

Pacific Journal of Mathematics

**ON THE EXISTENCE OF GLOBAL CLASSICAL SOLUTION OF
INITIAL-BOUNDARY VALUE PROBLEM FOR $\square u - u^3 = f$**

YUKIYOSHI EBIHARA, MITSUHIRO NAKAO AND TOKUMORI NANBU

ON THE EXISTENCE OF GLOBAL CLASSICAL
SOLUTION OF INITIAL-BOUNDARY VALUE
PROBLEM FOR $\square u - u^3 = f$

YUKIYOSHI EBIHARA, MITSUHIRO NAKAO AND TOKUMORI NANBU

In this paper we shall give a sufficient condition under which an initial-boundary value problem for $\square u - u^3 = f$ has a global classical solution.

1. Introduction. Let $\Omega \subset R^3$ be an open bounded domain with sufficiently smooth boundary $\partial\Omega$. In this note we are concerned with the existence of a global solution of the initial-boundary value problem:

$$\frac{\partial^2}{\partial t^2} u - \Delta u + \gamma u^3 = f(x, t) \quad \text{for } x \in \Omega, t > 0,$$

$$(*) \quad u(x, 0) = u_0(x), \frac{\partial}{\partial t} u(x, 0) = u_1(x), x \in \Omega,$$

and

$$u(x, t)|_{\partial\Omega} = 0 \quad \text{for } t \geq 0,$$

where Δ is the Laplacian in R^3 and $\gamma = -1$.

For the equation (*) with $\gamma = +1$ instead of -1 , as is well known, the existence of a global classical solution was proved by J. Sather [4]. His method, however, depends largely on the monotonically increasingness of the term u^3 , and is not applicable to our problem in its original form.

On the other hand D. H. Sattinger [5] introduced the concept of potential well (stable set) to show the existence of global but generalized solutions of the initial-boundary problems of hyperbolic equations with non-monotonic nonlinear terms, though in the case $f(x, t) \equiv 0$. The method of potential well is useful also for nonlinear partial differential equations of other types (Lions [2], Tsutsumi [6]).

Now, a local existence of a classical solution for (*) is known (Ebihara [1]), but that of a global one seems to be unknown and the aim of this note is to give it by combining the method of Sather's with the one of Sattinger's.

Roughly speaking our result is: Let $\{u_0, u_1\}$ belong to the stable set and be sufficiently smooth, and moreover let f have small norm and be smooth. Then (*) admits a global classical solution.

Though we treat only a typical equation with also typical nonlinear term, our method should be applicable to more general ones.

2. Preliminaries. Throughout this paper the functions considered are all real valued and the notations are as usual (e.g. Lions [2], Mizohata [3]). In this section we offer some lemmas which will be used later.

LEMMA 1 (Sobolev). (i) *If $1 \leq q \leq 6$, we have*

$$\|u\|_{L^q(\Omega)} \leq C_0(q, \Omega) \|u\|_{H^1(\Omega)} \quad \text{for } u \in \dot{H}^1(\Omega).$$

(ii) *If k is a nonnegative integer, we have*

$$\|u\|_{C^k(\bar{\Omega})} \leq C_1(k, \Omega) \|u\|_{H^{2+k}(\Omega)} \quad \text{for } u \in H^{2+k}(\Omega).$$

For brevity we use the notations $|\cdot|$, $\|\cdot\|$, $|\cdot|_q$ for $\|\cdot\|_{L^2(\Omega)}$, $\|\nabla \cdot\|_{L^2(\Omega)}$, $|\cdot|_{L^q(\Omega)}$, respectively.

We define ‘kinetic’ and ‘potential’ energies associated with our equation by the functionals

$$K(u) = \int_{\Omega} \frac{1}{2} |u_t(x, t)|^2 dx = \frac{1}{2} |u_t(t)|^2$$

and

$$J(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u(x, t)|^2 - \frac{1}{4} u^4 \right) dx = \frac{1}{2} \|u(t)\|^2 - \frac{1}{4} |u(t)|_4^4,$$

and according to [5] we put

$$d = \inf_{\substack{u \in \dot{H}^1(\Omega) \\ u \neq 0}} J(\lambda_1 u),$$

where $\lambda_1 = \lambda_1(u)$ (≥ 0) is the first value of $\lambda \geq 0$ at which $J(\lambda u)$ begins to decrease. Then with the aid of Lemma 1, we have (see also Tsutsumi [6]):

LEMMA 2. *The number d satisfies*

$$0 < \frac{1}{4C_0^4(4, \Omega)} \leq d < \infty.$$

Now the potential well W is defined as

$$W = \{u \in \dot{H}^1(\Omega) \mid 0 \leq J(\lambda u) < d \text{ for } 0 \leq \lambda \leq 1\}.$$

Then we have:

LEMMA 3 (Sattinger). *The set W is bounded in $\dot{H}^1(\Omega)$.*

For convenience we say the initial data $\{u_0, u_1\}$ belongs to the stable set if

$$u_0 \in W \quad \text{and} \quad K(u_1) + J(u_0) < d.$$

Here we state our hypotheses on the initial values u_0, u_1 , and inhomogeneous term f . For this, let us consider the eigenfunctions $\{\psi_k\}$ for the Laplacian Δ with zero boundary condition:

$$\psi_k \in \dot{H}^1(\Omega) \quad \text{and} \quad \Delta \psi_k = \mu_k \psi_k \quad \text{in } \Omega \quad (k = 1, 2, \dots),$$

where μ_k is the eigenvalue for ψ_k .

With respect to the regularity of ψ_k , it is well known that $\{\psi_k\}$ is involved in $H^6(\Omega)$ (recall $\partial\Omega$ is sufficiently smooth).

We introduce the spaces of the admissible initial data as follows:

$$V_j \equiv \{\text{closed linear extension of the eigenfunctions } \{\psi_k\} \text{ in } H^{6-2j}\},$$

$$j = 0, 1,$$

and assume

$$A_1. \quad u_0 \in V_0 \cap W \quad \text{and} \quad u_1 \in V_1.$$

Regarding the energy source function it is required that

$$A_2. \quad f \in C^4([0, \infty); L^2) \bigcap_{k=1}^4 C^{4-k}([0, \infty); H^k \cap \dot{H}^{k-1}).$$

Finally we assume $f \in L^1([0, \infty); L^2)$ and

$$A_3. \quad E_0 + 2 \sqrt{E_0 + \left(\int_0^\infty |f(t)| dt \right)^2} \int_0^\infty |f(t)| dt < d,$$

where

$$E_0 = K(u_1) + J(u_0) \quad (\text{total energy of the initial data}).$$

Note that A_1 and A_3 imply $\{u_0, u_1\}$ belongs to the stable set.

3. Theorem. In this section we prove:

THEOREM. *Under the assumption $A_1, A_2,$ and $A_3,$ the problem (*) has a classical solution $u(x, t) \in C^2(\bar{\Omega} \times [0, \infty))$.*

Proof. The Galerkin's method is employed. Let $\{u_{0m}\}$ and $\{u_{1m}\}$ be sequences such that

$$(1) \quad u_{0m} = \sum_{i=1}^m \alpha_{im} \psi_i \rightarrow u_0 \quad \text{in } H^6 \cap H^1,$$

and

$$u_{1m} = \sum_{i=1}^m \beta_{im} \psi_i \rightarrow u_1 \quad \text{in } H^4.$$

This is possible by the assumption A_1 . By $A_1, A_3,$ and the continuity of $K(u)$ and $J(u)$ with respect to H^1 -topology, we may assume

$$(2) \quad u_{0m} \in W$$

and

$$(3) \quad K(u_{1m}) + J(u_{0m}) + 2 \sqrt{K(u_{1m}) + J(u_{0m}) + \left(\int_0^\infty |f| dt \right)^2} \int_0^\infty |f| dt < d.$$

Let us consider the approximate solutions:

$$(4) \quad u_m(t) = u_m(\cdot, t) = \sum_{k=1}^m \lambda_k^m(t) \psi_k \quad (k = 1, 2, \dots, m)$$

which are determined by the following system of ordinary differential equations:

$$(5) \quad (D_t^2 u_m(t), \psi_K) + ((u_m(t), \psi_K)) - (u_m^3(t), \psi_K) = (f(t), \psi_K)$$

with initial values

$$(6) \quad \begin{cases} u_m(0) = u_{0m} \\ D_t u_m(0) = u_{1m}, \end{cases}$$

where (\cdot, \cdot) denotes L^2 -innerproduct and $((\cdot, \cdot))$ denotes $(\nabla \cdot, \nabla \cdot)$.

Clearly $u_m(t)$ exists in some interval, say, in $[0, t_m]$, $t_m > 0$. Multiplying (5) by $\lambda_k^m = D_t \lambda_k^m$ and summing over k from 1 to m , we obtain

$$(7) \quad K(u'_m(t)) + J(u_m(t)) = K(u_{1m}) + J(u_{0m}) + \int_0^t (f(\tau), u'_m(\tau)) d\tau$$

for $t \in [0, t_m]$,

where ' denotes D_t .

By use of this equation we verify:

$$(8) \quad u_m(t) \in W \quad \text{for } \forall t \in [0, t_m].$$

Indeed, suppose that (8) is false and let t^* be the smallest time for that $u_m(t^*) \notin W$. Then in virtue of the continuity of $u_m(t)$ we see $u_m(t^*) \in \partial W$ and hence we have ([2], [5], [6])

$$(9) \quad J(u_m(t^*)) = d.$$

On the other hand, setting $M = \sup_{t \in [0, t^*]} |u'_m(t)|$, (7) implies

$$\frac{1}{2} M^2 \leq K(u_m) + J(u_{0m}) + M \int_0^\infty |f(t)| dt.$$

Here we have used the fact that $J(u) \geq 0$ if $u \in W$. From this we have

$$M \leq 2 \sqrt{K(u_{1m}) + J(u_{0m}) + \left(\int_0^\infty |f(t)| dt \right)^2}.$$

Hence,

$$\begin{aligned} J(u_m(t^*)) &\leq K(u'_m(t^*)) + J(u_m(t^*)) \\ &= K(u_{1m}) + J(u_{0m}) + \int_0^{t^*} (f(\tau), u'_m(\tau)) d\tau \\ &\leq K(u_m) + J(u_{0m}) + M \int_0^\infty |f(t)| dt \\ &\leq K(u_{1m}) + J(u_{0m}) \\ &\quad + 2 \sqrt{K(u_{1m}) + J(u_{0m}) + \left(\int_0^\infty |f| dt \right)^2} \int_0^\infty |f| dt \\ &< d \quad (\text{by } A_3), \end{aligned}$$

which is a contradiction to (9). Thus (8) is valid.

By (8) and Lemma 3, $|u'_m|$ and $\|u_m\|_{\dot{H}^1}$ are in fact majorized by a constant independent of m and we conclude that $u_m(t)$ exists in $[0, \infty)$ and the inequality

$$(10) \quad |D_t u_m(t)|^2 + \|u_m(t)\|^2 \leq C_0 \quad \text{for } t \in [0, \infty)$$

holds.

This is the key estimate for our arguments and the estimations of higher derivatives of u_m are carried out on the basis of (10). For the problem (*) with $\gamma = 1$, we note, this is easily derived from the monotone increasingness of u^3 .

Now we proceed to consideration of higher derivatives of u_m , which is the same as Sather's [4] and sketched briefly.

For arbitrarily fixed $T > 0$, the estimations

$$(11) \quad |D_t^{k+1} u_m(t)|^2 + \|D_t^k u_m(t)\| \leq C_k(T) \\ \text{for } k = 1, 2, 3, 4 \quad t \in I = [0, T],$$

hold, where $C_k(T)$ are constants depending on T but independent of m . Indeed by the linearity of (5) with respect to ψ_k we obtain

$$(12) \quad (D_t^{2+j} u_m(t), D_t^{j+1} u_m(t)) + ((D_t^j u_m(t), D_t^{j+1} u_m(t)) \\ - (D_t^j u_m^3, D_t^{j+1} u_m)) = (D_t^j f, D_t^{j+1} u_m).$$

(12) with $j = 1$ implies

$$\begin{aligned} \frac{d}{dt} \{ |D_t^2 u_m(t)|^2 + \|D_t u_m\|^2 \} &= 2(3u_m^2 D_t u_m, D_t^2 u_m) + 2(D_t f, D_t^2 u_m) \\ &\leq 3(|u_m^2 D_t u_m|^2 + |D_t^2 u_m|^2) \\ &\quad + (|D_t f|^2 + |D_t^2 u_m|^2) \\ &\leq \text{const.} (|u_m|_6^4 |D_t u_m|_6^2 + |D_t^2 u_m|^2) + |D_t f|^2 \\ &\leq \text{const.} (\|u_m\|^4 \|D_t u_m\|^2 + |D_t^2 u_m|^2) + |D_t f|^2 \\ &\leq \text{const.} (|D_t^2 u_m|^2 + \|D_t u_m\|^2) + |D_t f|^2, \end{aligned}$$

here we used Hölder's inequality, Lemma 1 and (10). Applying the Gronwall's lemma we get

$$(13) \quad |D_t^2 u_m(t)|^2 + \|D_t u_m(t)\|^2 \\ \leq \left\{ |(D_t^2 u_m)_0|^2 + \|(D_t u_m)_0\|^2 + \int_0^T |D_t f|^2 dt \right\} \times e^{\text{const. } T} \quad \text{for } \forall t \in I,$$

where $(D_t^K u_m)_0$ denotes the value of $D_t^K u_m(x, t)$ at $t = 0$.

$\|(D_t u_m)_0\|$ is obviously uniformly bounded in m . For the bound of $\|(D_t^2 u_m)_0\|$, set $t = 0$ in (5) to get

$$((D_t^2 u_m)_0 - (\Delta u_m)_0 - (u_m^3)_0 - (f)_0, \psi_k) = 0, \quad 1 \leq k \leq m,$$

and hence

$$|(D_t^2 u_m)_0| = |\Delta u_{m0} + P_m f_0 + P_m u_{m0}^3|,$$

where P_m is the orthogonal projection onto the m -dimensional subspace of L^2 with basis $\{\psi_1, \psi_2, \dots, \psi_m\}$. This implies, with the aid of Lemma 1, the uniformly boundedness of $|(D_t^2 u_m)_0|$.

Combining these uniform estimates of initial values with (13), we obtain (11) for $k = 1$.

The succession of similar procedure gives (11) for $k = 2, 3, 4$.

Now by the standard arguments of the approximate solutions we conclude, after the extraction of suitable subsequence if necessary, the following:

$$D_t^k u_m \rightarrow D_t^k u \quad \text{in } L^2(\Omega \times I) \quad \text{for } 0 \leq k \leq 4,$$

$$D_t^k u_m(t) \rightarrow D_t^k u(t) \quad \text{in } L^2(\Omega) \quad \text{uniformly for } t \in I, \quad 0 \leq k \leq 3,$$

$$D_t D_t^k u_m(t) \rightarrow D_t D_t^k \quad \text{weakly in } L^2, \quad 0 \leq j, \quad k \leq 3, \quad t \in I,$$

$$\text{where } D_t \text{ denotes } \frac{\partial}{\partial x_j},$$

$$D_t^k u(t) \in \mathring{H}_1, \quad 0 \leq k \leq 3,$$

$$D_t^k u_m(t) \rightarrow D_t^k u(t) \quad \text{weakly in } L^2, \quad t \in I,$$

$$|D_t^k u(t) - D_t^k u(\tau)| \leq \text{const.} |t - \tau|,$$

$$|D_t D_t^k u(t) - D_t D_t^k u(\tau)| \leq \text{const.} |t - \tau|, \quad 0 \leq j, \quad k \leq 3,$$

$$|D_t^k u_m(x, t)| \leq \text{const.} \quad 0 \leq k \leq 3, \quad (x, t) \in \bar{\Omega} \times I,$$

and

$$D_t^k u_m^3(t) \rightarrow D_t^k u^3(t) \quad \text{in } L^2 \quad \text{uniformly in } t \in I, \quad 0 \leq k \leq 2.$$

The limit-function u satisfies of course:

$$(D_t^{2+j} u(t), v) + ((D_t^j u(t), v)) - (D_t^j u^3(t), v) = (D_t^j f, v)$$

$$\text{for } \forall v \in H^1, \quad \forall t \in I, \quad 0 \leq \forall j \leq 2.$$

Moreover applying the well-known regularity results concerning weak solution of elliptic equation, we obtain finally

$$u \in C^4(I, H_0) \bigcap_{k=1}^4 C^{4-k}(I, H^k \cap \dot{H}^1).$$

Since $\bigcap_{k=0}^2 C^k(I, H^{4-k}(\Omega)) \subset C^2(\bar{\Omega} \times I)$ holds (c.f. Lemma 1, (ii)), we conclude that u belongs to the class $C^2(\bar{\Omega} \times I)$ and is the classical solution on $\bar{\Omega} \times I$ of the problem (*).

From the arbitrariness of T and the uniqueness of the classical solution on $[0, T]$ (it is obvious) we can construct, as is usual, the classical solution u on $\bar{\Omega} \times [0, \infty)$. The proof of theorem is now completed.

REFERENCES

1. Y. Ebihara, *On the local classical solutions of the mixed problem for nonlinear wave equations*. (To appear).
2. J. L. Lions, *Quelques méthodes de résolution de problèmes aux limites non linéaires*, Dunod, Paris, (1969).
3. S. Mizohata, *The theory of partial differential equations*, Iwanami, Japan, 1965.
4. J. Sather, *The existence of a global classical solution of the initial boundary value problem for $\square u + u^3 = f$* , Arch. Rat. Mech. Anal., **22** (1966), 292-307.
5. D. H. Sattinger, *On global solution of nonlinear hyperbolic equations*, Arch. Rat. Mech. Anal., **30** (1968), 148-172.
6. M. Tsutsumi, *Existence and nonexistence of global solutions for nonlinear parabolic equations*, Publ. R.I.M.S., Kyoto Univ., **8** (1972), 211-229.

Received July 17, 1974.

FUKUOKA UNIVERSITY,
 KYUSHU UNIVERSITY,
 AND
 KYUSHU UNIVERSITY.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics
All Rights Reserved

Waleed A. Al-Salam and A. Verma, <i>A fractional Leibniz q-formula</i>	1
Robert A. Bekes, <i>Algebraically irreducible representations of $L_1(G)$</i>	11
Thomas Theodore Bowman, <i>Construction functors for topological semigroups</i>	27
Stephen LaVern Campbell, <i>Operator-valued inner functions analytic on the closed disc. II</i>	37
Leonard Eliezer Dor and Edward Wilfred Odell, Jr., <i>Monotone bases in L_p</i>	51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, <i>On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 = f$</i>	63
Y. Gordon, <i>Unconditional Schauder decompositions of normed ideals of operators between some l_p-spaces</i>	71
Gary Grefsrud, <i>Oscillatory properties of solutions of certain nth order functional differential equations</i>	83
Irvin Roy Hentzel, <i>Generalized right alternative rings</i>	95
Zensiro Goseki and Thomas Benny Rushing, <i>Embeddings of shape classes of compacta in the trivial range</i>	103
Emil Grosswald, <i>Brownian motion and sets of multiplicity</i>	111
Donald LaTorre, <i>A construction of the idempotent-separating congruences on a bisimple orthodox semigroup</i>	115
Pjek-Hwee Lee, <i>On subrings of rings with involution</i>	131
Marvin David Marcus and H. Minc, <i>On two theorems of Frobenius</i>	149
Michael Douglas Miller, <i>On the lattice of normal subgroups of a direct product</i>	153
Grattan Patrick Murphy, <i>A metric basis characterization of Euclidean space</i>	159
Roy Martin Rakestraw, <i>A representation theorem for real convex functions</i>	165
Louis Jackson Ratliff, Jr., <i>On Rees localities and H_i-local rings</i>	169
Simeon Reich, <i>Fixed point iterations of nonexpansive mappings</i>	195
Domenico Rosa, <i>B-complete and B_r-complete topological algebras</i>	199
Walter Roth, <i>Uniform approximation by elements of a cone of real-valued functions</i>	209
Helmut R. Salzmann, <i>Homogene kompakte projektive Ebenen</i>	217
Jerrold Norman Siegel, <i>On a space between BH and B_∞</i>	235
Robert C. Sine, <i>On local uniform mean convergence for Markov operators</i>	247
James D. Stafney, <i>Set approximation by lemniscates and the spectrum of an operator on an interpolation space</i>	253
Árpád Szász, <i>Convolution multipliers and distributions</i>	267
Kalathoor Varadarajan, <i>Span and stably trivial bundles</i>	277
Robert Breckenridge Warfield, Jr., <i>Countably generated modules over commutative Artinian rings</i>	289
John Yuan, <i>On the groups of units in semigroups of probability measures</i>	303