

# Pacific Journal of Mathematics

**BROWNIAN MOTION AND SETS OF MULTIPLICITY**

ROBERT P. KAUFMAN

# BROWNIAN MOTION AND SETS OF MULTIPLICITY

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$X(t)$  is Brownian motion on the axis  $-\infty < t < \infty$ , with paths in  $R^n$ ,  $n \geq 2$ .  $X(t)$  leads to composed mappings  $f \circ X$ , where  $f$  is a real-valued function of class  $\Lambda^\alpha(R^n)$ , whose gradient never vanishes. To define the class  $\Lambda^\alpha(R^n)$ , when  $\alpha > 1$ , take the integer  $p$  in the interval  $\alpha - 1 \leq p < \alpha$  and require that  $f$  have continuous partial derivatives of orders  $1, \dots, p$  and these fulfill a Lipschitz condition in exponent  $\alpha - p$  on each compact set; to specify further that  $\text{grad } f \neq 0$  throughout  $R^n$ , write  $\Lambda_+^\alpha$ . Then a closed set  $T$  is a set of " $\Lambda^\alpha$ -multiplicity" if every transform  $f(T) \subseteq R^1 (f \in \Lambda_+^\alpha)$  is a set of strict multiplicity—an  $M_0$ -set (see below). Henceforth we define  $b = \alpha^{-1}$  and take  $S$  to be a closed linear set.

**THEOREM 1.** *In order that  $X(S)$  be almost surely a set of  $\Lambda^\alpha$ -multiplicity, it is sufficient that the Hausdorff dimension of  $S$  exceed  $b$ . It is not sufficient that  $\dim S = b$ .*

An  $M_0$ -set in  $R$  is one carrying a measure  $\mu \neq 0$  whose Fourier-Stieltjes transform vanishes at infinity; the theory of  $M_0$ -sets is propounded in [1, p. 57] and [8, pp. 344, 348, 383] and Hausdorff dimension is treated in [1, II—III]. Theorem 1 reveals a difference between multi-dimensional Brownian motion and the linear process; for linear paths the critical point is  $\dim S = \frac{1}{2}b$  [5]. Theorem 2 below contains a sharper form of the sufficiency condition.

**THEOREM 2.** *Let  $S$  be a compact set, carrying a probability measure  $\mu$  for which*

$$h(u) \equiv \sup \mu(x, x + u) = o(u^b) \cdot |\log u|^{-1}.$$

*Then  $X(S)$  is almost surely a set of  $\Lambda^\alpha$ -multiplicity.*

1. (Proof of Theorem 2) We can assume that  $S$  is mapped by  $X$  entirely within some fixed ball  $B$  in  $R^n$  and that all elements  $f$  appearing below are bounded in  $\Lambda^\alpha$ -norm over  $B$  (defined in analogy with the norms in Banach spaces of Lipschitz functions). Moreover we can assume that all gradients fulfill an inequality  $\|\nabla\| \geq \delta > 0$  on all of  $B$ , and even on all of  $R^n$ .

(a) There is a function  $\xi(u) > 0$  of  $u$  so that  $\lim u^{-1}\xi(u) = +\infty$  and  $h(\xi(u)) = o(u^b) |\log u|^{-1}$  as  $u \rightarrow 0+$ . In proving that all sets  $f \circ X(S)$  are  $M_0$ -sets, we study integrals  $\int \exp -2\pi iy f \circ X(s) \cdot \mu(ds)$ , since these are the Fourier-Stieltjes transforms of probability measures carried by  $f \circ X(S)$ . Our plan is to estimate the probability of an event  $|\int| > \eta$  for an individual  $f$  and  $y$ , and then combine a large enough number of these inequalities to obtain a bound for *all* functions  $f$  in question. The individual estimations are obtained as in [5, pp. 60–61], using the independence of increments of  $X$ . To obtain a uniform estimate on the expected values, similar to that in [5], we divide  $S$  into intervals of length rather larger than  $y^{-2}$ . The expected values are then integral involving the normal density in  $R^n$ , and these are handled by integration first along straight lines approximately parallel to  $\nabla f$ . For each  $\eta > 0$  we find

$$P\{|\int \exp -2\pi iy f \circ X(s) \mu(ds)| > \eta\} < \exp -A(\eta)\psi(y)\log y \cdot y^{2b}$$

where  $A(\eta) > 0$  and  $\psi(y) \rightarrow +\infty$  with  $y$ .

(b) To each large  $y$  and  $\eta > 0$  we shall find a determinate set  $L(y)$  in  $\Lambda_+^\alpha$ , with this property: there is a random number  $y_0$ , almost surely finite, and a random set  $S^\#$  of  $\mu$ -measure  $1 - \eta$ ; to each function  $f$  in  $\Lambda_+^\alpha$  there is a function  $f_1$  in  $L(y)$ , such that  $|f - f_1| \leq \eta y^{-1}$  on  $X(S^\#)$ —all this for  $y > y_0$ . Moreover  $L(y)$  contains at most  $\exp A'(\eta)y^{2b} \log y$  elements  $f_1$ . When  $L(y)$  has been secured, we let  $y$  tend to  $+\infty$  along the sequence  $1, \sqrt{2}, \dots, k^{1/2}, \dots$  for example, and use the Borel-Cantelli Lemma to estimate the integrals involving  $f_1 \in L(y)$ . The properties of  $L(y)$  allow us to extend our almost-sure inequalities to all of  $\Lambda_+^\alpha$ .

At the corresponding stage in the treatment of linear Brownian motion, Kolmogorov's estimates of entropy in the space  $\Lambda^\alpha[-1, 1]$  are exploited; an interesting aspect of the argument below is the minor role of the dimension  $n$ . Compare [6, Ch. 9–10].

(c) In carrying out the program of (b) we let  $y$  increase through the sequence  $2^{k\alpha}$  ( $k = 1, 2, 3, \dots$ ) and observe that the sets  $L(2^{k\alpha})$  will serve for  $2^{(k-1)\alpha} \leq y \leq 2^{k\alpha}$ . To each  $\eta > 0$  we can find a constant  $C_1$  so large that the inequality  $\|X(t)\| \leq C_1$ ,  $0 \leq t \leq 1$ , is valid with  $P > 1 - \frac{1}{2}\eta$ . We divide the  $t$ -axis into adjacent intervals  $I$  of length  $4^{-k}$  and write  $\mu_k^*$  for the total  $\mu$ -measure of those  $t$ -intervals on which  $X(t)$  oscillates more than  $2C_1 \cdot 2^{-k}$ . By the scaling of  $X$ , and by independence of increments, we find upper bounds for the mean and variance of  $\mu_k^*$ , namely  $E(\mu_k^*) < \frac{1}{2}\eta$  and  $\sigma^2(\mu_k^*) \leq 0(1)h(4^{-k})$ . By Chebyshev's inequality,  $P\{\mu_k^* > \eta\} \leq 0(1)h(4^{-k})$ , and from  $\sum h(4^{-k}) < +\infty$  we conclude that  $\mu_k^* < \eta$  for large  $k$ , almost surely. The complementary intervals now form  $S^\#$ , so that  $X(S^\#)$  is contained in  $0(4^k)$  subsets of  $R^n$ , of diameter  $C_1 \cdot 2^{1-k}$ . (By our standing assumptions,  $\|X(S^\#)\| \leq B$ ). Let  $\eta_1$  be a small constant, depending on  $\eta$  and the Lipschitz constants of the

functions  $f$ , and let us cover the ball  $\|X\| \leq B$  with a grid of rectangles of side  $\eta_1 2^{-k}$ ; for large  $n$  the grid contains  $< 2^{(n+1)k}$  cells. Moreover  $X(S^*)$  is contained in  $C_2 4^k$  of these cells, and these cells can be chosen in at most  $\exp C_3 k 4^k$  different ways. For each set  $T_0$ , composed of  $C_2 4^k$  cells, we construct a "matching set"  $L(y, T_0) \subseteq \Lambda_\mp^\alpha$  of the proper cardinality. As the sets  $T_0$  are not too numerous, the join of all sets  $L(y, T_0)$  in  $\Lambda_\mp^\alpha$  will be our set  $L(y)$ .

On each cell we replace each  $f$  by its Taylor expansion about the center, up to derivatives of order  $p$ ; if  $\eta_1$  is sufficiently small, the Taylor expansion deviates from  $f$  by at most  $1/8 \eta \cdot 2^{-k\alpha}$ , and the totality of functions so constructed has dimension  $\leq (p+1)^n \cdot C_2 4^k$ . At points common to two or more cells in  $T$ , we replace the Taylor expansion by 0. Now we have a finite dimensional subspace of the Banach space of bounded functions on  $T$ —and by the inequality between "widths and entropy" [6, p. 164] the totality of approximating functions is contained in  $\exp C_4 k 4^k$  sets of diameter  $1/8 \eta 2^{-k\alpha}$ . From elementary inequalities in metric spaces, we can cover all the functions  $f$  by the same number of balls, of radius  $\frac{1}{2} \eta \cdot 2^{-k\alpha}$  in the uniform norm on  $T$ , centered at functions  $f$ . Now  $k 4^k = 0(1)y^{2b} \log y$  so the set  $L(y)$  is small enough to complete the proof of Theorem 2.

2. (Proof of Theorem 1). First we find a set  $S$  of Hausdorff dimension  $b_1$ , arbitrarily close to  $b$ , such that  $X(S)$  is not a set of  $\Lambda^\alpha$ -multiplicity.

Let  $\alpha_1$  and  $c$  be chosen so that  $b_1^{-1} > \alpha_1 > \alpha$  and  $1 < c < \alpha^{-1} \alpha_1$ . Then let  $M$  be a sequence of positive integers  $m$  such that each set  $\{m \in M, m \leq k\}$  has at least  $b_1 k$  elements; then the set  $S = S_M$  of all sums  $\Sigma \pm 2^{-m}$  has Hausdorff dimension at least  $b_1$ . In addition, we assume that  $M$  contains infinitely many pairs of consecutive elements  $q, q_1$  such that  $q_1 > \alpha_1 q$ . Sequences  $M$  exist because  $\alpha_1 b_1 < 1$ . Each number  $q$  of this type determines a division of  $S$  into at most  $2^q$  subsets  $S_p$ , based on the coordinates for  $m \leq q$ : each  $S_p$  has diameter  $< 4 \cdot 2^{-q_1}$ , and the sets  $S_p$  have mutual distances  $\geq 2^{-q-1}$ .

For large enough  $q$ , the sets  $X(S_p)$  are dispersed in a sense to be made precise in a moment. Taking an integer  $s > 1 + (c-1)^{-1}$  we investigate the event that  $s$  distinct sets  $S_p$  are mapped within  $d = 2^{-qc/2}$  of each other. By a famous inequality of Paul Lévy, the sets  $X(S_p)$  have diameters  $o(q_1 2^{-q_1/2}) = o(d)$  for large  $q$ , so we can simplify the calculation by taking  $t_p \in S_p$  and bounding the probability that  $s$  numbers  $t_p$  are mapped within  $2d$  of each other. We use the scaling property and independence of increments, with the observation that  $n = 2$  is the least favorable case. An  $s$ -tuple leads to an event of probability  $0(1) \cdot \Pi d^2 |u_{j+1} - u_j|^{-1}$ . We sum this for all  $s$ -tuples chosen from the numbers  $t_p$  and recall that  $u_1$  takes at most  $2^q$  values. Each factor  $d^2 |u_{j+1} - u_j|^{-1}$  adds a factor  $2^q \cdot d^2$  to the sum. From the formula

$d = 2^{-qc/2}$  and the inequality  $(s-1)c - (s-1) > 1$ , we find that the sum has magnitude  $2^{-\delta q}$  for some  $\delta > 0$ . The Borel–Cantelli Lemma then shows that the dispersion property holds for large  $q$ , with probability 1.

Now  $X(S)$  is a union of sets of diameter  $< d_1 = q_1 2^{-q_1 c}$  and at most  $s-1$  sets  $X(S_p)$  have mutual distances  $< d$ . Moreover  $d > d_1^\beta$  for some  $\beta < \alpha^{-1}$  because  $c < \alpha^{-1} \alpha_1$ . It is proved in [2, 5, p. 66] that  $f \circ X(S)$  is not an  $M_0$ -set (nor even an  $M$ -set) for all  $f$  in  $\Lambda^\alpha$  except a set of first category. Of course  $\Lambda_+^\alpha$  is an open subset of  $\Lambda^\alpha$  so the same is true of  $\Lambda_+^\alpha$ .

To finish the proof of the negative statement in Theorem 1, we let  $b_1$  increase to  $b$  along a sequence and choose a union of sets  $S_M$ , wherein  $M$  depends on  $b_1$ . As the union is countable, the union of the meager sets obtained for each  $S_M$  is again meager, and it is classical that, for measures  $\mu$  such that  $\hat{\mu}(\infty) = 0$ , the entire space  $L^1(\mu)$  inherits this property. This completes the proof of the second assertion in Theorem 1.

The positive assertion is a consequence of Theorem 2: by a theorem of Frostman [1, II–III] any closed set of Hausdorff dimension  $> b$  carries a measure  $\mu$  fulfilling the inequalities of Theorem 2.

A problem that appears much more difficult is the behavior of sets  $S$  with “strong dimension”  $b$ :  $S$  is not the union of a sequence  $US_m$ ,  $\dim S_m < b$ . These sets can be characterized in the theory of Hausdorff measures [7]. Some of the analysis is done in [3,4].

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