OSCILLATORY PROPERTIES OF SOLUTIONS OF CERTAIN $n$TH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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With \( n \) even and \( \int_{-\infty}^{\infty} t^{n-1} a(t) dt < \infty \), necessary conditions for \( x^{(n)}(t) + a(t)f(x(g(t))) = 0 \) to have a bounded nonoscillatory solution are given. If \( n = 2 \), sufficient conditions are also given. Conditions which insure that solutions of \( x^{(n)}(t) + f(t, x(g(t))) = 0 \) are oscillatory or tend monotonically to zero are also presented in this paper.

Let \( g(t) \) and \( f(t, y) \) be real valued functions. In this paper we prove several oscillation theorems associated with solutions of the following two \( n \)th order functional differential equations:

\[
\begin{align*}
(1) & \quad x^{(n)}(t) + a(t)f(x(g(t))) = 0, \\
(2) & \quad x^{(n)}(t) + f(t, x(g(t))) = 0.
\end{align*}
\]

We use the “normal” definition of oscillatory, that is, \( x(t) \) is an oscillatory solution of (1) or (2) if \( x(t) \) satisfies (1) or (2) for large \( t \) and \( x(t) \) has arbitrarily large zeros \( (x(t) \neq 0) \).

Theorems 4 and 5 are generalizations of results proved by Ryder and Wend [6], associated with the equation \( x^{(n)} + f(t, x) = 0 \). In fact the proof of theorem 5 has been omitted because of its similarity with the corresponding result in [6].

Before stating our main results we give the following lemmas.

**Lemma 1.** Suppose \( f(t) \in C^k[a, \infty), f(t) \geq 0 \) and \( f^{(k)}(t) \) is monotone. Then exactly one of the following is true:

(i) \( \lim_{t \to \infty} f^{(k)}(t) = 0 \),

(ii) \( \lim_{t \to \infty} f^{(k)}(t) > 0 \) and \( f(t), \ldots, f^{(k-1)}(t) \) tend to \( \infty \) as \( t \to \infty \).

**Lemma 2.** If \( y(t) \in C^n[a, \infty), y(t) \geq 0 \) and \( y^{(n)}(t) \leq 0 \) on \([a, \infty)\), then exactly one of the following is true:

(I) \( y'(t), \ldots, y^{(n-1)}(t) \) tend monotonically to zero as \( t \to \infty \).

(II) There is an odd integer \( k, 1 \leq k \leq n-1 \), such that \( \lim_{t \to \infty} y^{(n-j)}(t) = 0 \) for \( 1 \leq j \leq k - 1 \), \( \lim_{t \to \infty} y^{(n-k)}(t) \geq 0 \), \( \lim_{t \to \infty} y^{(n-k-1)}(t) > 0 \) and \( y(t), y'(t), \ldots, y^{(n-k-2)}(t) \) tend to \( \infty \) as \( t \to \infty \).
Analogous statements can be made if \( y(t) \leq 0 \) and \( y^{(n)}(t) \leq 0 \) on \([a, \infty)\).

The results of Lemma's 1 and 2, given in [6], will be used throughout this paper.

**Theorem 1.** Suppose that \( n \) is even and

(i) \( a(t) \equiv 0 \) for \( t \) sufficiently large,

(ii) \( \lim_{t \to \infty} g(t) = +\infty \),

(iii) \( yf(y) > 0 \) (\( y \neq 0 \)), \( f(y) \) continuous on \((\infty, \infty)\).

Then a necessary condition for equation (1) to have a bounded nonoscillatory solution is

\[
\int_{t}^{\infty} t^{-1} a(t) dt < \infty.
\]

**Proof.** Let \( x(t) \) be a bounded nonoscillatory solution of (1). Suppose \( x(t) > 0 \) for \( t \) sufficiently large. Thus, since \( \lim_{t \to \infty} g(t) = +\infty \), we have that \( x(g(t)) > 0 \) for \( t \) sufficiently large. Hence, pick \( T \) large enough so that \( a(t) \equiv 0 \), \( x(t) > 0 \) and \( x(g(t)) > 0 \) for \( t \geq T \). We have (for \( t \geq T \)), using Lemma 2, \( x^{(n-1)}(t) \equiv 0 \),

\[
x^{(n-2)}(t) \equiv 0, \cdots, \dot{x}(t) \equiv 0: \lim_{t \to \infty} x^{(i)}(t) = 0, \quad i = 1, \cdots, n - 1.
\]

Thus, \( x(t) \) is a nondecreasing function and since \( x(t) > 0 \) and is bounded we have, \( \lim_{t \to \infty} x(t) = \lim_{t \to \infty} x(g(t)) = L > 0 \).

From (1),

\[ (3) \quad x^{(n-1)}(s) \equiv \int_{s}^{\infty} a(u) f(x(g(u))) du. \]

An integration of (3) \( n - 2 \) times from \( t \) to \( \infty \) yields

\[ (4) \quad (-1)^{n} \dot{x}(t) \equiv \int_{t}^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} a(u) f(x(g(u))) du \]
and integrating (4) from \( s \) to \( t \) where \( T \leq s \leq t \) we have

\[
x(t) - x(s) \equiv \int_{s}^{t} \frac{(u-s)^{n-1}}{(n-1)!} a(u) f(x(g(u))) du.
\]

Now using the continuity of \( f \) we may choose \( T_{1} \geq T \) such that for \( t \geq T_{1} \), \( f(x(g(t))) \leq \frac{1}{2} f(L) = M \). Hence for \( T \leq T_{1} \leq s \leq t \) we have

\[ (5) \quad x(t) - x(s) \leq \frac{M}{(n-1)!} \int_{s}^{t} (u-s)^{n-1} a(u) du. \]
Letting $t \to \infty$ in (5) we have

$$\int_{s}^{\infty} (u-s)^{-1} a(u) du < \infty.$$ 

Then for $t \geq 2s$ we have

$$\int_{s}^{\infty} \left(\frac{u}{2}\right)^{-1} a(u) du < \int_{s}^{\infty} (u-s)^{-1} a(u) du < \infty.$$ 

i.e. $\int_{s}^{\infty} u^{-1} a(u) du < \infty$.

If $x(t) < 0$ for $t$ sufficiently large a similar proof yields the desired result. Q.E.D.

When $n = 2$, we establish sufficient conditions for equation (1) to have a bounded nonoscillatory solution.

**Theorem 2.** With $n = 2$ and

(i) there exists $t_1 > 0$ such that $g(t) \equiv t_1$ for all $t \geq t_1$,

(ii) $g(t)$ is continuous on $[0, \infty)$,

(iii) $f(y)$ is continuous on $(-\infty, \infty)$ with $yf(y) > 0$ for $y \neq 0$,

(iv) $|f(y_1)| \leq |f(y_2)|$ if $|y_1| \leq |y_2|$,

(v) for each $\beta > 0$, there is a $t > 0$ that satisfies the inequality $f(t) \leq \beta t$,

(vi) $a(t) \geq 0$ and locally integrable on $[0, \infty)$ with $a(t)$ not identically zero on any subinterval of $[0, \infty)$, if

$$\int_{t_1}^{\infty} ta(t) dt < \infty,$$

then there exists a bounded nonoscillatory solution of (1).

**Proof.** Assuming that $\int_{t_1}^{\infty} ta(t) dt < \infty$, we note that (v) implies the existence of some number $M > 0$ such that

$$\int_{t_1}^{\infty} sa(s) ds \leq \frac{M}{2f(M)},$$

where $t_1$ is chosen to satisfy (i). Consider now the integral equation

$$x(t) = \frac{M}{2} + t \int_{t_1}^{\infty} a(s)f(x(g(s))) ds + \int_{t_1}^{t} sa(s)f(x(g(s))) ds.$$
We now define a sequence \( \{x_k(t)\} \) by

\[
x_0(t) = \frac{M}{2}
\]

(9) \[
x_k(t) = \frac{M}{2} + t \int_{t}^{\infty} a(s)f(x_{k-1}(g(s)))ds
\]

\[+ \int_{t}^{T} sa(s)f(x_{k-1}(g(s)))ds.\]

One concludes that \( x_k(t) \), \( k = 0, 1, 2, \cdots \), is defined and continuous and, in fact, is positive on \([t_1, \infty)\). By induction we have

(10) \[
\frac{M}{2} \leq x_k(t) \leq M, \quad k = 0, 1, 2, \cdots, \quad \text{and}
\]

(11) \[
x_k(t) \geq x_{k-1}(t).
\]

Thus the sequence \( \{x_k(t)\} \) converges to some function \( x(t) \) for \( t \geq t_1 \) and indeed

\[
\frac{M}{2} \leq x(t) \leq M \quad \left( \frac{M}{2} \leq x(g(t)) \leq M \right)
\]

for \( t \geq t_1 \).

We now must establish that \( x(t) \) is a solution of the integral equation (8) and thus a solution (nonoscillatory) of (1). For any \( \epsilon > 0 \), choose \( T \) large enough so that \( \int_{t}^{\infty} sa(s)ds < \epsilon/2f(M) \). Then we have

\[
\left| x_k(t) - \frac{M}{2} - t \int_{t}^{\infty} a(s)f(x(g(s)))ds - \int_{t}^{T} sa(s)f(x(g(s)))ds \right|
\]

\[\leq t \int_{t}^{\infty} a(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds
\]

\[+ \int_{t}^{T} sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds
\]

\[\leq \int_{t}^{T} sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds
\]

\[+ \int_{t}^{T} sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds
\]

\[+ \int_{T}^{\infty} sa(s)f(x_{k-1}(g(s)))ds + \int_{T}^{\infty} sa(s)f(x(g(s)))ds
\]

\[\leq \int_{t_1}^{T} sa(s)|f(x_{k-1}(g(s))) - f(x(g(s)))|ds + \epsilon.
\]
Letting \( k \to \infty \) we obtain
\[
\left| x(t) - \frac{M}{2} - t \int_{t}^{\infty} a(s)f(x(g(s)))ds - \int_{t}^{t} sa(s)f(x(g(s)))ds \right| \leq \varepsilon.
\]
Thus \( x(t) \) is a bounded nonoscillatory solution of (1). Q.E.D.

Restricting our attention now to equation (2), we make the following assumptions:

(12)

(i) \( g(t) \geq t - c \) for \( t \) sufficiently large, \( c > 0 \), constant,
(ii) \( f(t, y) \) is continuous in \( S = [0, \infty) \times (-\infty, \infty) \),
(iii) \( a(t)\Phi(y) \leq f(t, y) \) if \( y > 0 \) and \( f(t, y) \leq b(t)\psi(y) \) if \( y < 0 \),
(\( t, y) \in S \), where
(iv) \( a(t) \) and \( b(t) \) are nonnegative and locally integrable on \([0, \infty)\) and neither \( a(t) \) nor \( b(t) \) is identically zero on any subinterval of \([0, \infty)\),
(v) \( \Phi(y) \) and \( \psi(y) \) are nondecreasing with \( y\Phi(y) > 0 \) and \( y\psi(y) > 0 \) on \((-\infty, \infty)\) for \( y \neq 0 \).
(vi) there exist positive constants \( \beta \) and \( \delta \) such that \( \Phi(\lambda y) = \lambda^\beta \Phi(y) \), \( \psi(\lambda y) = \lambda^\delta \psi(y) \), \( \lambda \) constant,
(vii) for some \( \alpha > 0 \)
\[
\int_{-\infty}^{\infty} \frac{du}{\Phi(u)} < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{du}{\psi(u)} < \infty.
\]

**Theorem 3.** Let \( x(t) \) be a solution of (2), valid for large \( t \), which is nonoscillatory. If \( n \) is odd, assume \( \lim_{t \to \infty} x(t) \neq 0 \). Suppose conditions (i)-(vi) of (12) are satisfied. Then there exists a positive number \( k \) such that \( \Phi(x(g(t)))/\Phi(x(t)) \leq k \) if \( x(t) \) is eventually positive and \( \psi(x(g(t)))/\psi(x(t)) \leq k \) if \( x(t) \) is eventually negative for \( t \) sufficiently large.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of (2). Suppose \( x(t) > 0 \) for \( t \) sufficiently large. Pick \( T \) large enough so that \( x(t - c) > 0 \) for \( t \geq T \). From (2) we have
\[
(13) \quad x^{(n)}(t) = -f(t, x(g(t))) - a(t)\Phi(x(g(t))) \leq 0 \quad \text{if} \quad t \geq T.
\]
Thus from Lemmas 1 and 2, \( x(t) \) satisfies one of the following:

(1) \( \dot{x}(t) \geq 0, \; \dot{x}(t) \leq 0 \) for \( t \) sufficiently large,
\[
\lim_{t \to \infty} \dot{x}(t) = 0, \quad \lim_{t \to \infty} x(t) = L > 0.
\]

(2) \( \dot{x}(t) \leq 0, \; \dot{x}(t) \geq 0 \) for \( t \) sufficiently large.
(3) \( \dot{x}(t) \geq 0, \; \ddot{x}(t) \geq 0 \) for \( t \) sufficiently large, with \( x(t), \dot{x}(t), \ldots, x^{(n-k-2)}(t) \) tending to \( \infty \) as \( t \to \infty \), \( x^{(n-k)}(t) \) increasing to \( L \) \( (0 < L \leq \infty) \), \( x^{(n-k)}(t) \) decreasing to \( M(M \geq 0) \), and \( x^{(n-k+1)}(t), \ldots, x^{(n-1)}(t) \), tending to zero as \( t \to \infty \).

If case (1) applies we trivially have \( x(g(t))/x(t) \geq \frac{1}{2} \) for \( t \) sufficiently large.

In either case (2) or (3) we have, since \( x(t) \geq 0 \), \( x(g(t)) \geq x(t - c) \) and thus \( x(g(t))/x(t) \geq x(t - c)/x(t) \).

If case (2) applies, then exactly as in [1], we find \( x(g(t))/x(t) \geq k_1(k_1 > 0) \) for \( t \) large.

Now suppose case (3) applies. Consider \( \lim_{t \to \infty} x(t - c)/x(t) \) which is of the form \( \infty/\infty \). Using L'Hopital's rule a sufficient number of times we obtain

\[
\lim_{t \to \infty} \frac{x(t-c)}{x(t)} = \cdots = \lim_{t \to \infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)}.
\]

If \( L \) (in case 3) is finite we are done since then

\[
\lim_{t \to \infty} \frac{x(t-c)}{x(t)} = \frac{L}{L} = 1.
\]

When \( L = \infty \), then again using L'Hopital's rule we have

\[
\lim_{t \to \infty} \frac{x(t-c)}{x(t)} = \cdots = \lim_{t \to \infty} \frac{x^{(n-k)}(t-c)}{x^{(n-k)}(t)}.
\]

If \( M \) (in case 3) is positive again we are done since

\[
\lim_{t \to \infty} \frac{x(t-c)}{x(t)} = \frac{M}{M} = 1.
\]

However, if \( M \) is zero we then claim that

\[
\lim_{t \to \infty} \frac{x^{(n-k-1)}(t-c)}{x^{(n-k-1)}(t)} = 1
\]

since

\[
\lim_{t \to \infty} [x^{(n-k-1)}(t) - x^{(n-k-1)}(t - c)] = \lim_{t \to \infty} x^{(n-k)}(\xi)c = 0, \quad t - c < \xi < t.
\]

Thus
where \( t_1 \geq T \), is such that \( x^{(n-k-1)}(t_1) > 0 \). Summarizing we have \( \lim_{t \to \infty} x(t-c)/x(t) = 1 \). Thus for \( t \) large enough, \( x(g(t))/x(t) \geq x(t-c)/x(t) > \frac{1}{2} \).

Now letting \( k_2 = \min\{\frac{1}{2}, k_1\} \), we have \( x(g(t))/x(t) \geq k_2 \) for \( t \geq T_1 \geq T \) and

\[
\frac{\Phi(x(g(t)))}{\Phi(x(t))} \geq \frac{\Phi(k_2 x(t))}{\Phi(x(t))} = k_2 \frac{\Phi(x(t))}{\Phi(x(t))} = k_2 = k.
\]

Now suppose \( x(t) \) is a nonoscillatory solution of (2) which is negative for \( t \geq T \). Again, pick \( T \) large enough so that \( x(t-c) < 0 \) for \( t \geq T \). Then (13) becomes

\[
\left(15\right) \quad x^{(n)}(t) = -f(t, x(g(t))) \geq -b(t) \psi(x(g(t))) \geq 0 \quad \text{if} \quad t \geq T,
\]

and we find that \( x(t) \) must satisfy one of the following:

(1) \( \dot{x}(t) \leq 0 \), \( x(t) \geq 0 \) for \( t \) sufficiently large,

\[
\lim_{t \to \infty} \dot{x}(t) = 0, \quad \lim_{t \to \infty} x(t) = L < 0,
\]

(2) \( \dot{x}(t) \geq 0 \), \( \dot{x}(t) \leq 0 \) for \( t \) sufficiently large, with \( x(t), \dot{x}(t), \cdots, x^{(n-k-2)}(t) \) tending to \( -\infty \) as \( t \to \infty \), \( x^{(n-k-1)}(t) \) decreasing to \( L (-\infty \leq L < 0) \), \( x^{(n-k)}(t) \) increasing to \( M (M \leq 0) \), and \( n^{(n-k+1)}(t), \cdots, x^{(n-1)}(t) \) tending to zero as \( t \to \infty \).

If case (1) applies, we have that \( \lim_{t \to \infty} x(g(t)) = L \) since \( g(t) \geq t - c \) and \( x(t) \) is decreasing to \( L < 0 \). Thus

\[
\lim_{t \to \infty} \frac{x(g(t))}{x(t)} = \frac{L}{L} = 1.
\]

In either case (2) or (3), \( g(t) \geq t - c \) implies \( x(g(t)) \leq x(t-c) \) and \( |x(g(t))| \geq |x(t-c)| \) with

\[
\frac{x(g(t))}{x(t)} = \left| \frac{x(g(t))}{x(t)} \right| \geq \left| \frac{x(t-c)}{x(t)} \right| = \frac{x(t-c)}{x(t)}.
\]

If we now use arguments similar to those used when \( x(t) > 0 \), we obtain the desired conclusion.
THEOREM 4. If \( g(t) \) is nondecreasing and satisfies (i) and \( f(t,y) \) satisfies (ii)–(vii) of (12) and in addition

\[
\int_0^\infty t^{n-1}a(t)dt = \int_0^\infty t^{n-1}b(t)dt = +\infty,
\]

then if \( n \) is even each solution of (2), valid for large \( t \), is oscillatory, while if \( n \) is odd each solution of (2), valid for large \( t \), is either oscillatory or it tends monotonically to zero together with its first \( n-1 \) derivatives.

Proof. Suppose \( x(t) \) is a nonoscillatory solution of (2), valid for large \( t \). Assume \( x(t) \) is eventually positive. Thus \( x(t) > 0 \) and \( x(g(t)) > 0 \) for \( t \geq T \). From (2)

\[
x^{(n)}(t) = -f(t,x(g(t))) \leq -a(t)\Phi(x(g(t))) \leq 0.
\]

Thus by Lemma 1 \( x^{(n-1)}(t) \) decreases to a nonnegative limit, so from (17) we obtain

\[
x^{(n-1)}(s) \geq \int_s^\infty a(u)\Phi(x(g(u)))du.
\]

Suppose case I of Lemma 2 holds. Then an integration of (18) \( n-2 \) times from \( t \) to \( \infty \) yields

\[
(-1)^{n-2}\dot{x}(t) \geq \int_t^\infty \frac{(u-t)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u)))du.
\]

If \( n \) is even, integrating (19) from \( T \) to \( t \geq T \), we have

\[
x(t) \geq \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du.
\]

Since \( \Phi \) is nondecreasing

\[
\Phi(x(r))/\Phi \left[ \int_T^t \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du \right] \geq 1.
\]

If we now multiply (20) by

\[
\frac{(t-T)^{n-1}}{(n-1)!} a(t) \frac{\Phi(x(g(t)))}{\Phi(x(t))}
\]

and integrate from \( r \) to \( s \) we get, after a change of variable on the left
\[
\int_{r}^{s} \frac{du}{\Phi(u)} \geq \int_{r}^{s} \frac{(t-T)^{n-1}}{(n-1)!} a(t) \frac{\Phi(x(g(t)))}{\Phi(x(t))} dt \\
\geq k \int_{r}^{s} \frac{(t-T)^{n-1}}{(n-1)!} a(t) dt
\]

(21)

where

\[
R = \int_{T}^{T'} \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du
\]

and

\[
S = \int_{T}^{s} \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du.
\]

Now if by an appropriate choice of \( r \), we can make \( R \geq \alpha \), then the left hand side of (21) is bounded above for all \( s > r \), hence \( \int_{0}^{\infty} t^{n-1}a(t)dt < \infty \).

If this is not possible then for all \( r \leq T \)

\[
\alpha > \int_{T}^{T'} \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du \\
\geq \Phi(x(g(T))) \int_{T}^{T'} \frac{(u-T)^{n-1}}{(n-1)!} a(u)du
\]

and the result again follows.

If \( n \) is odd, then (19) becomes

\[
-x(t) \geq \int_{T}^{\infty} (u-t)^{n-2} \frac{(n-2)!}{(n-1)!} a(u)\Phi(x(g(u)))du \geq 0.
\]

(22)

So \( x(t) \) decreases to a limit \( L \geq 0 \). Suppose \( L > 0 \). Then integrating (22) from \( T \) to \( \infty \),

\[
x(T) - x(T) - L \geq \int_{T}^{\infty} \frac{(u-T)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du \\
\geq \Phi(L) \int_{T}^{\infty} \frac{(u-T)^{n-1}}{(n-1)!} a(u)du,
\]

using the monotonicity of \( \Phi \). But this implies \( \int_{0}^{\infty} t^{n-1}a(t)dt < \infty \).

Now suppose that case II of Lemma 2 holds. Integrating (17) a sufficient number of times we have
(23) \[ x^{(n-k)}(t) \geq \int_{t_1}^{t} \frac{(u-t)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))du. \]

Since \( x^{(j)}(t) \) increases to \( \infty \), \( j < n - k - 1 \), there exists \( t_1 \geq T \) such that \( x^{(j)}(t) > 0 \) for \( t \geq t_1, \ j = 0, \ldots, n - k - 1 \). Integrating (23) from \( t_1 \) to \( t > t_1 \),

\[ x^{(n-k-1)}(t) \geq \int_{t_1}^{t} \int_{s}^{t} \frac{(u-s)^{k-1}}{(k-1)!} a(u)\Phi(x(g(u)))du ds \]

\[ \equiv \int_{t_1}^{t} \frac{t-t_1)^{k} - (u - t)^{k}}{k!} a(u)\Phi(x(g(u)))du. \]

So

(24) \[ x^{(n-k-1)}(t) > \int_{t_1}^{t} \frac{(t-t_1)^{k}}{k!} a(u)\Phi(x(g(u)))du. \]

Integrating (24) successively \( n - k - 2 \) times from \( t_1 \) to \( t \) we obtain

(25) \[ \dot{x}(t) > \int_{t_1}^{t} \frac{(t-t_1)^{n-2}}{(n-2)!} a(u)\Phi(x(g(u)))du \]

and integrating (25) from \( t_1 \) to \( t \) gives

\[ x(t) > \int_{t_1}^{t} \frac{(u-t_1)^{n-1}}{(n-1)!} a(u)\Phi(x(g(u)))du. \]

Now the proof proceeds as in case I.

If \( x(t) \) is a solution of (2), valid for large \( t \), such that \( x(t) < 0 \) for \( t \geq T \), the proof is the same except \( a(t) \) and \( \Phi(u) \) are replaced respectively by \( b(t) \) and \( \psi(u) \), and the sense of appropriate inequalities are changed. Q.E.D.

In the next theorem, condition (vi) of (12) is changed so that equation (2) includes the special case

\[ x^{(n)} + a(t)x^\alpha(g(t)) = 0, \quad 0 \leq \alpha < 1, \]

the ratio of odd integers.

**Theorem 5.** Let \( g(t) \) satisfy (i) and \( f(t, y) \) satisfy (ii)–(v) of (12). In addition suppose \( f(t, y) \) satisfies (vii) there exist positive constants \( \lambda_0, M, N \) and constants \( \beta, \gamma \), where \( 0 \leq \beta < 1, 0 \leq \gamma < 1 \), such that
\[ \Phi(\lambda y) \geq M\lambda^\beta \Phi(y), \quad y > 0, \]
\[ \psi(\lambda y) \leq N\lambda^\gamma \psi(y), \quad y < 0, \quad \lambda \geq \lambda_0 > 0. \]

Then if

\[ \int_t^\infty t^{(n-1)\beta} a(t) dt = \int_t^\infty t^{(n-1)\gamma} b(t) dt = +\infty, \]

each solution of (2), valid for large \( t \), is oscillatory when \( n \) is even and is either oscillatory or tends to zero with its first \( n - 1 \) derivative if \( n \) is odd.

REFERENCES


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