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# OSCILLATORY PROPERTIES OF SOLUTIONS OF CERTAIN *n*th ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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**With $n$ even and $\int^{\infty} t^{n-1}a(t)dt < \infty$, necessary conditions for $x^{(n)}(t) + a(t)f(x(g(t))) = 0$ to have a bounded nonoscillatory solution are given. If $n = 2$, sufficient conditions are also given. Conditions which insure that solutions of $x^{(n)}(t) + f(t, x(g(t))) = 0$ are oscillatory or tend monotonically to zero are also presented in this paper.**

Let $g(t)$ and $f(t, y)$ be real valued functions. In this paper we prove several oscillation theorems associated with solutions of the following two $n$th order functional differential equations:

$$x^{(n)}(t) + a(t)f(x(g(t))) = 0, \quad \text{and} \tag{1}$$

$$x^{(n)}(t) + f(t, x(g(t))) = 0. \tag{2}$$

We use the "normal" definition of oscillatory, that is, $x(t)$ is an oscillatory solution of (1) or (2) if $x(t)$ satisfies (1) or (2) for large $t$ and $x(t)$ has arbitrarily large zeros ($x(t) \not\equiv 0$).

Theorems 4 and 5 are generalizations of results proved by Ryder and Wend **[6]**, associated with the equation $x^{(n)} + f(t, x) = 0$. In fact the proof of theorem 5 has been omitted because of its similarity with the corresponding result in **[6]**.

Before stating our main results we give the following lemmas.

LEMMA 1. *Suppose $f(t) \in C^k[a, \infty)$, $f(t) \geqq 0$ and $f^{(k)}(t)$ is monotone. Then exactly one of the following is true:*

(i) $\lim_{t\to\infty} f^{(k)}(t) = 0$,

(ii) $\lim_{t\to\infty} f^{(k)}(t) > 0$ *and* $f(t), \cdots, f^{(k-1)}(t)$ *tend to* $\infty$ *as* $t \to \infty$.

LEMMA 2. *If $y(t) \in C^n[a, \infty)$, $y(t) \geqq 0$ and $y^{(n)}(t) \leqq 0$ on $[a, \infty)$, then exactly one of the following is true:*

(I) $y'(t), \cdots, y^{(n-1)}(t)$ *tend monotonically to zero as* $t \to \infty$.

(II) *There is an odd integer* $k$, $1 \leqq k \leqq n-1$, *such that* $\lim_{t\to\infty} y^{(n-j)}(t) = 0$ for $1 \leqq j \leqq k-1$, $\lim_{t\to\infty} y^{(n-k)}(t) \geqq 0$, $\lim_{t\to\infty} y^{(n-k-1)}(t) > 0$ *and* $y(t), y'(t), \cdots, y^{(n-k-2)}(t)$ *tend to* $\infty$ *as* $t \to \infty$.

Analogous statements can be made if $y(t) \leqq 0$ and $y^{(n)}(t) \geqq 0$ on $[a, \infty)$.

The results of Lemma's 1 and 2, given in [6], will be used throughout this paper.

THEOREM 1. *Suppose that n is even and*

(i) $a(t) \geqq 0$ *for t sufficiently large,*

(ii) $\lim_{t\to\infty} g(t) = +\infty$,

(iii) $yf(y) > 0$ $(y \neq 0)$, $f(y)$ *continuous on* $(-\infty, \infty)$.

Then a necessary condition for equation (1) to have a *bounded* nonoscillatory solution is $\int^{\infty} t^{n-1} a(t)dt < \infty$.

*Proof.* Let $x(t)$ be a bounded nonoscillatory solution of (1). Suppose $x(t) > 0$ for $t$ sufficiently large. Thus, since $\lim_{t\to\infty} g(t) = +\infty$, we have that $x(g(t)) > 0$ for $t$ sufficiently large. Hence, pick $T$ large enough so that $a(t) \geqq 0$, $x(t) > 0$ and $x(g(t)) > 0$ for $t \geqq T$. We have (for $t \geqq T$), using Lemma 2, $x^{(n-1)}(t) \geqq 0$,

$$x^{(n-2)}(t) \leqq 0, \cdots, \dot{x}(t) \geqq 0: \lim_{t\to\infty} x^{(i)}(t) = 0, \qquad i = 1, \cdots, n-1.$$

Thus, $x(t)$ is a nondecreasing function and since $x(t) > 0$ and is bounded we have, $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x(g(t)) = L > 0$.

From (1),

$$x^{(n-1)}(s) \geqq \int_s^{\infty} a(u) f(x(g(u)))du. \tag{3}$$

An integration of (3) $n-2$ times from $t$ to $\infty$ yields

$$(-1)^n \dot{x}(t) \geqq \int_t^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} a(u) f(x(g(u)))du \tag{4}$$

and integrating (4) from $s$ to $t$ where $T \leqq s \leqq t$ we have

$$x(t) - x(s) \geqq \int_s^t \frac{(u-s)^{n-1}}{(n-1)!} a(u) f(x(g(u)))du.$$

Now using the continuity of $f$ we may choose $T_1 \geqq T$ such that for $t \geqq T_1$, $f(x(g(t))) \geqq \frac{1}{2} f(L) = M$. Hence for $T \leqq T_1 \leqq s \leqq t$ we have

$$x(t) - x(s) \geqq \frac{M}{(n-1)!} \int_s^t (u-s)^{n-1} a(u)du. \tag{5}$$

Letting $t \to \infty$ in (5) we have

$$\int_s^\infty (u-s)^{n-1} a(u)du < \infty.$$

Then for $t \geqq 2s$ we have

$$\int_t^\infty \left(\frac{u}{2}\right)^{n-1} a(u)du < \int_t^\infty (u-s)^{n-1} a(u)du < \infty.$$

i.e. $\int_t^\infty u^{n-1} a(u)du < \infty$.

If $x(t) < 0$ for $t$ sufficiently large a similar proof yields the desired result. Q.E.D.

When $n = 2$, we establish sufficient conditions for equation (1) to have a bounded nonoscillatory solution.

THEOREM 2. *With $n = 2$ and*

(i) *there exists $t_1 > 0$ such that $g(t) \geqq t_1$ for all $t \geqq t_1$,*

(ii) *$g(t)$ is continuous on $[0, \infty)$,*

(iii) *$f(y)$ is continuous on $(-\infty, \infty)$ with $yf(y) > 0$ for $y \neq 0$,*

(iv) *$|f(y_1)| \leqq |f(y_2)|$ if $|y_1| \leqq |y_2|$,*

(v) *for each $\beta > 0$, there is a $t > 0$ that satisfies the inequality $f(t) \leqq \beta t$,*

(vi) *$a(t) \geqq 0$ and locally integrable on $[0, \infty)$ with $a(t)$ not identically zero on any subinterval of $[0, \infty)$, if*

(6) $$\int^\infty ta(t)dt < \infty,$$

*then there exists a bounded nonoscillatory solution of* (1).

*Proof.* Assuming that $\int^\infty ta(t)dt < \infty$, we note that (v) implies the existence of some number $M > 0$ such that

(7) $$\int_{t_1}^\infty sa(s)ds \leqq \frac{M}{2f(M)},$$

where $t_1$ is chosen to satisfy (i). Consider now the integral equation

(8) $$x(t) = \frac{M}{2} + t\int_t^\infty a(s)f(x(g(s)))ds + \int_{t_1}^t sa(s)f(x(g(s)))ds.$$

We now define a sequence $\{x_k(t)\}$ by

$$x_0(t) = \frac{M}{2}$$

(9)
$$x_k(t) = \frac{M}{2} + t\int_t^\infty a(s)f(x_{k-1}(g(s)))ds + \int_{t_1}^t sa(s)f(x_{k-1}(g(s)))ds.$$

One concludes that $x_k(t)$, $k = 0, 1, 2, \cdots$, is defined and continuous and, in fact, is positive on $[t_1, \infty)$. By induction we have

(10)
$$\frac{M}{2} \leqq x_k(t) \leqq M, \qquad k = 0, 1, 2, \cdots, \qquad \text{and}$$

(11)
$$x_k(t) \geqq x_{k-1}(t).$$

Thus the sequence $\{x_k(t)\}$ converges to some function $x(t)$ for $t \geqq t_1$ and indeed

$$\frac{M}{2} \leqq x(t) \leqq M \left( \frac{M}{2} \leqq x(g(t)) \leqq M \right)$$

for $t \geqq t_1$.

We now must establish that $x(t)$ is a solution of the integral equation (8) and thus a solution (nonoscillatory) of (1). For any $\epsilon > 0$, choose $T$ large enough so that $\int_T^\infty sa(s)ds < \epsilon/2f(M)$. Then we have

$$\left| x_k(t) - \frac{M}{2} - t\int_t^\infty a(s)f(x(g(s)))ds - \int_{t_1}^t sa(s)f(x(g(s)))ds \right|$$

$$\leqq t\int_t^\infty a(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|\,ds + \int_{t_1}^t sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|\,ds$$

$$\leqq \int_t^T sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|\,ds + \int_{t_1}^t sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|\,ds + \int_T^\infty sa(s)f(x_{k-1}(g(s)))ds + \int_T^\infty sa(s)f(x(g(s)))ds$$

$$\leqq \int_{t_1}^T sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|\,ds + \epsilon.$$

Letting $k \to \infty$ we obtain

$$\left| x(t) - \frac{M}{2} - t\int_t^\infty a(s)f(x(g(s)))ds - \int_{t_1}^t sa(s)f(x(g(s)))ds \right| \leqq \epsilon.$$

Thus $x(t)$ is a bounded nonoscillatory solution of (1). Q.E.D.

Restricting our attention now to equation (2), we make the following assumptions:

(12)

(i) $g(t) \geqq t - c$ for $t$ sufficiently large, $c > 0$, constant,

(ii) $f(t, y)$ is continuous in $S = [0, \infty)x(-\infty, \infty)$,

(iii) $a(t)\Phi(y) \leqq f(t, y)$ if $y > 0$ and $f(t, y) \leqq b(t)\psi(y)$ if $y < 0$, $(t, y) \in S$, where

(iv) $a(t)$ and $b(t)$ are nonnegative and locally integrable on $[0, \infty)$ and neither $a(t)$ nor $b(t)$ is identically zero on any subinterval of $[0, \infty)$,

(v) $\Phi(y)$ and $\psi(y)$ are nondecreasing with $y\Phi(y) > 0$ and $y\psi(y) > 0$ on $(-\infty, \infty)$ for $y \neq 0$.

(vi) there exist positive constants $\beta$ and $\delta$ such that $\Phi(\lambda y) = \lambda^\beta \Phi(y)$, $\psi(\lambda y) = \lambda^\delta \psi(y)$, $\lambda$ constant,

(vii) for some $\alpha > 0$

$$\int_\alpha^\infty \frac{du}{\Phi(u)} < \infty \qquad \text{and} \qquad \int_{-\alpha}^{-\infty} \frac{du}{\psi(u)} < \infty.$$

THEOREM 3. *Let $x(t)$ be a solution of* (2), *valid for large $t$, which is nonoscillatory. If $n$ is odd, assume* $\lim_{t\to\infty} x(t) \neq 0$. *Suppose conditions* (i)–(vi) *of* (12) *are satisfied. Then there exists a positive number $k$ such that $\Phi(x(g(t)))/\Phi(x(t)) \geqq k$ if $x(t)$ is eventually positive and $\psi(x(g(t)))/\psi(x(t)) \geqq k$ if $x(t)$ is eventually negative for $t$ sufficiently large.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of (2). Suppose $x(t) > 0$ for $t$ sufficiently large. Pick $T$ large enough so that $x(t - c) > 0$ for $t \geqq T$. From (2) we have

$$(13) \qquad x^{(n)}(t) = -f(t, x(g(t))) \leqq -a(t)\Phi(x(g(t))) \leqq 0 \quad \text{if} \quad t \geqq T.$$

Thus from Lemmas 1 and 2, $x(t)$ satisfies one of the following:

(1) $\ddot{x}(t) \geqq 0$, $\dot{x}(t) \leqq 0$ for $t$ sufficiently large,

$$\lim_{t\to\infty} \dot{x}(t) = 0, \qquad \lim_{t\to\infty} x(t) = L > 0.$$

(2) $\ddot{x}(t) \leqq 0$, $\dot{x}(t) \geqq 0$ for $t$ sufficiently large.

(3) $\ddot{x}(t) \geqq 0$, $\dot{x}(t) \geqq 0$ for $t$ sufficiently large, with $x(t), \dot{x}(t), \cdots, x^{(n-k-2)}(t)$ tending to $\infty$ as $t \to \infty$, $x^{(n-k-1)}(t)$ increasing to $L$ $(0 < L \leqq \infty)$, $x^{(n-k)}(t)$ decreasing to $M (M \geqq 0)$, and $x^{(n-k+1)}(t)$, $\cdots, x^{(n-1)}(t)$, tending to zero as $t \to \infty$.

If case (1) applies we trivially have $x(g(t))/x(t) \geqq \frac{1}{2}$ for $t$ sufficiently large.

In either case (2) or (3) we have, since $\dot{x}(t) \geqq 0$, $x(g(t)) \geqq x(t-c)$ and thus $x(g(t))/x(t) \geqq x(t-c)/x(t)$.

If case (2) applies, then exactly as in **[1]**, we find $x(g(t))/x(t) \geqq k_1 (k_1 > 0)$ for $t$ large.

Now suppose case (3) applies. Consider $\lim_{t\to\infty} x(t-c)/x(t)$ which is of the form $\infty/\infty$. Using L'Hopital's rule a sufficient number of times we obtain

$$\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \cdots = \lim_{t\to\infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} .$$

If $L$ (in case 3) is finite we are done since then

$$\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \frac{L}{L} = 1.$$

When $L = \infty$, then again using L'Hopital's rule we have

$$\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \cdots = \lim_{t\to\infty} \frac{x^{(n-k)}(t-c)}{x^{(n-k)}(t)} .$$

If $M$ (in case 3) is positive again we are done since

$$\lim_{t\to\infty} \frac{x(t-c)}{x(t)} = \frac{M}{M} = 1.$$

However, if $M$ is zero we then claim that

$$\lim_{t\to\infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} = 1$$

since

$$\lim_{t\to\infty} [x^{(n-k-1)}(t) - x^{(n-k-1)}(t-c)] = \lim_{t\to\infty} x^{(n-k)}(\xi)c = 0, \quad t-c < \xi < t.$$

Thus

$$\left|\frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)}-1\right| = \left|\frac{x^{(n-k-1)}(t-c)-x^{(n-k-1)}(t)}{x^{(n-k-1)}(t)}\right| < \frac{\epsilon x^{(n-k-1)}(t_1)}{x^{(n-k-1)}(t_1)} < \epsilon$$

where $t_1 \geqq T$, is such that $x^{(n-k-1)}(t_1) > 0$. Summarizing we have $\lim_{t\to\infty} x(t-c)/x(t) = 1$. Thus for $t$ large enough, $x(g(t))/x(t) \geqq x(t-c)/x(t) > \frac{1}{2}$.

Now letting $k_2 = \min\{\frac{1}{2}, k_1\}$, we have $x(g(t))/x(t) \geqq k_2$ for $t \geqq T_1 \geqq T$ and

$$\frac{\Phi(x(g(t)))}{\Phi(x(t))} \geqq \frac{\Phi(k_2 x(t))}{\Phi(x(t))} = k_2^{\beta}\frac{\Phi(x(t))}{\Phi(x(t))} = k_2^{\beta} = k.$$

Now suppose $x(t)$ is a nonoscillatory solution of (2) which is negative for $t \geqq T$. Again, pick $T$ large enough so that $x(t-c) < 0$ for $t \geqq T$. Then (13) becomes

$$(15) \qquad x^{(n)}(t) = -f(t, x(g(t))) \geqq -b(t)\psi(x(g(t))) \geqq 0 \quad \text{if} \quad t \geqq T,$$

and we find that $x(t)$ must satisfy one of the following:

(1) $\ddot{x}(t) \leqq 0$, $\dot{x}(t) \geqq 0$ for $t$ sufficiently large,

$$\lim_{t\to\infty} \dot{x}(t) = 0, \qquad \lim_{t\to\infty} x(t) = L < 0,$$

(2) $\ddot{x}(t) \geqq 0$, $\dot{x}(t) \leqq 0$ for $t$ sufficiently large,

(3) $\ddot{x}(t) \leqq 0$, $\dot{x}(t) \leqq 0$ for $t$ sufficiently large, with $x(t)$, $\dot{x}(t), \cdots, x^{(n-k-2)}(t)$ tending to $-\infty$ as $t \to \infty$, $x^{(n-k-1)}(t)$ decreasing to $L(-\infty \leqq L < 0)$, $x^{(n-k)}(t)$ increasing to $M$ $(M \leqq 0)$, and $n^{(n-k+1)}(t), \cdots, x^{(n-1)}(t)$ tending to zero as $t \to \infty$.

If case (1) applies, we have that $\lim_{t\to\infty} x(g(t)) = L$ since $g(t) \geqq t - c$ and $x(t)$ is decreasing to $L < 0$. Thus

$$\lim_{t\to\infty} \frac{x(g(t))}{x(t)} = \frac{L}{L} = 1.$$

In either case (2) or (3), $g(t) \geqq t - c$ implies $x(g(t)) \leqq x(t-c)$ and $|x(g(t))| \geqq |x(t-c)|$ with

$$\frac{x(g(t))}{x(t)} = \left|\frac{x(g(t))}{x(t)}\right| \geqq \left|\frac{x(t-c)}{x(t)}\right| = \frac{x(t-c)}{x(t)}.$$

If we now use arguments similar to those used when $x(t) > 0$, we obtain the desired conclusion.

THEOREM 4. *If $g(t)$ is nondecreasing and satisfies* (i) *and $f(t, y)$ satisfies* (ii)–(vii) *of* (12) *and in addition*

$$\int_0^\infty t^{n-1}a(t)dt = \int_0^\infty t^{n-1}b(t)dt = +\infty, \tag{16}$$

*then if $n$ is even each solution of* (2), *valid for large $t$, is oscillatory, while if $n$ is odd each solution of* (2), *valid for large $t$, is either oscillatory or it tends monotonically to zero together wth its first $n-1$ derivatives.*

*Proof.* Suppose $x(t)$ is a nonoscillatory solution of (2), valid for large $t$. Assume $x(t)$ is eventually positive. Thus $x(t) > 0$ and $x(g(t)) > 0$ for $t \geqq T$. From (2)

$$x^{(n)}(t) = -f(t, x(g(t))) \leqq -a(t)\Phi(x(g(t))) \leqq 0. \tag{17}$$

Thus by Lemma 1 $x^{(n-1)}(t)$ decreases to a nonnegative limit, so from (17) we obtain

$$x^{(n-1)}(s) \geqq \int_s^\infty a(u)\Phi(x(g(u)))du. \tag{18}$$

Suppose case I of Lemma 2 holds. Then an integration of (18) $n-2$ times from $t$ to $\infty$ yields

$$(-1)^{(n-2)}\dot{x}(t) \geqq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u)))du. \tag{19}$$

If $n$ is even, integrating (19) from $T$ to $t \geqq T$, we have

$$x(t) \geqq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du.$$

Since $\Phi$ is nondecreasing

$$\Phi(x(t))/\Phi\left[\int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du\right] \geqq 1. \tag{20}$$

If we now multiply (20) by

$$\frac{(t-T)^{n-1}}{(n-1)!} a(t) \frac{\Phi(x(g(t)))}{\Phi(x(t))}$$

and integrate from $r$ to $s$ we get, after a change of variable on the left

$$\int_R^S \frac{du}{\Phi(u)} \geqq \int_r^s \frac{(t-T)^{n-1}}{(n-1)!}\, a(t)\, \frac{\Phi(x(g(t)))}{\Phi(x(t))}\, dt \tag{21}$$
$$\geqq k \int_r^s \frac{(t-T)^{n-1}}{(n-1)!}\, a(t) dt$$

where

$$R = \int_T^r \frac{(u-T)^{n-1}}{(n-1)!}\, a(u)\Phi(x(g(u)))du$$

and

$$S = \int_T^s \frac{(u-T)^{n-1}}{(n-1)!}\, a(u)\Phi(x(g(u)))du.$$

Now if by an appropriate choice of $r$, we can make $R \geqq \alpha$, then the left hand side of (21) is bounded above for all $s > r$, hence $\int_0^\infty t^{n-1} a(t) dt < \infty$. If this is not possible then for all $r \geqq T$

$$\alpha > \int_T^r \frac{(u-T)^{n-1}}{(n-1)!}\, a(u)\Phi(x(g(u)))du$$
$$\geqq \Phi(x(g(T))) \int_T^r \frac{(u-T)^{n-1}}{(n-1)!}\, a(u) du$$

and the result again follows.

If $n$ is odd, then (19) becomes

$$-\dot{x}(t) \geqq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!}\, a(u)\Phi(x(g(u)))du \geqq 0. \tag{22}$$

So $x(t)$ decreases to a limit $L \geqq 0$. Suppose $L > 0$. Then integrating (22) from $T$ to $\infty$,

$$x(T) > x(T) - L \geqq \int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!}\, a(u)\Phi(x(g(u)))du$$
$$\geqq \Phi(L) \int_T^\infty \frac{(u-T)^{n-1}}{(n-1)!}\, a(u) du,$$

using the monotonicity of $\Phi$. But this implies $\int_0^\infty t^{n-1} a(t) dt < \infty$.

Now suppose that case II of Lemma 2 holds. Integrating (17) a sufficient number of times we have

$$x^{(n-k)}(t) \geqq \int_t^\infty \frac{(u-t)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))du. \tag{23}$$

Since $x^{(j)}(t)$ increases to $\infty$, $j < n-k-1$, there exists $t_1 \geqq T$ such that $x^{(j)}(t) > 0$ for $t \geqq t_1$, $j = 0, \cdots, n-k-1$. Integrating (23) from $t_1$ to $t > t_1$,

$$\begin{aligned} x^{(n-k-1)}(t) &\geqq \int_{t_1}^t \int_s^\infty \frac{(u-s)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))duds \\ &\geqq \int_t^\infty \frac{(u-t_1)^k - (u-t)^k}{k!} a(u)\Phi(x(g(u)))du. \end{aligned}$$

So

$$x^{(n-k-1)}(t) > \int_t^\infty \frac{(t-t_1)^k}{k!} a(u)\Phi(x(g(u)))du. \tag{24}$$

Integrating (24) successively $n-k-2$ times from $t_1$ to $t$ we obtain

$$\dot{x}(t) > \int_t^\infty \frac{(t-t_1)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u)))du \tag{25}$$

and integrating (25) from $t_1$ to $t$ gives

$$x(t) > \int_{t_1}^t \frac{(u-t_1)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du.$$

Now the proof proceeds as in case I.

If $x(t)$ is a solution of (2), valid for large $t$, such that $x(t) < 0$ for $t \geqq T$, the proof is the same except $a(t)$ and $\Phi(u)$ are replaced respectively by $b(t)$ and $\psi(u)$, and the sense of appropriate inequalities are changed. Q.E.D.

In the next theorem, condition (vi) of (12) is changed so that equation (2) includes the special case

$$x^{(n)} + a(t)x^{\alpha}(g(t)) = 0, \qquad 0 \leqq \alpha < 1,$$

the ratio of odd integers.

THEOREM 5. *Let $g(t)$ satisfy* (i) *and $f(t, y)$ satisfy* (ii)–(v) *of* (12). *In addition suppose $f(t, y)$ satisfies* (vii) *there exist positive constants $\lambda_0$, $M$, $N$ and constants $\beta$, $\gamma$, where $0 \leqq \beta < 1$, $0 \leqq \gamma < 1$, such that*

$$\Phi(\lambda y) \geqq M\lambda^{\beta}\Phi(y), \qquad y > 0,$$
$$\psi(\lambda y) \leqq N\lambda^{\gamma}\psi(y), \qquad y < 0, \quad \lambda \geqq \lambda_0 > 0.$$

Then if

$$\int_t^\infty t^{(n-1)\beta} a(t)dt = \int_t^\infty t^{(n-1)\gamma} b(t)dt = +\infty, \tag{26}$$

each solution of (2), valid for large $t$, is oscillatory when $n$ is even and is either oscillatory or tends to zero with its first $n-1$ derivative if $n$ is odd.

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