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**OSCILLATORY PROPERTIES OF SOLUTIONS OF CERTAIN
 n TH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS**

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With n even and $\int^{\infty} t^{n-1} a(t) dt < \infty$, necessary conditions for $x^{(n)}(t) + a(t)f(x(g(t))) = 0$ to have a bounded nonoscillatory solution are given. If $n = 2$, sufficient conditions are also given. Conditions which insure that solutions of $x^{(n)}(t) + f(t, x(g(t))) = 0$ are oscillatory or tend monotonically to zero are also presented in this paper.

Let $g(t)$ and $f(t, y)$ be real valued functions. In this paper we prove several oscillation theorems associated with solutions of the following two n th order functional differential equations:

- (1) $x^{(n)}(t) + a(t)f(x(g(t))) = 0$, and
 (2) $x^{(n)}(t) + f(t, x(g(t))) = 0$.

We use the "normal" definition of oscillatory, that is, $x(t)$ is an oscillatory solution of (1) or (2) if $x(t)$ satisfies (1) or (2) for large t and $x(t)$ has arbitrarily large zeros ($x(t) \neq 0$).

Theorems 4 and 5 are generalizations of results proved by Ryder and Wend [6], associated with the equation $x^{(n)} + f(t, x) = 0$. In fact the proof of theorem 5 has been omitted because of its similarity with the corresponding result in [6].

Before stating our main results we give the following lemmas.

LEMMA 1. *Suppose $f(t) \in C^k[a, \infty)$, $f(t) \geq 0$ and $f^{(k)}(t)$ is monotone. Then exactly one of the following is true:*

- (i) $\lim_{t \rightarrow \infty} f^{(k)}(t) = 0$,
 (ii) $\lim_{t \rightarrow \infty} f^{(k)}(t) > 0$ and $f(t), \dots, f^{(k-1)}(t)$ tend to ∞ as $t \rightarrow \infty$.

LEMMA 2. *If $y(t) \in C^n[a, \infty)$, $y(t) \geq 0$ and $y^{(n)}(t) \leq 0$ on $[a, \infty)$, then exactly one of the following is true:*

- (I) $y'(t), \dots, y^{(n-1)}(t)$ tend monotonically to zero as $t \rightarrow \infty$.
 (II) *There is an odd integer k , $1 \leq k \leq n - 1$, such that $\lim_{t \rightarrow \infty} y^{(n-j)}(t) = 0$ for $1 \leq j \leq k - 1$, $\lim_{t \rightarrow \infty} y^{(n-k)}(t) \geq 0$, $\lim_{t \rightarrow \infty} y^{(n-k-1)}(t) > 0$ and $y(t), y'(t), \dots, y^{(n-k-2)}(t)$ tend to ∞ as $t \rightarrow \infty$.*

Analogous statements can be made if $y(t) \leq 0$ and $y^{(n)}(t) \geq 0$ on $[a, \infty)$.

The results of Lemma's 1 and 2, given in [6], will be used throughout this paper.

THEOREM 1. *Suppose that n is even and*

- (i) $a(t) \geq 0$ for t sufficiently large,
- (ii) $\lim_{t \rightarrow \infty} g(t) = +\infty$,
- (iii) $yf(y) > 0$ ($y \neq 0$), $f(y)$ continuous on $(-\infty, \infty)$.

Then a necessary condition for equation (1) to have a *bounded* nonoscillatory solution is $\int_{-\infty}^{\infty} t^{n-1} a(t) dt < \infty$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1). Suppose $x(t) > 0$ for t sufficiently large. Thus, since $\lim_{t \rightarrow \infty} g(t) = +\infty$, we have that $x(g(t)) > 0$ for t sufficiently large. Hence, pick T large enough so that $a(t) \geq 0$, $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq T$. We have (for $t \geq T$), using Lemma 2, $x^{(n-1)}(t) \geq 0$,

$$x^{(n-2)}(t) \leq 0, \dots, \dot{x}(t) \geq 0: \lim_{t \rightarrow \infty} x^{(i)}(t) = 0, \quad i = 1, \dots, n-1.$$

Thus, $x(t)$ is a nondecreasing function and since $x(t) > 0$ and is bounded we have, $\lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} x(g(t)) = L > 0$.

From (1),

$$(3) \quad x^{(n-1)}(s) \geq \int_s^{\infty} a(u) f(x(g(u))) du.$$

An integration of (3) $n-2$ times from t to ∞ yields

$$(4) \quad (-1)^n \dot{x}(t) \geq \int_t^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} a(u) f(x(g(u))) du$$

and integrating (4) from s to t where $T \leq s \leq t$ we have

$$x(t) - x(s) \geq \int_s^t \frac{(u-s)^{n-1}}{(n-1)!} a(u) f(x(g(u))) du.$$

Now using the continuity of f we may choose $T_1 \geq T$ such that for $t \geq T_1$, $f(x(g(t))) \geq \frac{1}{2} f(L) = M$. Hence for $T \leq T_1 \leq s \leq t$ we have

$$(5) \quad x(t) - x(s) \geq \frac{M}{(n-1)!} \int_s^t (u-s)^{n-1} a(u) du.$$

Letting $t \rightarrow \infty$ in (5) we have

$$\int_s^\infty (u - s)^{n-1} a(u) du < \infty.$$

Then for $t \geq 2s$ we have

$$\int_t^\infty \left(\frac{u}{2}\right)^{n-1} a(u) du < \int_t^\infty (u - s)^{n-1} a(u) du < \infty.$$

i.e. $\int_t^\infty u^{n-1} a(u) du < \infty.$

If $x(t) < 0$ for t sufficiently large a similar proof yields the desired result. Q.E.D.

When $n = 2$, we establish sufficient conditions for equation (1) to have a bounded nonoscillatory solution.

THEOREM 2. *With $n = 2$ and*

- (i) *there exists $t_1 > 0$ such that $g(t) \geq t_1$ for all $t \geq t_1$,*
- (ii) *$g(t)$ is continuous on $[0, \infty)$,*
- (iii) *$f(y)$ is continuous on $(-\infty, \infty)$ with $yf(y) > 0$ for $y \neq 0$,*
- (iv) *$|f(y_1)| \leq |f(y_2)|$ if $|y_1| \leq |y_2|$,*
- (v) *for each $\beta > 0$, there is a $t > 0$ that satisfies the inequality $f(t) \leq \beta t$,*
- (vi) *$a(t) \geq 0$ and locally integrable on $[0, \infty)$ with $a(t)$ not identically zero on any subinterval of $[0, \infty)$, if*

$$(6) \quad \int_t^\infty ta(t)dt < \infty,$$

then there exists a bounded nonoscillatory solution of (1).

Proof. Assuming that $\int_t^\infty ta(t)dt < \infty$, we note that (v) implies the existence of some number $M > 0$ such that

$$(7) \quad \int_{t_1}^\infty sa(s)ds \leq \frac{M}{2f(M)},$$

where t_1 is chosen to satisfy (i). Consider now the integral equation

$$(8) \quad x(t) = \frac{M}{2} + t \int_{t_1}^\infty a(s)f(x(g(s)))ds + \int_{t_1}^t sa(s)f(x(g(s)))ds.$$

We now define a sequence $\{x_k(t)\}$ by

$$(9) \quad \begin{aligned} x_0(t) &= \frac{M}{2} \\ x_k(t) &= \frac{M}{2} + t \int_t^\infty a(s) f(x_{k-1}(g(s))) ds \\ &\quad + \int_{t_1}^t sa(s) f(x_{k-1}(g(s))) ds. \end{aligned}$$

One concludes that $x_k(t)$, $k = 0, 1, 2, \dots$, is defined and continuous and, in fact, is positive on $[t_1, \infty)$. By induction we have

$$(10) \quad \frac{M}{2} \leq x_k(t) \leq M, \quad k = 0, 1, 2, \dots, \quad \text{and}$$

$$(11) \quad x_k(t) \geq x_{k-1}(t).$$

Thus the sequence $\{x_k(t)\}$ converges to some function $x(t)$ for $t \geq t_1$ and indeed

$$\frac{M}{2} \leq x(t) \leq M \left(\frac{M}{2} \leq x(g(t)) \leq M \right)$$

for $t \geq t_1$.

We now must establish that $x(t)$ is a solution of the integral equation (8) and thus a solution (nonoscillatory) of (1). For any $\epsilon > 0$, choose T large enough so that $\int_T^\infty sa(s) ds < \epsilon/2f(M)$. Then we have

$$\begin{aligned} & \left| x_k(t) - \frac{M}{2} - t \int_t^\infty a(s) f(x(g(s))) ds - \int_{t_1}^t sa(s) f(x(g(s))) ds \right| \\ & \leq t \int_t^\infty a(s) |f(x_{k-1}(g(s))) - f(x(g(s)))| ds \\ & \quad + \int_{t_1}^t sa(s) |f(x_{k-1}(g(s))) - f(x(g(s)))| ds \\ & \leq \int_t^T sa(s) |f(x_{k-1}(g(s))) - f(x(g(s)))| ds \\ & \quad + \int_{t_1}^t sa(s) |f(x_{k-1}(g(s))) - f(x(g(s)))| ds \\ & \quad + \int_T^\infty sa(s) f(x_{k-1}(g(s))) ds + \int_T^\infty sa(s) f(x(g(s))) ds \\ & \leq \int_{t_1}^T sa(s) |f(x_{k-1}(g(s))) - f(x(g(s)))| ds + \epsilon. \end{aligned}$$

Letting $k \rightarrow \infty$ we obtain

$$\left| x(t) - \frac{M}{2} - t \int_t^\infty a(s)f(x(g(s)))ds - \int_{t_1}^t sa(s)f(x(g(s)))ds \right| \leq \epsilon.$$

Thus $x(t)$ is a bounded nonoscillatory solution of (1). Q.E.D.

Restricting our attention now to equation (2), we make the following assumptions:

(12)

- (i) $g(t) \geq t - c$ for t sufficiently large, $c > 0$, constant,
- (ii) $f(t, y)$ is continuous in $S = [0, \infty) \times (-\infty, \infty)$,
- (iii) $a(t)\Phi(y) \leq f(t, y)$ if $y > 0$ and $f(t, y) \leq b(t)\psi(y)$ if $y < 0$, $(t, y) \in S$, where
- (iv) $a(t)$ and $b(t)$ are nonnegative and locally integrable on $[0, \infty)$ and neither $a(t)$ nor $b(t)$ is identically zero on any subinterval of $[0, \infty)$,
- (v) $\Phi(y)$ and $\psi(y)$ are nondecreasing with $y\Phi(y) > 0$ and $y\psi(y) > 0$ on $(-\infty, \infty)$ for $y \neq 0$.
- (vi) there exist positive constants β and δ such that $\Phi(\lambda y) = \lambda^\beta \Phi(y)$, $\psi(\lambda y) = \lambda^\delta \psi(y)$, λ constant,
- (vii) for some $\alpha > 0$

$$\int_\alpha^\infty \frac{du}{\Phi(u)} < \infty \quad \text{and} \quad \int_{-\alpha}^{-\infty} \frac{du}{\psi(u)} < \infty.$$

THEOREM 3. *Let $x(t)$ be a solution of (2), valid for large t , which is nonoscillatory. If n is odd, assume $\lim_{t \rightarrow \infty} x(t) \neq 0$. Suppose conditions (i)–(vi) of (12) are satisfied. Then there exists a positive number k such that $\Phi(x(g(t)))/\Phi(x(t)) \geq k$ if $x(t)$ is eventually positive and $\psi(x(g(t)))/\psi(x(t)) \geq k$ if $x(t)$ is eventually negative for t sufficiently large.*

Proof. Let $x(t)$ be a nonoscillatory solution of (2). Suppose $x(t) > 0$ for t sufficiently large. Pick T large enough so that $x(t - c) > 0$ for $t \geq T$. From (2) we have

$$(13) \quad x^{(n)}(t) = -f(t, x(g(t))) \leq -a(t)\Phi(x(g(t))) \leq 0 \quad \text{if } t \geq T.$$

Thus from Lemmas 1 and 2, $x(t)$ satisfies one of the following:

- (1) $\ddot{x}(t) \geq 0$, $\dot{x}(t) \leq 0$ for t sufficiently large,

$$\lim_{t \rightarrow \infty} \dot{x}(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = L > 0.$$

- (2) $\ddot{x}(t) \leq 0$, $\dot{x}(t) \geq 0$ for t sufficiently large.

(3) $\ddot{x}(t) \geq 0$, $\dot{x}(t) \geq 0$ for t sufficiently large, with $x(t), \dot{x}(t), \dots, x^{(n-k-2)}(t)$ tending to ∞ as $t \rightarrow \infty$, $x^{(n-k-1)}(t)$ increasing to L ($0 < L \leq \infty$), $x^{(n-k)}(t)$ decreasing to M ($M \geq 0$), and $x^{(n-k+1)}(t), \dots, x^{(n-1)}(t)$, tending to zero as $t \rightarrow \infty$.

If case (1) applies we trivially have $x(g(t))/x(t) \geq \frac{1}{2}$ for t sufficiently large.

In either case (2) or (3) we have, since $\dot{x}(t) \geq 0$, $x(g(t)) \geq x(t-c)$ and thus $x(g(t))/x(t) \geq x(t-c)/x(t)$.

If case (2) applies, then exactly as in [1], we find $x(g(t))/x(t) \geq k_1$ ($k_1 > 0$) for t large.

Now suppose case (3) applies. Consider $\lim_{t \rightarrow \infty} x(t-c)/x(t)$ which is of the form ∞/∞ . Using L'Hopital's rule a sufficient number of times we obtain

$$\lim_{t \rightarrow \infty} \frac{x(t-c)}{x(t)} = \dots = \lim_{t \rightarrow \infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)}.$$

If L (in case 3) is finite we are done since then

$$\lim_{t \rightarrow \infty} \frac{x(t-c)}{x(t)} = \frac{L}{L} = 1.$$

When $L = \infty$, then again using L'Hopital's rule we have

$$\lim_{t \rightarrow \infty} \frac{x(t-c)}{x(t)} = \dots = \lim_{t \rightarrow \infty} \frac{x^{(n-k)}(t-c)}{x^{(n-k)}(t)}.$$

If M (in case 3) is positive again we are done since

$$\lim_{t \rightarrow \infty} \frac{x(t-c)}{x(t)} = \frac{M}{M} = 1.$$

However, if M is zero we then claim that

$$\lim_{t \rightarrow \infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} = 1$$

since

$$\lim_{t \rightarrow \infty} [x^{(n-k-1)}(t) - x^{(n-k-1)}(t-c)] = \lim_{t \rightarrow \infty} x^{(n-k)}(\xi)c = 0, \quad t-c < \xi < t.$$

Thus

$$\left| \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} - 1 \right| = \left| \frac{x^{(n-k-1)}(t-c) - x^{(n-k-1)}(t)}{x^{(n-k-1)}(t)} \right| < \frac{\epsilon x^{(n-k-1)}(t_1)}{x^{(n-k-1)}(t_1)} < \epsilon$$

where $t_1 \geq T$, is such that $x^{(n-k-1)}(t_1) > 0$. Summarizing we have $\lim_{t \rightarrow \infty} x(t-c)/x(t) = 1$. Thus for t large enough, $x(g(t))/x(t) \geq x(t-c)/x(t) > \frac{1}{2}$.

Now letting $k_2 = \min\{\frac{1}{2}, k_1\}$, we have $x(g(t))/x(t) \geq k_2$ for $t \geq T_1 \geq T$ and

$$\frac{\Phi(x(g(t)))}{\Phi(x(t))} \geq \frac{\Phi(k_2 x(t))}{\Phi(x(t))} = k_2^\beta \frac{\Phi(x(t))}{\Phi(x(t))} = k_2^\beta = k.$$

Now suppose $x(t)$ is a nonoscillatory solution of (2) which is negative for $t \geq T$. Again, pick T large enough so that $x(t-c) < 0$ for $t \geq T$. Then (13) becomes

$$(15) \quad x^{(n)}(t) = -f(t, x(g(t))) \geq -b(t)\psi(x(g(t))) \geq 0 \quad \text{if } t \geq T,$$

and we find that $x(t)$ must satisfy one of the following:

$$(1) \quad \ddot{x}(t) \leq 0, \dot{x}(t) \geq 0 \text{ for } t \text{ sufficiently large,}$$

$$\lim_{t \rightarrow \infty} \dot{x}(t) = 0, \quad \lim_{t \rightarrow \infty} x(t) = L < 0,$$

$$(2) \quad \ddot{x}(t) \geq 0, \dot{x}(t) \leq 0 \text{ for } t \text{ sufficiently large,}$$

$$(3) \quad \ddot{x}(t) \leq 0, \dot{x}(t) \leq 0 \text{ for } t \text{ sufficiently large, with } x(t), \dot{x}(t), \dots, x^{(n-k-2)}(t) \text{ tending to } -\infty \text{ as } t \rightarrow \infty, x^{(n-k-1)}(t) \text{ decreasing to } L (-\infty \leq L < 0), x^{(n-k)}(t) \text{ increasing to } M (M \leq 0), \text{ and } x^{(n-k+1)}(t), \dots, x^{(n-1)}(t) \text{ tending to zero as } t \rightarrow \infty.$$

If case (1) applies, we have that $\lim_{t \rightarrow \infty} x(g(t)) = L$ since $g(t) \geq t-c$ and $x(t)$ is decreasing to $L < 0$. Thus

$$\lim_{t \rightarrow \infty} \frac{x(g(t))}{x(t)} = \frac{L}{L} = 1.$$

In either case (2) or (3), $g(t) \geq t-c$ implies $x(g(t)) \leq x(t-c)$ and $|x(g(t))| \geq |x(t-c)|$ with

$$\frac{x(g(t))}{x(t)} = \left| \frac{x(g(t))}{x(t)} \right| \geq \left| \frac{x(t-c)}{x(t)} \right| = \frac{x(t-c)}{x(t)}.$$

If we now use arguments similar to those used when $x(t) > 0$, we obtain the desired conclusion.

THEOREM 4. *If $g(t)$ is nondecreasing and satisfies (i) and $f(t, y)$ satisfies (ii)–(vii) of (12) and in addition*

$$(16) \quad \int_0^\infty t^{n-1} a(t) dt = \int_0^\infty t^{n-1} b(t) dt = +\infty,$$

then if n is even each solution of (2), valid for large t , is oscillatory, while if n is odd each solution of (2), valid for large t , is either oscillatory or it tends monotonically to zero together with its first $n - 1$ derivatives.

Proof. Suppose $x(t)$ is a nonoscillatory solution of (2), valid for large t . Assume $x(t)$ is eventually positive. Thus $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq T$. From (2)

$$(17) \quad x^{(n)}(t) = -f(t, x(g(t))) \leq -a(t)\Phi(x(g(t))) \leq 0.$$

Thus by Lemma 1 $x^{(n-1)}(t)$ decreases to a nonnegative limit, so from (17) we obtain

$$(18) \quad x^{(n-1)}(s) \geq \int_s^\infty a(u)\Phi(x(g(u))) du.$$

Suppose case I of Lemma 2 holds. Then an integration of (18) $n - 2$ times from t to ∞ yields

$$(19) \quad (-1)^{(n-2)} \dot{x}(t) \geq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u))) du.$$

If n is even, integrating (19) from T to $t \geq T$, we have

$$x(t) \geq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u))) du.$$

Since Φ is nondecreasing

$$(20) \quad \Phi(x(t))/\Phi \left[\int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u))) du \right] \geq 1.$$

If we now multiply (20) by

$$\frac{(t-T)^{n-1}}{(n-1)!} a(t) \frac{\Phi(x(g(t)))}{\Phi(x(t))}$$

and integrate from r to s we get, after a change of variable on the left

$$(21) \quad \int_R^s \frac{du}{\Phi(u)} \cong \int_r^s \frac{(t-T)^{n-1}}{(n-1)!} a(t) \frac{\Phi(x(g(t)))}{\Phi(x(t))} dt$$

$$\cong k \int_r^s \frac{(t-T)^{n-1}}{(n-1)!} a(t) dt$$

where

$$R = \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u) \Phi(x(g(u))) du$$

and

$$S = \int_T^s \frac{(u-T)^{n-1}}{(n-1)!} a(u) \Phi(x(g(u))) du.$$

Now if by an appropriate choice of r , we can make $R \cong \alpha$, then the left hand side of (21) is bounded above for all $s > r$, hence $\int_0^\infty t^{n-1} a(t) dt < \infty$. If this is not possible then for all $r \cong T$

$$\alpha > \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u) \Phi(x(g(u))) du$$

$$\cong \Phi(x(g(T))) \int_T^r \frac{(u-T)^{n-1}}{(n-1)!} a(u) du$$

and the result again follows.

If n is odd, then (19) becomes

$$(22) \quad -\dot{x}(t) \cong \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u) \Phi(x(g(u))) du \cong 0.$$

So $x(t)$ decreases to a limit $L \cong 0$. Suppose $L > 0$. Then integrating (22) from T to ∞ ,

$$x(T) > x(T) - L \cong \int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!} a(u) \Phi(x(g(u))) du$$

$$\cong \Phi(L) \int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!} a(u) du,$$

using the monotonicity of Φ . But this implies $\int_0^\infty t^{n-1} a(t) dt < \infty$.

Now suppose that case II of Lemma 2 holds. Integrating (17) a sufficient number of times we have

$$(23) \quad x^{(n-k)}(t) \cong \int_t^\infty \frac{(u-t)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))du.$$

Since $x^{(j)}(t)$ increases to ∞ , $j < n - k - 1$, there exists $t_1 \cong T$ such that $x^{(j)}(t) > 0$ for $t \cong t_1$, $j = 0, \dots, n - k - 1$. Integrating (23) from t_1 to $t > t_1$,

$$\begin{aligned} x^{(n-k-1)}(t) &\cong \int_{t_1}^t \int_s^\infty \frac{(u-s)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))duds \\ &\cong \int_t^\infty \frac{(u-t_1)^k - (u-t)^k}{k!} a(u)\Phi(x(g(u)))du. \end{aligned}$$

So

$$(24) \quad x^{(n-k-1)}(t) > \int_t^\infty \frac{(t-t_1)^k}{k!} a(u)\Phi(x(g(u)))du.$$

Integrating (24) successively $n - k - 2$ times from t_1 to t we obtain

$$(25) \quad \dot{x}(t) > \int_t^\infty \frac{(t-t_1)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u)))du$$

and integrating (25) from t_1 to t gives

$$x(t) > \int_{t_1}^t \frac{(u-t_1)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du.$$

Now the proof proceeds as in case I.

If $x(t)$ is a solution of (2), valid for large t , such that $x(t) < 0$ for $t \cong T$, the proof is the same except $a(t)$ and $\Phi(u)$ are replaced respectively by $b(t)$ and $\psi(u)$, and the sense of appropriate inequalities are changed. Q.E.D.

In the next theorem, condition (vi) of (12) is changed so that equation (2) includes the special case

$$x^{(n)} + a(t)x^\alpha(g(t)) = 0, \quad 0 \leq \alpha < 1,$$

the ratio of odd integers.

THEOREM 5. *Let $g(t)$ satisfy (i) and $f(t, y)$ satisfy (ii)–(v) of (12). In addition suppose $f(t, y)$ satisfies (vii) there exist positive constants λ_0, M, N and constants β, γ , where $0 \leq \beta < 1, 0 \leq \gamma < 1$, such that*

$$\begin{aligned}\Phi(\lambda y) &\geq M\lambda^\beta \Phi(y), & y > 0, \\ \psi(\lambda y) &\leq N\lambda^\gamma \psi(y), & y < 0, \quad \lambda \geq \lambda_0 > 0.\end{aligned}$$

Then if

$$(26) \quad \int_t^\infty t^{(n-1)\beta} a(t) dt = \int_t^\infty t^{(n-1)\gamma} b(t) dt = +\infty,$$

each solution of (2), valid for large t , is oscillatory when n is even and is either oscillatory or tends to zero with its first $n - 1$ derivative if n is odd.

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