where, as usual, } \var{p_n} = \sum_{n=1}^{\infty} |p_n - p_{n+1}|\right).
\]
Proof. Let \( e_{n,i} = \chi_{A_n}/m(A_n). \) We have:

\[
(3) \quad e_{n,1} = e_{n-1,1} + 2c_n h_{n,1}, \quad \text{where}
\]

\[
(4) \quad c_n = m(A_{n,2})/m(A_{n-1,1}).
\]

(check their integrals on \( A_{n-1,1} \) and on \( A_{n,2} \)).

Thus for any \( k \leq n - 1 \) and \( i \leq 2^k \), we have:

\[
\int_{B_{k,i}} T e_{n,1} = \int_{B_{k,i}} T e_{n-1,1} + 2c_n \int_{B_{k,i}} g_{n,1} = \int_{B_{k,i}} T e_{n-1,1} = \int_{B_{k,i}} T e_{k,1},
\]

and so

\[
\int_{B_{k,2}} T e_{n,1} = \int_{B_{k-1,1}} T e_{n,1} - \int_{B_{k,1}} T e_{n,1} = p_{k-1} - p_k.
\]

Now, \( T e_{n,1} \) is constant on \( B_{k,2} \) (\( k \leq n \)), and \( B_{n,1} \), so that:

\[
\| T e_{n,1} \| = \sum_{k=1}^{n} \left| \int_{B_{k,2}} T e_{n,1} \right| + \left| \int_{B_{n,1}} T e_{n,1} \right| = \sum_{k=1}^{n} |p_{k-1} - p_k| + |p_n|.
\]

Finally, \( e_{n,2} = e_{n-1,1} - 2(1 - c_n) h_{n,1} \), similarly to (3), so \( \| T e_{n,2} \| \leq \| T e_{n-1,1} \| + 2 \), and the unit ball of \( [h_{n,1}, n = 0, 1, \cdots] \) is the closed convex hull of the set \( \{ \pm e_{n,2}, n = 1, 2, \cdots \} \).

From (3) and the definition of the Haar functions \( g_{n,i} \) we get that

\[
(5) \quad p_n = m(B_{n,1}) \left\{ 1 + \sum_{k=1}^{n} c_k / m(B_{k,1}) \right\}.
\]

Applying Stolz's theorem (i.e. the discrete version of L'Hospital's rule, cf. [8] p. 77, Remark 5) to \( p_n \) and putting:

\[
(6) \quad d_n = m(B_{n,2})/m(B_{n-1,1}),
\]

we see that if \( \lim_n c_n/d_n = \lambda \) exists then \( \lim_n p_n = \lambda \). Given a sequence \( (c_n) \), there is a generalized Haar system \( (h_{n,i}) \) for which (4) holds provided that:

\[
(7) \quad 0 < c_n < 1 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n = \infty
\]

(The latter condition ensures that \( m(A_{n,1}) = \prod_{n=1}^{\infty} (1 - c_i)^{n} \to 0 \)). In particular, if we take \( c_n = (n + 1)^{-\alpha} \), for fixed \( 0 < \alpha \leq 1 \), then different values of \( \alpha \) give mutually non-equivalent generalized Haar bases.
The considerations above motivate:

**Theorem 3.3.** Let \((h_n, t), (g_n, \iota)\) be two generalized Haar systems, built on the dyadic trees \((A_n, t), (B_n, \iota)\) respectively. Let \(c_n = m(A_n, t)/m(A_{n-1}, t), d_n = m(B_n, \iota)/m(B_{n-1}, \iota)\). If

\[
\text{var} \left( \frac{c_n}{d_n} \right), \text{var} \left( \frac{d_n}{c_n} \right) \leq M < \infty
\]

then the chains \((h_n, t)\) and \((g_n, \iota)\) are equivalent (with constant \(\leq 2M + 3\)).

**Proof.** In formula (5), putting: \(h_{k, t} = g_{k, \iota}\) we get

\[
1 = m(B_n, \iota) \left\{ 1 + \sum_{k=1}^{n} d_k / m(B_k, \iota) \right\},
\]

so

\[
p_n = \left\{ 1 + \sum_{k=1}^{n} c_k / m(B_k, \iota) \right\} / \left\{ 1 + \sum_{k=1}^{n} d_k / m(B_k, \iota) \right\}.
\]

It is enough therefore to apply the following:

**Lemma.** Let \((a_n), (b_n)\) be sequences of reals with all \(b_n > 0\), and let

\[
A_n = \sum_{k=1}^{n} a_k, \quad B_n = \sum_{k=1}^{n} b_k.
\]

Then

\[
\text{var} \left( \frac{A_n}{B_n} \right) \leq \text{var} \left( \frac{a_n}{b_n} \right).
\]

**Proof.** Let \(a_k = t_kb_k\). Using Abel's transform, we have:

\[
\frac{A_n}{B_n} = B_n^{-1} \sum_{k=1}^{n} t_kb_k = B_n^{-1} \sum_{k=1}^{n-1} (t_k - t_{k+1})B_k + t_n, \text{ which gives:}
\]

\[
\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} = \left( \frac{1}{B_n} - \frac{1}{B_{n+1}} \right) \sum_{k=1}^{n} (t_{k+1} - t_k)B_k, \text{ so that:}
\]

\[
\text{var} \left( \frac{A_n}{B_n} \right) = \sum_{n=1}^{\infty} \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right|
\]

\[
\leq \sum_{n=1}^{\infty} \left( \frac{1}{B_n} - \frac{1}{B_{n+1}} \right) \sum_{k=1}^{n} |t_{k+1} - t_k|B_k
\]

\[
= \sum_{k=1}^{\infty} |t_{k+1} - t_k| \cdot B_k \sum_{n=k}^{\infty} \left( \frac{1}{B_n} - \frac{1}{B_{n+1}} \right) \leq \text{var} (t_k).
\]
REMARKS. (1) It is conceivable that the condition in Theorem 3.3 is also necessary. We can prove only that if \((h_n, \gamma)\) and \((g_n, \delta)\) are equivalent and either \(\inf c_n > 0\) or \(\inf d_n > 0\), then \(\var(c_n/d_n) < \infty\).

(2) If \((h_n, \gamma)\) is a generalized Haar basis for \(L_p\), then the chain \((h_n; n = 0, 1, \ldots)\) spans a space isometric to \(L_p\). In \(L_1\) these chains are conditional bases for \(l_1\) (by (7), (3) and [6], Lemma 2). As, shown above, there is an uncountable family of mutually non-equivalent such chains.

For \(1 < p \neq 2\), we do not know if all chains in \(L_p\) are equivalent.

REFERENCES


Received July 17, 1974.

THE OHIO STATE UNIVERSITY
PACIFIC JOURNAL OF MATHEMATICS
EDITORS
RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS
E. F. BECKENBACH      B. H. NEUMANN      F. WOLF      K. YOSHIDA

SUPPORTING INSTITUTIONS
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics
All Rights Reserved
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Waleed A. Al-Salam and A. Verma, A fractional Leibniz q-formula</td>
<td>1</td>
</tr>
<tr>
<td>Robert A. Bekes, Algebraically irreducible representations of $L_1(G)$</td>
<td>11</td>
</tr>
<tr>
<td>Thomas Theodore Bowman, Construction functors for topological semigroups</td>
<td>27</td>
</tr>
<tr>
<td>Stephen LaVern Campbell, Operator-valued inner functions analytic on the closed disc. II</td>
<td>37</td>
</tr>
<tr>
<td>Leonard Eliezer Dor and Edward Wilfred Odell, Jr., Monotone bases in $L_p$</td>
<td>51</td>
</tr>
<tr>
<td>Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 = f$</td>
<td>63</td>
</tr>
<tr>
<td>Y. Gordon, Unconditional Schauder decompositions of normed ideals of operators between some $l_p$-spaces</td>
<td>71</td>
</tr>
<tr>
<td>Gary Grefsrud, Oscillatory properties of solutions of certain nth order functional differential equations</td>
<td>83</td>
</tr>
<tr>
<td>Irvin Roy Hentzel, Generalized right alternative rings</td>
<td>95</td>
</tr>
<tr>
<td>Zensiro Goseki and Thomas Benny Rushing, Embeddings of shape classes of compacta in the trivial range</td>
<td>103</td>
</tr>
<tr>
<td>Emil Grosswald, Brownian motion and sets of multiplicity</td>
<td>111</td>
</tr>
<tr>
<td>Donald LaTorre, A construction of the idempotent-separating congruences on a bisimple orthodox semigroup</td>
<td>115</td>
</tr>
<tr>
<td>Pjek-Hwee Lee, On subrings of rings with involution</td>
<td>131</td>
</tr>
<tr>
<td>Marvin David Marcus and H. Minc, On two theorems of Frobenius</td>
<td>149</td>
</tr>
<tr>
<td>Michael Douglas Miller, On the lattice of normal subgroups of a direct product</td>
<td>153</td>
</tr>
<tr>
<td>Grattan Patrick Murphy, A metric basis characterization of Euclidean space</td>
<td>159</td>
</tr>
<tr>
<td>Roy Martin Rakestraw, A representation theorem for real convex functions</td>
<td>165</td>
</tr>
<tr>
<td>Louis Jackson Ratliff, Jr., On Rees localities and $H_i$-local rings</td>
<td>169</td>
</tr>
<tr>
<td>Simeon Reich, Fixed point iterations of nonexpansive mappings</td>
<td>195</td>
</tr>
<tr>
<td>Domenico Rosa, $B$-complete and $B_r$-complete topological algebras</td>
<td>199</td>
</tr>
<tr>
<td>Walter Roth, Uniform approximation by elements of a cone of real-valued functions</td>
<td>209</td>
</tr>
<tr>
<td>Helmut R. Salzmann, Homogene kompakte projektive Ebenen</td>
<td>217</td>
</tr>
<tr>
<td>Jerrold Norman Siegel, On a space between $BH$ and $B_\infty$</td>
<td>235</td>
</tr>
<tr>
<td>Robert C. Sine, On local uniform mean convergence for Markov operators</td>
<td>247</td>
</tr>
<tr>
<td>James D. Stafney, Set approximation by lemniscates and the spectrum of an operator on an interpolation space</td>
<td>253</td>
</tr>
<tr>
<td>Árpád Száz, Convolution multipliers and distributions</td>
<td>267</td>
</tr>
<tr>
<td>Kalathoor Varadarajan, Span and stably trivial bundles</td>
<td>277</td>
</tr>
<tr>
<td>Robert Breckenridge Warfield, Jr., Countably generated modules over commutative Artinian rings</td>
<td>289</td>
</tr>
<tr>
<td>John Yuan, On the groups of units in semigroups of probability measures</td>
<td>303</td>
</tr>
</tbody>
</table>