Pacific Journal of Mathematics

BROWNIAN MOTION AND SETS OF MULTIPLICITY

ROBERT P. KAUFMAN

Vol. 60, No. 2 October 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

BROWNIAN MOTION AND SETS OF MULTIPLICITY

ROBERT KAUFMAN

$X(t)$ is Brownian motion on the axis $-\infty < t < \infty$, with paths in R^n, $n \geqq 2$. $X(t)$ leads to composed mappings $f \circ X$, where f is a real-valued function of class $\Lambda^\alpha(R^n)$, whose gradient never vanishes. To define the class $\Lambda^\alpha(R^n)$, when $\alpha > 1$, take the integer p in the interval $\alpha - 1 \leqq p < \alpha$ and require that f have continuous partial derivatives of orders $1, \cdots, p$ and these fulfill a Lipschitz condition in exponent $\alpha - p$ on each compact set; to specify further that grad $f \neq 0$ throughout R^n, write Λ_+^α. Then a closed set T is a set of "Λ^α-multiplicity" if every transform $f(T) \subseteq R^1 (f \in \Lambda_+^\alpha)$ is a set of strict multiplicity—an M_0-set (see below). Henceforth we define $b = \alpha^{-1}$ and take S to be a closed linear set.

THEOREM 1. *In order that $X(S)$ be almost surely a set of Λ^α-multiplicity, it is sufficient that the Hausdorff dimension of S exceed b. It is not sufficient that $\dim S = b$.*

An M_0-set in R is one carrying a measure $\mu \neq 0$ whose Fourier-Stieltjes transform vanishes at infinity; the theory of M_0-sets is propounded in [**1**, p. 57] and [**8**, pp. 344, 348, 383] and Hausdorff dimension is treated in [**1**, II—III]. Theorem 1 reveals a difference between multi-dimensional Brownian motion and the linear process; for linear paths the critical point is dim $S = \frac{1}{2}b$ [**5**]. Theorem 2 below contains a sharper form of the sufficiency condition.

THEOREM 2. *Let S be a compact set, carrying a probability measure μ for which*

$$h(u) \equiv \sup \mu(x, x+u) = o(u^b) \cdot |\log u|^{-1}.$$

Then $X(S)$ is almost surely a set of Λ^α-multiplicity.

1. (Proof of Theorem 2) We can assume that S is mapped by X entirely within some fixed ball B in R^n and that all elements f appearing below are bounded in Λ^α-norm over B (defined in analogy with the norms in Banach spaces of Lipschitz functions). Moreover we can assume that all gradients fulfill an inequality $\|\nabla\| \geqq \delta > 0$ on all of B, and even on all of R^n.

(a) There is a function $\xi(u) > 0$ of u so that $\lim u^{-1}\xi(u) = +\infty$ and $h(\xi(u)) = o(u^b)\ |\log u|^{-1}$ as $u \to 0+$. In proving that all sets $f \circ X(S)$ are M_0-sets, we study integrals $\int \exp -2\pi i y f \circ X(s) \cdot \mu(ds)$, since these are the Fourier-Stieltjes transforms of probability measures carried by $f \circ X(S)$. Our plan is to estimate the probability of an event $|\int| > \eta$ for an individual f and y, and then combine a large enough number of these inequalities to obtain a bound for *all* functions f in question. The individual estimations are obtained as in **[5**, pp. 60–61], using the independence of increments of X. To obtain a uniform estimate on the expected values, similar to that in **[5]**, we divide S into intervals of length rather larger than y^{-2}. The expected values are then integral involving the normal density in R^n, and these are handled by integration first along straight lines approximately parallel to ∇f. For each $\eta > 0$ we find

$$P\{|\int \exp -2\pi i y f \circ X(s)\, \mu(ds)| > \eta\} < \exp - A(\eta)\psi(y) \log y \cdot y^{2b}$$

where $A(\eta) > 0$ and $\psi(y) \to +\infty$ with y.

(b) To each large y and $\eta > 0$ we shall find a determinate set $L(y)$ in Λ^α_+, with this property: there is a random number y_0, almost surely finite, and a random set $S^\#$ of μ-measure $1 - \eta$; to each function f in Λ^α_+ there is a function f_1 in $L(y)$, such that $|f - f_1| \leqq \eta y^{-1}$ on $X(S^\#)$—all this for $y > y_0$. Moreover $L(y)$ contains at most $\exp A'(\eta) y^{2b} \log y$ elements f_1. When $L(y)$ has been secured, we let y tend to $+\infty$ along the sequence $1, \sqrt{2}, \cdots, k^{1/2}, \cdots$ for example, and use the Borel-Cantelli Lemma to estimate the integrals involving $f_1 \in L(y)$. The properties of $L(y)$ allow us to extend our almost-sure inequalities to all of Λ^α_+.

At the corresponding stage in the treatment of linear Brownian motion, Kolmogorov's estimates of entropy in the space $\Lambda^\alpha[-1, 1]$ are exploited; an interesting aspect of the argument below is the minor role of the dimension n. Compare **[6**, Ch. 9–10].

(c) In carrying out the program of (b) we let y increase through the sequence $2^{k\alpha}$ $(k = 1, 2, 3, \cdots)$ and observe that the sets $L(2^{k\alpha})$ will serve for $2^{(k-1)\alpha} \leqq y \leqq 2^{k\alpha}$. To each $\eta > 0$ we can find a constant C_1 so large that the inequality $\|X(t)\| \leqq C_1$, $0 \leqq t \leqq 1$, is valid with $P > 1 - \frac{1}{2}\eta$. We divide the t-axis into adjacent intervals I of length 4^{-k} and write μ^*_k for the total μ-measure of those t-intervals on which $X(t)$ oscillates more than $2C_1 \cdot 2^{-k}$. By the scaling of X, and by independence of increments, we find upper bounds for the mean and variance of μ^*_k, namely $E(\mu^*_k) < \frac{1}{2}\eta$ and $\sigma^2(\mu^*_k) \leqq 0(1) h(4^{-k})$. By Chebyshev's inequality, $P\{\mu^*_k > \eta\} \leqq 0(1) h(4^{-k})$, and from $\Sigma\, h(4^{-k}) < +\infty$ we conclude that $\mu^*_k < \eta$ for large k, almost surely. The complementary intervals now form $S^\#$, so that $X(S^\#)$ is contained in $0(4^k)$ subsets of R^n, of diameter $C_1 \cdot 2^{1-k}$. (By our standing assumptions, $\|X(S^\#)\| \leqq B$). Let η_1 be a small constant, depending on η and the Lipschitz constants of the functions f, and let us cover the ball $\| X \| \leqq B$ with a grid of rectangles of side $\eta_1 2^{-k}$; for large n the grid contains $< 2^{(n+1)k}$ cells. Moreover $X(S^*)$ is contained in $C_2 4^k$ of these cells, and these cells can be chosen in at most $\exp C_3 k 4^k$ different ways. For each set T_0, composed of $C_2 4^k$ cells, we construct a "matching set" $L(y, T_0) \subseteq \Lambda_+^\alpha$ of the proper cardinality. As the sets T_0 are not too numerous, the join of all sets $L(y, T_0)$ in Λ_+^α will be our set $L(y)$.

On each cell we replace each f by its Taylor expansion about the center, up to derivatives of order p; if η_1 is sufficiently small, the Taylor expansion deviates from f by at most $1/8\ \eta \cdot 2^{-k\alpha}$, and the totality of functions so constructed has dimension $\leqq (p+1)^n \cdot C_2 4^k$. At points common to two or more cells in T, we replace the Taylor expansion by 0. Now we have a finite dimensional subspace of the Banach space of bounded functions on T—and by the inequality between "widths and entropy" [**6**, p. 164] the totality of approximating functions is contained in $\exp C_4 k 4^k$ sets of diameter $1/8\ \eta 2^{-k\alpha}$. From elementary inequalities in metric spaces, we can cover all the functions f by the same number of balls, of radius $\frac{1}{2}\eta \cdot 2^{-k\alpha}$ in the uniform norm on T, centered at functions f. Now $k 4^k = 0(1) y^{2b} \log y$ so the set $L(y)$ is small enough to complete the proof of Theorem 2.

2. (Proof of Theorem 1). First we find a set S of Hausdorff dimension b_1, arbitrarily close to b, such that $X(S)$ is not a set of Λ^α-multiplicity.

Let α_1 and c be chosen so that $b_1^{-1} > \alpha_1 > \alpha$ and $1 < c < \alpha^{-1}\alpha_1$. Then let M be a sequence of positive integers m such that each set $\{m \in M, m \leqq k\}$ has at least $b_1 k$ elements; then the set $S = S_M$ of all sums $\Sigma \pm 2^{-m}$ has Hausdorff dimension at least b_1. In addition, we assume that M contains infinitely many pairs of consecutive elements q, q_1 such that $q_1 > \alpha_1 q$. Sequences M exist because $\alpha_1 b_1 < 1$. Each number q of this type determines a division of S into at most 2^q subsets S_p, based on the coordinates for $m \leqq q$: each S_p has diameter $< 4 \cdot 2^{-q_1}$, and the sets S_p have mutual distances $\geqq 2^{-q-1}$.

For large enough q, the sets $X(S_p)$ are dispersed in a sense to be made precise in a moment. Taking an integer $s > 1 + (c-1)^{-1}$ we investigate the event that s distinct sets S_p are mapped within $d = 2^{-qc/2}$ of each other. By a famous inequality of Paul Lévy, the sets $X(S_p)$ have diameters $o(q_1 2^{-q_1/2}) = o(d)$ for large q, so we can simplify the calculation by taking $t_p \in S_p$ and bounding the probability that s numbers t_p are mapped within $2d$ of each other. We use the scaling property and independence of increments, with the observation that $n = 2$ is the least favorable case. An s-tuple leads to an event of probability $0(1) \cdot \Pi d^2 | u_{j+1} - u_j |^{-1}$. We sum this for all s-tuples chosen from the numbers t_p and recall that u_1 takes at most 2^q values. Each factor $d^2 | u_{j+1} - u_j |^{-1}$ adds a factor $2^q q \cdot d^2$ to the sum. From the formula

$d = 2^{-qc/2}$ and the inequality $(s-1)c-(s-1)>1$, we find that the sum has magnitude $2^{-\delta q}$ for some $\delta > 0$. The Borel–Cantelli Lemma then shows that the dispersion property holds for large q, with probability 1.

Now $X(S)$ is a union of sets of diameter $< d_1 = q_1 2^{-q_1/2}$ and at most $s-1$ sets $X(S_p)$ have mutual distances $< d$. Moreover $d > d_1^\beta$ for some $\beta < \alpha^{-1}$ because $c < \alpha^{-1}\alpha_1$. It is proved in [2, 5, p. 66] that $f \circ X(S)$ is not an M_0-set (nor even an M-set) for all f in Λ^α except a set of first category. Of course Λ_+^α is an open subset of Λ^α so the same is true of Λ_+^α.

To finish the proof of the negative statement in Theorem 1, we let b_1 increase to b along a sequence and choose a union of sets S_M, wherein M depends on b_1. As the union is countable, the union of the meager sets obtained for each S_M is again meager, and it is classical that, for measures μ such that $\hat{\mu}(\infty) = 0$, the entire space $L^1(\mu)$ inherits this property. This completes the proof of the second assertion in Theorem 1.

The positive assertion is a consequence of Theorem 2: by a theorem of Frostman [1, II–III] any closed set of Hausdorff dimension $> b$ carries a measure μ fulfilling the inequalities of Theorem 2.

A problem that appears much more difficult is the behavior of sets S with "strong dimension" b: S is not the union of a sequence US_m, $\dim S_m < b$. These sets can be characterized in the theory of Hausdorff measures [7]. Some of the analysis is done in [3,4].

References

1. J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, Hermann, Paris, 1963.

2. R. Kaufman, *A functional method for linear sets*, Israel J. Math. **5** (1967), 785–787.

3. R. Kaufman, *Une propriété métrique du mouvement brownien*, C.R. Acad. Sci. Paris **268 A** (1969), 727–728.

4. R. Kaufman, *Brownian motion and dimension of perfect sets*, Canad. J. Math. **22** (1970), 674–680.

5. R. Kaufman, *Brownian motion, approximation of functions, and Fourier analysis*, Studia Math. (to appear).

6. G. G. Lorentz, *Approximation of Functions*, Holt, New York, 1966.

7. C. A. Rogers, *Sets non-σ-finite for Hausdorff measures*, Mathematika **9** (1962), 95–103.

8. A. Zygmund, *Trigonometric Series I.* Cambridge, 1959 and 1966.

Received December 11, 1973 and in revised form March 15, 1974. Alfred P. Sloan Fellow.

University of Illinois

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $ 72.00 a year (6 Vols., 12 issues). Special rate: $ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics
Vol. 60, No. 2 October, 1975

Waleed A. Al-Salam and A. Verma, *A fractional Leibniz q-formula* 1
Robert A. Bekes, *Algebraically irreducible representations of $L_1(G)$* 11
Thomas Theodore Bowman, *Construction functors for topological semigroups* 27
Stephen LaVern Campbell, *Operator-valued inner functions analytic on the closed disc. II* 37
Leonard Eliezer Dor and Edward Wilfred Odell, Jr., *Monotone bases in L_p* 51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, *On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 = f$* 63
Y. Gordon, *Unconditional Schauder decompositions of normed ideals of operators between some l_p-spaces* 71
Gary Grefsrud, *Oscillatory properties of solutions of certain nth order functional differential equations* 83
Irvin Roy Hentzel, *Generalized right alternative rings* 95
Zensiro Goseki and Thomas Benny Rushing, *Embeddings of shape classes of compacta in the trivial range* 103
Emil Grosswald, *Brownian motion and sets of multiplicity* 111
Donald LaTorre, *A construction of the idempotent-separating congruences on a bisimple orthodox semigroup* 115
Pjek-Hwee Lee, *On subrings of rings with involution* 131
Marvin David Marcus and H. Minc, *On two theorems of Frobenius* 149
Michael Douglas Miller, *On the lattice of normal subgroups of a direct product* 153
Grattan Patrick Murphy, *A metric basis characterization of Euclidean space* 159
Roy Martin Rakestraw, *A representation theorem for real convex functions* 165
Louis Jackson Ratliff, Jr., *On Rees localities and H_i-local rings* 169
Simeon Reich, *Fixed point iterations of nonexpansive mappings* 195
Domenico Rosa, *B-complete and B_r-complete topological algebras* 199
Walter Roth, *Uniform approximation by elements of a cone of real-valued functions* 209
Helmut R. Salzmann, *Homogene kompakte projektive Ebenen* 217
Jerrold Norman Siegel, *On a space between BH and B_∞* 235
Robert C. Sine, *On local uniform mean convergence for Markov operators* 247
James D. Stafney, *Set approximation by lemniscates and the spectrum of an operator on an interpolation space* 253
Árpád Száz, *Convolution multipliers and distributions* 267
Kalathoor Varadarajan, *Span and stably trivial bundles* 277
Robert Breckenridge Warfield, Jr., *Countably generated modules over commutative Artinian rings* 289
John Yuan, *On the groups of units in semigroups of probability measures* 303