# Pacific Journal of Mathematics

EMBEDDINGS OF SHAPE CLASSES OF COMPACTA IN THE TRIVIAL RANGE

JOHN G. HOLLINGSWORTH AND THOMAS BENNY RUSHING

# EMBEDDINGS OF SHAPE CLASSES OF COMPACTA IN THE TRIVIAL RANGE

## J. G. HOLLINGSWORTH AND T. B. RUSHING

We show that for compacta X,  $Y \,\subset R^n$ ,  $n \ge 5$ , satisfying the small loops condition and having dimensions in the trivial range with respect to n, Sh(X) = Sh(Y) if and only if  $R^n - X \approx R^n - Y$ . As a corollary we obtain the following result whose statement is void of shape: If  $X, Y \subset R^n$ ,  $n \ge 5$ , are homeomorphic compacta satisfying the small loops condition and having dimensions in the trivial range with respect to n, then  $R^n - X \approx R^n - Y$ .

1. Main results and introduction. In this paper we are concerned with the general problem of classifying the collection  $C_Z$  of compacta in a space Z for which the following property holds: Sh(X) = Sh(Y) is equivalent to  $Z - X \approx Z - Y$  ( $\approx$  means "is homeomorphic to") for all X,  $Y \in C_Z$ . Our results apply to compacta in  $R^n$  whose dimensions are in the trivial range with respect to n. After defining a fundamental homotopy condition and stating our main results, we will discuss some related work.

Let X be a compactum in a manifold M. We say that X satisfies the small loops condition (SLC) if for any neighborhood U of X, there is a neighborhood V of X in U and an  $\epsilon > 0$  such that each map of S<sup>1</sup> into V - X of diameter less than  $\epsilon$  is null homotopic in U - X. We say that k is in the trivial range with respect to (w.r.t.) n if  $2k + 2 \le n$  (or equivalently  $k \le \lfloor n/2 \rfloor - 1$ ).

THEOREM 1. Let X,  $Y \subset \mathbb{R}^n$ ,  $n \ge 5$ , be compact satisfying SLC whose dimensions are in the trivial range w.r.t. n. Then, Sh(X) = Sh(Y) if and only if  $\mathbb{R}^n - X \approx \mathbb{R}^n - Y$ .

Theorem 1 generalizes the main result of [6] which we recapture in the following corollary. (The paper [6] improved upon [4] which was the first trivial range work of the nature of Theorem 1.) The 1-ULC hypothesis of Geoghegan and Summerhill is a local condition that traditionally has been used to show that topological embeddings are flat or unknotted, whereas the SLC, the cellularity criterion, and the global 1-alg property are three intimately related (see Proposition 1.5 of [5]) global conditions that have traditionally been used to show that topological embeddings have nice complements (are weakly flat) or homeomorphic complements. We will give an example illustrating this point following Corollary 3. COROLLARY 1. (Geoghegan and Summerhill) Let X,  $Y \subset R^n$ ,  $n \ge 5$ , be compacta whose dimensions are in the trivial range w.r.t. n such that  $R^n - X$  and  $R^n - Y$  are 1-ULC. Then, Sh(X) = Sh(Y) if and only if  $R^n - X \approx R^n - Y$ .

The next two corollaries are purely topological, i.e., their statements are void of shape. However, it is interesting to note that an attempt to prove these corollaries directly seems to naturally lead one into shape theory!

COROLLARY 2. Let X,  $Y \subset \mathbb{R}^n$ ,  $n \ge 5$ , be compact ANR's whose dimensions are in the trivial range w.r.t. n and that satisfy the SLC. Then, X and Y have the same homotopy type if and only if their complements are homeomorphic.

COROLLARY 3. Let X,  $Y \subset \mathbb{R}^n$ ,  $n \ge 5$ , be homeomorphic compacta whose dimensions are in the trivial range and that satisfy SLC. Then,  $\mathbb{R}^n - X \approx \mathbb{R}^n - Y$ .

It is now known that two cells in  $\mathbb{R}^n$  with 1-ULC complements are equivalently embedded, and that two cells in  $\mathbb{R}^n$  satisfying the cellularity criterion [8] have homeomorphic complements. (In fact, [8] establishes Theorem 1 for compacta with trivial shape.) Analogously, Bryant [2], [3] and Štan'ko [13] have shown that two trivial range compacta in  $\mathbb{R}^n$ with 1-ULC complements are equivalently embedded, and Corollary 3 shows that two trivial range compacta in  $\mathbb{R}^n$  satisfying the SLC have homeomorphic complements.

By considering embeddings of shape classes, we are able to prove "weak flattening" theorems for a wide class of compacta. Also, shape theory allows us to obtain converses for "weak flattening" theorems, i.e., under certain conditions, homeomorphic complements implies equal shape. In our opinion, this is one important justification for the theory of shape.

The following trivial range theorem can be proved by using Theorem 2.4 of [5] and some work of Wall [14]: If  $X, Y \subset \mathbb{R}^n, n \ge 5$ , are compacta in the trivial range that satisfy the SLC and that have the shape of a finite complex, then Sh(X) = Sh(Y) implies that  $\mathbb{R}^n - X \approx \mathbb{R}^n - Y$ . (Ross Geoghegan pointed out the relevance of [14] to this result.) Although this theorem replaces the 1-ULC hypothesis of Geoghegan and Summerhill with the more desirable SLC, it adds the condition that the compacta have the shape of a finite complex. It is the purpose of this paper to show that that condition is unnecessary and to prove the converse.

Notice that one direction of Corollary 2 follows from the theorem

mentioned in the last paragraph combined with the recent result of West [15] that every compact metric ANR has finite type. Our proof will not use [15].

The authors would like to thank John Bryant for a key observation concerning the proof of Theorem 1 for the case n = 5.

2. Proof of Theorem 1. We begin by making a definition. Let  $X \subset \mathbb{R}^n$  be a compactum whose dimension k is in the trivial range w.r.t. n. Then, X is homotopically stable in  $\mathbb{R}^n$  if given a neighborhood U of X, there is a neighborhood  $V \subset U$  of X such that regardless of base point  $\pi_i(V, V - X) \approx 0$  for  $0 \le i \le n - \lfloor n/2 \rfloor$ . (We will ignore the base point in this paper.)

Before proving our first lemma, let us state a couple of sublemmas. The first sublemma follows from Theorem 1.3 of [5].

SUBLEMMA 1. If  $X \subset \mathbb{R}^n$ , dim  $X \leq n-3$ , is a compactum which satisfies the SLC, then  $\pi_i(U, U-X) \approx 0$ , i = 0, 1, 2, for each neighborhood U of X.

The next sublemma follows from Theorem 10 (page 342) of [10]. ( $H_*$  denotes singular homology with Z coefficients.)

SUBLEMMA 2. Let V be an orientable n-manifold and let  $X \subset V$  be a closed subset such that dim X = k. Then,  $H_i(V, V - X) \approx 0$  for  $i \leq n - k - 1$ . (Thus, if  $k \leq \lfloor n/2 \rfloor - 1$ , then  $H_i(V, V - X) \approx 0$  for  $i \leq n - \lfloor n/2 \rfloor$ .)

LEMMA 1. Let  $X \subset \mathbb{R}^n$  be a compactum, dim  $X = k \leq n-3$ , that satisfies the SLC. Then, X has arbitrarily small neighborhoods V such that  $\pi_i(V, V - X) \approx 0$  for  $i \leq n - k - 1$ . (Thus, if  $k \leq \lfloor n/2 \rfloor - 1$ , then X is homotopically stable.)

**Proof.** Choose an arbitrarily small neighborhood V of X satisfying Sublemma 1. Let  $\tilde{V}$  denote the universal cover of V with projection  $p: \tilde{V} \to V$ . Denote  $p^{-1}(X)$  by  $\bar{X}$ . Since  $\pi_2(V, V - X) \approx 0$ , it follows from the homotopy lifting property that  $\tilde{V} - \bar{X}$  (denoted by V - X) is the universal cover of V - X with projection  $p \mid (\tilde{V} - \bar{X})$ . It follows from Sublemma 1 and the homotopy lifting property that  $\pi_i(\tilde{V}, V - X) \approx$ 0 for i = 0, 1, 2. Sublemma 2 implies that  $H_i(\tilde{V}, V - X) \approx 0$  for  $i \leq n - k - 1$ . Hence, the relative Hurewicz Theorem yields  $\pi_i(\tilde{V}, V - X) \approx$ 0 for  $i \leq n - k - 1$ . For  $i = 3, 4, \dots, n - k - 1$ , consider the diagram

An application of the five lemma to this diagram finishes the proof.

For completeness, we will briefly sketch a classical type proof, based on some remarks in [2], of the next lemma. Alternately, one can prove Lemma 2 by techniques in §3 of [6] and we prefer that method to the more classical one outlined below. Let  $M_n^{n-3}$  denote the set of points in  $R^n$  at most n-3 of whose coordinates are rational.

LEMMA 2. Let  $X \subset \mathbb{R}^n$  be a k-dimensional compactum,  $2k + 1 \leq n$ . Then, given  $\epsilon > 0$ , every map of X into  $\mathbb{R}^n$  can be  $\epsilon$ -approxin ated by an embedding g such that  $\mathbb{R}^n - g(X)$  is 1-ULC.

**Proof.** Let M(X, Y) [E(X, Y)] denote the set of all mappings [embeddings] of X into Y. Then,  $E(X, R^n)$  is a dense  $G_{\delta}$  subset of  $M(X, R^n)$ , c.f., page 57 of [7]. The proof of Theorem V5 of [7] shows that  $M(X, M_n^{n-3})$  is a dense  $G_{\delta}$  subset of  $M(X, R^n)$ . Thus,  $E(X, M_n^{n-3}) =$  $E(X, R^n) \cap M(X, M_n^{n-3})$  is a dense  $G_{\delta}$  subset of  $M(X, R^n)$ . But if  $g \in E(X, M_n^{n-3})$ , then  $R^n - g(X)$  is 1-ULC by the proof of Theorem 2 of [2].

REMARK. M. A. Štan'ko [12] has shown that every embedding of a k-dimensional compactum  $X, k \leq n-3$ , into  $R^n$  can be approximated with an embedding g such that  $R^n - g(X)$  is 1-ULC. This result could be used to replace the last part of the above proof.

LEMMA 3. Let X be a homotopically stable compactum in  $\mathbb{R}^n$ ,  $n \ge 5$ , and let  $f: X \to \mathbb{R}^n$  be a map such that dist $(x, f(x)) < \epsilon$  for each  $x \in X$ . Let 0 be an open set containing f(X). Then, there is a homeomorphism  $h: \mathbb{R}^n \to \mathbb{R}^n$  such that

- (1) h = identity outside of the  $\epsilon$ -neighborhood of X,
- (2)  $h(X) \subset 0$ , and in fact
- (3)  $h | X \simeq f$  in 0 (the homotopy may oscillate badly in 0).

**Proof.** Case 1.  $(n \ge 6)$ : (The first part of this is similar to Lemma 2 of [2], however, our hypothesis is too weak to insure that h can be chosen so that  $h \mid X$  approximates f. Consequently, the existence of the homotopy specified by (3) is crucial.)

Extend f to a map  $\overline{f}$  of  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Let N be a small polyhedral neighborhood of X equipped with a triangulation T of small mesh. (We require that  $\overline{f}(N) \subset 0$ .) Let  $T_*$  be the  $(\lfloor n/2 \rfloor - 1)$ -skeleton of T and  $T^*$ be the dual  $(n - \lfloor n/2 \rfloor)$ -skeleton. Approximate  $\overline{f} \mid N$  with a PL map  $\hat{f}$ that is in general position with respect to T (cf. Theorem 1.6.10 of [13]). Then,  $\hat{f} \mid X$  is homotopic to f inside of 0 and  $\hat{f} \mid T_*$  is a PL embedding. Now there is an  $\epsilon$ -push  $h_1$  of  $(\mathbb{E}^n, T_*)$  by [1] such that  $h_1 \mid T_* = \hat{f} \mid T_*$ . Since X is homotopically stable, we may employ Stallings' engulfing Theorem [11] to shift X off of  $T^*$  with a homeomorphism  $h_2$ , where  $h_2$  is ambient isotopic to the identity by an isotopy that is fixed outside of a small neighborhood  $W \subset N$  of X. (Of course this isotopy may oscillate fiercely inside of W.) Now we may use the local join structure between  $T^*$  and  $T_*$  to obtain a small push  $h_3$  of  $(\mathbb{R}^n, h_2(X))$  such that  $h_3 h_2(X)$  is very close to  $T_*$ . Let  $h = h_1 h_3 h_2$ . By construction, h satisfies (1) and (2). It remains to see that  $h \mid X = f$  in 0.

We have already observed that  $f \approx \hat{f} | X$  inside of 0 and so it will suffice to show  $\hat{f} | X \approx h | X$  inside of 0. First notice that the identity  $1_X: X \to X$  is homotopic (in fact, pseudo-isotopic) to a map  $p: X \to T_*$ by a homotopy  $r_t$  whose track lies in N. In particular, let  $r_t, 0 \le t \le \frac{1}{2}$ , be the restriction to X of the ambient isotopy that realizes  $h_2$ , and let  $r_t$ ,  $\frac{1}{2} \le t \le 1$ , be the pseudo-isotopy obtained by pushing  $h_2(X)$  into  $T_*$ straight down the local join structure between  $T^*$  and  $T_*$ . Now, notice that  $\hat{f} | X$  is homotopic to  $\hat{f}p = h_1p$  by the homotopy  $\hat{f}r_t$  that takes place in  $\hat{f}(N) \subset 0$ .

We will be through if we also show that h | X is homotopic to  $\hat{f}p = h_1 p$  inside 0. Well,  $h_3 h_2 | X$  is certainly homotopic to p via the pseudo-isotopy  $g_i$  which slides  $h_3 h_2(X)$  straight down the local join structure. But then  $h_1 h_3 h_2 | X = h | X$  is homotopic (inside 0) to  $h_1 p$  by the homotopy  $h_1 g_i$  as desired.

Case 2. (n = 5): (the proof of this case involves a modification of the techniques of proof of Lemmas 1, 2 and 3 of [3]. Again we cannot hope to have h | X approximate f, consequently the existence of the homotopy specified by (3) is important.)

Extend f to a map  $\overline{f}$  of  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Let N be a polyhedral neighborhood of X such that  $f(N) \subset 0$ . We now make a claim.

CLAIM A. There exists a cover of X of order 1 by simply connected open sets  $U_1, \dots, U_r$  such that  $U_i \cap U_j$  is connected for each  $i, j = 1, \dots, r$ . Furthermore,  $U_i \subset N$ ,  $i = 1, \dots, r$ .

The proof of Claim A follows the lines of proof of Lemma 1 of [3], with one modification necessary. This modification occurs at the point where we want to kill  $\pi_1(\partial M_i)$  by the surgery trick of trading 2-handles. In particular, suppose we have reached the stage that we have a PL simple closed curve C in  $\partial M_1$  which represents a generator of  $\pi_1(\partial M_1)$  and suppose we have the PL 2-cell D in N such that  $\partial D = C$ . At this point, Bryant uses the fact that  $E^n - X$  is 1-ULC to shift D off of X with a small shift. We may use our homotopically stable hypothesis and Stallings' engulfing to shift D off of X by a shift which takes place in a small neighborhood of X. The proof now proceeds as in Lemma 1 of [3]. Although we cannot conclude that the resulting  $U_i$ 's are small, as in [3], we nevertheless establish Claim A.

Consider the cover  $\{U_1, \dots, U_r\}$  of X given by Claim A. One may piecewise linearly embed the nerve P of  $\{U_i\}$  in  $U = \bigcup_{i=1}^r U_i$  as in [3]. Approximate  $\overline{f} | N$  with a PL map  $\hat{f}$  that is in general position with respect to P. Then,  $\hat{f} | X$  is homotopic to f inside of 0 and  $\hat{f} | P$  is a PL embedding. Now there is an  $\epsilon$ -push  $h_1$  of  $(E^n, P)$  by [1] such that  $h_1 | P = \hat{f} | P$ .

Let W be a regular neighborhood of P in N such that  $h_1(W) \subset 0$ . (Then W deformation retracts to P.) This brings us to another claim.

CLAIM B. Suppose that  $\{U_1, \dots, U_r\}$ , P, and W are as above. Then, there is an ambient isotopy  $e_i$  of  $\mathbb{R}^n$  which is the identity outside U such that  $e_1(X) \subset W$ .

The proof of Claim B is very much like the proof of Lemma 2 of [3]; however, a few comments are in order. Notice that it is not necessary that the  $V_{ij}$ 's be simply connected (only connected). We would be in trouble if this were necessary! We can construct the homotopy  $g: M^2 \times I$  $\rightarrow U$  even though the U's are not small; however, g will not necessarily be an  $\epsilon$ -homotopy. The properties of  $\{U_i\}$  allow us to apply Zeeman piping to get an ambient isotopy  $e_t^1$  of  $E^n$  such that  $e_t^1$  is the identity outside  $\boldsymbol{U}$ and  $e_1^1(W) \supset g(M^2 \times 0)$ . The homotopically stable hypothesis and Stallings' engulfing suffice to yield and ambient isotopy  $e_t^2$ that moves only in a small neighborhood of X and that shifts X off of  $\tilde{M}^{n-3}$ . Let  $e_t^3$  be the isotopy pushing across the join structure between  $M^2$  and  $\tilde{M}^{n-3}$  that carries  $e_1^2(X)$  into  $e_1^1(W)$ . Then,  $e_i$  may be taken to be the composition of the three isotopies  $e_t^2$ ,  $e_t^3$  and  $e_t^1$ , respectively.

We now define h to be  $h_1 e_1$ . It is clear that h satisfies (1) and (2); hence, it remains only to establish (3). We have already observed that  $f \approx \hat{f} | X$  inside of 0 and so it will suffice to show that  $\hat{f} | X \approx h | X$  inside of 0. First notice that the identity  $1_X : X \to X$  is homotopic to a map  $p: X \to P$  by a homotopy  $r_t$  whose track lies in N. In particular,  $r_t$  is the homotopy  $e_t | X$  followed by a deformation retraction of W to P. Notice that  $\hat{f} | X$  is homotopic to  $\hat{f}p = h_1p$  by the homotopy  $\hat{f}r_t$  which takes place in  $\hat{f}(N) \subset 0$ .

We will be through if we also show that h | X is homotopic to  $\hat{f}p = h_1 p$  inside 0. Well,  $e_1 | X$  is certainly homotopic to p via  $g_i e_1$  where  $g_i$  is the deformation retraction of W to P. But then  $h_1 e_1 | X = h | X$  is homotopic (inside 0) to  $h_1 p$  by the homotopy  $h_1 g_i e_1$  as desired.

LEMMA 4. Let X be a homotopically stable compactum in  $\mathbb{R}^n$ ,  $n \ge 5$ , and let  $f: X \to \mathbb{R}^n$  be a map which is homotopic to  $1_X$  in an open neighborhood U of  $X \cup f(X)$ . Then, given an open set V where  $f(X) \subset$  $V \subset U$ , there is a homeomorphism  $h: \mathbb{R}^n \to \mathbb{R}^n$  such that

(1) h = identity outside of U,

(2) 
$$h(X) \subset V$$
, and in fact

(3) 
$$h \mid X \simeq f \text{ in } V.$$

**Proof.** Let  $e_t$  denote the homotopy connecting  $1_X$  and f. Let  $\epsilon$  be one-half the distance between the track of  $e_t$  and  $R^n - U$ . Then, by using Lemma 2 to approximate f and enough levels of  $e_t$ , we may obtain a sequence of embeddings  $f_1, f_2, \dots, f_k$  of X into  $R^n$  satisfying the following properties.

$$(1) \quad f_1 = 1_X$$

- (2)  $f_k \simeq f$  in V
- (3)  $E^n f_i(X)$  is 1-ULC for  $i = 1, 3, \dots, k$
- (4)  $d(f_i, f_{i+1}) < \epsilon$  for  $i = 1, 2, \dots, k-1$ , and
- (5)  $d(f_i(X), \mathbb{R}^n U) < \epsilon$  for  $i = 1, 2, \dots, k$ .

Now, by applying [2] or [3] (k-2)-times, we obtain a homeomorphism  $h_1: \mathbb{R}^n \to \mathbb{R}^n$  such that

- $(1) \quad h_1 f_2 = f_k$
- (2)  $h_1 | R^n U =$ identity.

But now we are in position to apply Lemma 3 where  $(X, f, R^n, 0, \epsilon)$  of that lemma corresponds to  $(X, f_2, R^n, h_1^{-1}(V), \epsilon)$ . Let  $h_2$  be the resulting homeomorphism of  $R^n$ . Then, it is clear that  $h = h_1 h_2$  is the desired homeomorphism.

The following lemma corresponds to Lemma 4.1 of [6]. (It is perhaps worth mentioning here that the statement of Lemma 4.1 of [6] as well as the statements of several other results in [6] are redundant in that the requirement "strong  $Z_{n-k-2}$ -set ( $k \ge 0$ ,  $n \ge 2k + 2$ )" could be replaced by "strong  $Z_{[n/2]-1}$ -set.")

LEMMA 5. Let  $X, Y \subset \mathbb{R}^n$ ,  $n \ge 5$ , be homotopically stable compacta whose dimensions are in the trivial range such that Sh(X) = Sh(Y). Let  $\{f_i, X, B\}$  and  $\{f'_i, Y, A\}$  be fundamental sequences (in  $\mathbb{R}^n$ ) which are homotopy inverses to one another. Let  $U_0$  be a neighborhood of X and let h be a homeomorphism of  $\mathbb{R}^n$  such that  $Y \subset h(U_0)$  and  $h^{-1}|Y$  is homotopic to  $f'_i|Y$  in  $U_0$  for almost all i. Then, for every neighborhood  $V_0$  such that  $Y \subset V_0 \subset h(U_0)$ , there is a homeomorphism q of  $\mathbb{R}^n$  such that q = houtside  $U_0, X \subset q^{-1}(V_0)$  and q |X is homotopic to  $f_i|X$  in  $V_0$  for almost all i.

**Proof.** Construct  $\hat{f}: X \to V$  such that  $\hat{f} \simeq h | X$  in  $h(U_0)$  just as is done in the proof of Lemma 4.1 of [6]. Now appeal to Lemma 4 to obtain the desired q.

Proof of Theorem 1. First assume that Sh(X) = Sh(Y). Lemma 1 assures us that X and Y are homotopically stable. One can show that  $R^n - X \approx R^n - Y$  by meshing a basic system  $\{T_i\}$  of neighborhoods for X

with a basic system  $\{W_i\}$  of neighborhoods for Y just as is done in the proof of Lemma 4.2 of [6]. In that process, our Lemma 3 plays the role of Lemma 4.1 of [6].

Now assume that  $R^n - X \approx R^n - Y$ . By Lemma 2, we can find copies  $\tilde{X}$  and  $\tilde{Y}$  of X and Y, respectively, in  $R^n$  such that  $R^n - \tilde{X}$  and  $R^n - \tilde{Y}$  are 1-ULC (hence satisfy the SLC). Now the direction of Theorem 1 already proved implies that  $R^n - \tilde{X} \approx R^n - X$  and  $R^n - \tilde{Y} \approx$  $R^n - Y$ . Theorem 1.2 of [6] implies that  $Sh(\tilde{X}) = Sh(\tilde{Y})$ . Therefore, Sh(X) = Sh(Y) as desired.

#### REFERENCES

1. R. H. Bing and J. M. Kister, Taming complexes in hyperspaces, Duke Math. J., 31 (1964), 491-511.

2. J. L. Bryant, On embeddings of compacta in Euclidean space, Proc. Amer. Math. Soc., 23 (1969), 46-51.

3. ——, On embeddings of 1-dimensional compacta in E<sup>5</sup>, Duke Mat. J., 38 (1971), 265-270.

4. T. A. Chapman, Shapes of finite-dimensional compacta, Fund. Math., 76 (1972), 261-276.

5. D. Coram, R. J. Daverman and P. F. Duvall, Jr., A loop condition for embedded compacta, Trans. Amer. Math. Soc. (to appear).

6. Ross Geoghegan and R. Richard Summerhill, Concerning the shapes of finite-dimensional comapcta, Trans. Amer. Math. Soc., 179 (1973), 281-292.

7. Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton University Press, Princeton, 1948.

8. D. R. McMillan, Jr., A criterion for cellularity in a manifold, Ann. of Math., 79 (1964), 327-337.

9. T. B. Rushing, Topological Embeddings, Academic Press, New York, 1973.

10. E. Spanier, Algebraic Topology, McGraw Hill, New York, 1966.

11. J. R. Stallings, *The piecewise linear structure of Euclidean space*, Proc. Cambridge Philos. Soc., **58** (1962), 481–488.

12. M. A. Štan'ko, Approximation of imbeddings of compacta in codimensions greater than two, Dokl. Akad. Nauk USSR, 198 (1971) = Soviet Math. Dokl., 12 (1971), 906-909.

13. \_\_\_\_\_, The embedding of compacta in Euclidean space, Mat. Sb., 83 (125) (1970) = Math USSR Sb., 12 (1970), 234–255.

14. C. T. C. Wall, Finiterness conditions for CW-complexes, Ann. of Math., 77 (1963), 490-503.

15. James E. West, Compact ANR's have finite type, Bull. Amer. Math. Soc., 81 (1975), 163-165.

Received April 14, 1975 and in revised form July 18, 1975. Presented March, 1975 to the A.M.S. "Special Session on Shape Theory," Mobile, Alabama (organized by B. J. Ball). This paper improves upon the result of the second author announced in Notices A.M.S., Feb., 1975, Abstract #721-G16. First author was supported by N.S.F. Grant GP-29585 A-2. Second author was supported by N.S.F. Grant NPS 70-01667 A04. Much of the research by the second author was performed while this author was visiting the University of Florida.

UNIVERSITY OF GEORGIA AND UNIVERSITY OF UTAH

# PACIFIC JOURNAL OF MATHEMATICS

## **EDITORS**

### **RICHARD ARENS (Managing Editor)**

University of California Los Angeles, California 90024

R. A. BEAUMONT

University of Washington Seattle, Washington 98105

J. DUGUNDJI

Department of Mathematics University of Southern California Los Angeles, California 90007

D. GILBARG AND J. MILGRAM Stanford University Stanford, California 94305

#### ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. Wolf

K. YOSHIDA

#### SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA CALIFORNIA INSTITUTE OF TECHNOLOGY UNIVERSITY OF CALIFORNIA MONTANA STATE UNIVERSITY UNIVERSITY OF NEVADA NEW MEXICO STATE UNIVERSITY OREGON STATE UNIVERSITY UNIVERSITY OF OREGON OSAKA UNIVERSITY UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON

• •

AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

> Copyright © 1975 Pacific Journal of Mathematics All Rights Reserved

# Pacific Journal of Mathematics Vol. 60, No. 2 October, 1975

Waleed A. Al-Salam and A. Verma, A fractional Leibniz q-formula	1
Robert A. Bekes, Algebraically irreducible representations of $L_1(G)$	11
Thomas Theodore Bowman, Construction functors for topological	27
Stephen LaVern Comphell, Operator valued inner functions analytic on the	21
closed disc. II	37
Leonard Eliezer Dor and Edward Wilfred Odell Ir Monotone bases in L.	51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 - f$	63
Y. Gordon, Unconditional Schauder decompositions of normed ideals of	05
operators between some $l_p$ -spaces	71
Gary Grefsrud, Oscillatory properties of solutions of certain nth order functional	
differential equations	83
Irvin Roy Hentzel, Generalized right alternative rings	95
Zensiro Goseki and Thomas Benny Rushing, Embeddings of shape classes of	
compacta in the trivial range	103
Emil Grosswald, Brownian motion and sets of multiplicity	111
Donald LaTorre, A construction of the idempotent-separating congruences on a bisimple orthodox semigroup	115
Piek-Hwee Lee. On subrings of rings with involution	131
Marvin David Marcus and H. Minc. On two theorems of Frobenius	149
Michael Douglas Miller, On the lattice of normal subgroups of a direct	112
product	153
Grattan Patrick Murphy, <i>A metric basis characterization of Euclidean space</i>	159
Roy Martin Rakestraw, A representation theorem for real convex functions	165
Louis Jackson Ratliff, Jr., On Rees localities and H <sub>i</sub> -local rings	169
Simeon Reich, <i>Fixed point iterations of nonexpansive mappings</i>	195
Domenico Rosa, <i>B</i> -complete and $B_r$ -complete topological algebras	199
Walter Roth, Uniform approximation by elements of a cone of real-valued	
functions	209
Helmut R. Salzmann, Homogene kompakte projektive Ebenen	217
Jerrold Norman Siegel, On a space between BH and $B_{\infty}$	235
Robert C. Sine, On local uniform mean convergence for Markov operators	247
James D. Stafney, Set approximation by lemniscates and the spectrum of an	
operator on an interpolation space	253
Arpád Száz, Convolution multipliers and distributions	267
Kalathoor Varadarajan, Span and stably trivial bundles	277
Robert Breckenridge Warfield, Jr., Countably generated modules over	200
commutative Artinian rings	289
Jonn Yuan, On the groups of units in semigroups of probability measures	303