BROWNIAN MOTION AND SETS OF MULTIPLICITY

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\[ X(t) \] is Brownian motion on the axis \(-\infty < t < \infty, \) with paths in \( R^n, n \geq 2. \) \( X(t) \) leads to composed mappings \( f \circ X, \) where \( f \) is a real-valued function of class \( \Lambda^\alpha(R^n), \) whose gradient never vanishes. To define the class \( \Lambda^\alpha(R^n), \) when \( \alpha > 1, \) take the integer \( p \) in the interval \( \alpha - 1 \leq p < \alpha \) and require that \( f \) have continuous partial derivatives of orders \( 1, \ldots, p \) and these fulfill a Lipschitz condition in exponent \( \alpha - p \) on each compact set; to specify further that \( \text{grad } f \neq 0 \) throughout \( R^n, \) write \( \Lambda^\alpha. \) Then a closed set \( T \) is a set of "\( \Lambda^\alpha \)-multiplicity" if every transform \( f(T) \subseteq R^1(\ f \in \Lambda^\alpha) \) is a set of strict multiplicity—an \( M_0 \)-set (see below). Henceforth we define \( b = \alpha^{-1} \) and take \( S \) to be a closed linear set.

**THEOREM 1.** In order that \( X(S) \) be almost surely a set of \( \Lambda^\alpha \)-multiplicity, it is sufficient that the Hausdorff dimension of \( S \) exceed \( b. \) It is not sufficient that \( \text{dim } S = b. \)

An \( M_0 \)-set in \( R \) is one carrying a measure \( \mu \neq 0 \) whose Fourier-Stieltjes transform vanishes at infinity; the theory of \( M_0 \)-sets is propounded in [1, p. 57] and [8, pp. 344, 348, 383] and Hausdorff dimension is treated in [1, II—III]. Theorem 1 reveals a difference between multi-dimensional Brownian motion and the linear process; for linear paths the critical point is \( \text{dim } S = \frac{1}{2} b \) [5]. Theorem 2 below contains a sharper form of the sufficiency condition.

**THEOREM 2.** Let \( S \) be a compact set, carrying a probability measure \( \mu \) for which

\[ h(u) = \sup \mu(x, x + u) = o(u^b) \cdot |\log u|^{-1}. \]

Then \( X(S) \) is almost surely a set of \( \Lambda^\alpha \)-multiplicity.

1. (Proof of Theorem 2) We can assume that \( S \) is mapped by \( X \) entirely within some fixed ball \( B \) in \( R^n \) and that all elements \( f \) appearing below are bounded in \( \Lambda^\alpha \)-norm over \( B \) (defined in analogy with the norms in Banach spaces of Lipschitz functions). Moreover we can assume that all gradients fulfill an inequality \( ||\nabla|| \geq \delta > 0 \) on all of \( B, \) and even on all of \( R^n. \)
(a) There is a function $\xi(u) > 0$ of $u$ so that $\lim u^{-1} \xi(u) = +\infty$ and $h(\xi(u)) = o(u^b) |\log u|^{-1}$ as $u \to 0 +$. In proving that all sets $f \circ X(S)$ are $M_0$-sets, we study integrals $\int \exp -2\pi iy f \circ X(s) \cdot \mu(ds)$, since these are the Fourier-Stieltjes transforms of probability measures carried by $f \circ X(S)$. Our plan is to estimate the probability of an event $|f| > \eta$ for an individual $f$ and $y$, and then combine a large enough number of these inequalities to obtain a bound for all functions $f$ in question. The individual estimations are obtained as in [5, pp. 60–61], using the independence of increments of $X$. To obtain a uniform estimate on the expected values, similar to that in [5], we divide $S$ into intervals of length rather larger than $y^{-2}$. The expected values are then integral involving the normal density in $\mathbb{R}^n$, and these are handled by integration first along straight lines approximately parallel to $\nabla f$. For each $\eta > 0$ we find

$$P \{|\int \exp -2\pi iy f \circ X(s) \cdot \mu(ds)| > \eta\} < \exp -A(\eta) \psi(y) \log y \cdot y^{2b}$$

where $A(\eta) > 0$ and $\psi(y) \to +\infty$ with $y$.

(b) To each large $y$ and $\eta > 0$ we shall find a determinate set $L(y)$ in $\Lambda_\eta^*$, with this property: there is a random number $y_0$, almost surely finite, and a random set $S^\ast$ of $\mu$-measure $1 - \eta$; to each function $f$ in $\Lambda_\eta^*$ there is a function $f_1$, in $L(y)$, such that $|f - f_1| \leq \eta y^{-1}$ on $X(S^\ast)$—all this for $y > y_0$. Moreover $L(y)$ contains at most $\exp A'(\eta) y^{2b} \log y$ elements $f_1$. When $L(y)$ has been secured, we let $y$ tend to $+\infty$ along the sequence $1, \sqrt{2}, \ldots, k^{1/2}, \ldots$ for example, and use the Borel-Cantelli Lemma to estimate the integrals involving $f_1 \in L(y)$. The properties of $L(y)$ allow us to extend our almost-sure inequalities to all of $\Lambda_\eta^*$.

At the corresponding stage in the treatment of linear Brownian motion, Kolmogorov's estimates of entropy in the space $\Lambda_\eta^*[−1,1]$ are exploited; an interesting aspect of the argument below is the minor role of the dimension $n$. Compare [6, Ch. 9–10].

(c) In carrying out the program of (b) we let $y$ increase through the sequence $2^{ka}$ ($k = 1, 2, 3, \ldots$) and observe that the sets $L(2^{ka})$ will serve for $2^{(k-1)a} \leq y \leq 2^{ka}$. To each $\eta > 0$ we can find a constant $C_1$ so large that the inequality $\|X(t)\| \leq C_1$, $0 \leq t \leq 1$, is valid with $P > 1 - \frac{1}{2} \eta$. We divide the $t$-axis into adjacent intervals $I$ of length $4^{-k}$ and write $\mu^*_t$ for the total $\mu$-measure of those $t$-intervals on which $X(t)$ oscillates more than $2C_1 \cdot 2^{-k}$. By the scaling of $X$, and by independence of increments, we find upper bounds for the mean and variance of $\mu^*_t$, namely $E(\mu^*_t) \leq \frac{1}{2} \eta$ and $\sigma^2(\mu^*_t) \leq 0(1) h(4^{-k})$. By Chebyshev's inequality, $P\{\mu^*_t > \eta\} \leq 0(1) h(4^{-k})$, and from $\sum h(4^{-k}) < +\infty$ we conclude that $\mu^*_t < \eta$ for large $k$, almost surely. The complementary intervals now form $S^\ast$, so that $X(S^\ast)$ is contained in $0(4^\ast)$ subsets of $\mathbb{R}^n$, of diameter $C_1 \cdot 2^{1-k}$. (By our standing assumptions, $\|X(S^\ast)\| \leq B$). Let $\eta_1$ be a small constant, depending on $\eta$ and the Lipschitz constants of the
functions \( f \), and let us cover the ball \( \|X\| \leq B \) with a grid of rectangles of side \( \eta_1 2^{-k} \); for large \( n \) the grid contains \( < 2^{(n+1)k} \) cells. Moreover \( X(S^*) \) is contained in \( C_2 4^k \) of these cells, and these cells can be chosen in at most \( \exp C_1 k 4^k \) different ways. For each set \( T_0 \), composed of \( C_2 4^k \) cells, we construct a “matching set” \( L(y, T_0) \subseteq \Lambda^* \) of the proper cardinality. As the sets \( T_0 \) are not too numerous, the join of all sets \( L(y, T_0) \in \Lambda^* \) will be our set \( L(y) \).

On each cell we replace each \( f \) by its Taylor expansion about the center, up to derivatives of order \( p \); if \( \eta_1 \) is sufficiently small, the Taylor expansion deviates from \( f \) by at most \( \frac{1}{8} \eta_1 2^{-ka} \), and the totality of functions so constructed has dimension \( \leq (p + 1)^n \cdot C_2 4^k \). At points common to two or more cells in \( T \), we replace the Taylor expansion by 0. Now we have a finite dimensional subspace of the Banach space of bounded functions on \( T \) — and by the inequality between “widths and entropy” [6, p. 164] the totality of approximating functions is contained in \( \exp C_4 k 4^k \) sets of diameter \( \frac{1}{8} \eta_1 2^{-ka} \). From elementary inequalities in metric spaces, we can cover all the functions \( f \) by the same number of balls, of radius \( \frac{1}{2} \eta_1 2^{-ka} \) in the uniform norm on \( T \), centered at functions \( f \). Now \( k 4^k = 0(1) y^{2b} \log y \) so the set \( L(y) \) is small enough to complete the proof of Theorem 2.

2. (Proof of Theorem 1). First we find a set \( S \) of Hausdorff dimension \( b_1 \), arbitrarily close to \( b \), such that \( X(S) \) is not a set of \( \Lambda^* \)-multiplicity.

Let \( \alpha \) and \( c \) be chosen so that \( b_1^{-1} > \alpha > \alpha^{-1} \alpha_1 \). Then let \( M \) be a sequence of positive integers \( m \) such that each set \( \{m \in M, m \leq k\} \) has at least \( b_1 k \) elements; then the set \( S = S_m \) of all sums \( \Sigma \pm 2^{-m} \) has Hausdorff dimension at least \( b_1 \). In addition, we assume that \( M \) contains infinitely many pairs of consecutive elements \( q, q_1 \) such that \( q_1 > \alpha_1 q \). Sequences \( M \) exist because \( \alpha_1 b_1 < 1 \). Each number \( q \) of this type determines a division of \( S \) into at most \( 2^q \) subsets \( S_p \), based on the coordinates for \( m \in q \): each \( S_p \) has diameter \( < 4 \cdot 2^{-c} \), and the sets \( S_p \) have mutual distances \( \geq 2^{-q-1} \).

For large enough \( q \), the sets \( X(S_p) \) are dispersed in a sense to be made precise in a moment. Taking an integer \( s > 1 + (c - 1)^{-1} \) we investigate the event that \( s \) distinct sets \( S_p \) are mapped within \( d = 2^{-\sigma/\sigma} \) of each other. By a famous inequality of Paul Lévy, the sets \( X(S_p) \) have diameters \( o(q_12^{-q_1/2}) = o(d) \) for large \( q \), so we can simplify the calculation by taking \( t_p \in S_p \) and bounding the probability that \( s \) numbers \( t_p \) are mapped within \( 2d \) of each other. We use the scaling property and independence of increments, with the observation that \( n = 2 \) is the least favorable case. An \( s \)-tuple leads to an event of probability \( 0(1) \cdot \Pi d^2 |u_{i+1} - u_i|^{-1} \). We sum this for all \( s \)-tuples chosen from the numbers \( t_p \) and recall that \( u_1 \) takes at most \( 2^q \) values. Each factor \( d^2 |u_{i+1} - u_i|^{-1} \) adds a factor \( 2^q q \cdot d^2 \) to the sum. From the formula
\[ d = 2^{-\alpha/2} \] and the inequality \((s - 1)c - (s - 1) > 1\), we find that the sum has magnitude \(2^{\delta q} \) for some \(\delta > 0\). The Borel–Cantelli Lemma then shows that the dispersion property holds for large \(q\), with probability 1.

Now \(X(S)\) is a union of sets of diameter \(< d_1 = q_12^{-\alpha_1} \) and at most \(s - 1\) sets \(X(S_p)\) have mutual distances \(< d\). Moreover \(d > d_1^6\) for some \(\beta < \alpha^{-1}\) because \(c < \alpha^{-1}a_1\). It is proved in [2, 5, p. 66] that \(f \circ X(S)\) is not an \(M_0\)-set (nor even an \(M\)-set) for all \(f\) in \(\Lambda^a\) except a set of first category. Of course \(\Lambda^a\) is an open subset of \(\Lambda^a\) so the same is true of \(\Lambda^a\).

To finish the proof of the negative statement in Theorem 1, we let \(b_1\) increase to \(b\) along a sequence and choose a union of sets \(S_M\), wherein \(M\) depends on \(b_1\). As the union is countable, the union of the meager sets obtained for each \(S_M\) is again meager, and it is classical that, for measures \(\mu\) such that \(\hat{\mu}(\infty) = 0\), the entire space \(L^1(\mu)\) inherits this property. This completes the proof of the second assertion in Theorem 1.

The positive assertion is a consequence of Theorem 2: by a theorem of Frostman [1, II–III] any closed set of Hausdorff dimension \(> b\) carries a measure \(\mu\) fulfilling the inequalities of Theorem 2.

A problem that appears much more difficult is the behavior of sets \(S\) with "strong dimension" \(b\): \(S\) is not the union of a sequence \(U S_m\), \(\dim S_m < b\). These sets can be characterized in the theory of Hausdorff measures [7]. Some of the analysis is done in [3,4].

**References**


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