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We give a systematic account on the relationship between a ring R with involution and its subrings $\bar{S}$ and $\bar{K}$, which are generated by all its symmetric elements or skew elements respectively.

I. Introduction. Let R be a ring with involution * and $\bar{S}$ the subring generated by the set S of all symmetric elements in R. The relationship between R and $\bar{S}$ has been studied by various authors. In [3] Dieudonné showed that if R is a division ring of characteristic not 2, then either $\bar{S} = R$ or $\bar{S} \subseteq Z(R)$, the center of R. Later Herstein [4] extended this result by proving $\bar{S} = R$ for any simple ring R with $\dim_Z R > 4$ and char.$R \neq 2$. The restriction on characteristic was removed by Montgomery [12]. Recently, Lanski [9] proved that if R is prime or semi-prime, so is $\bar{S}$. In §2 of this paper, we show that $\bar{S}$ can inherit a number of ring-theoretic properties such as primitivity, semi-simplicity, absence of nonzero nil ideals etc.. In doing so, a notion called *symmetric subring*, which is a generalization of $\bar{S}$ and its *-homomorphic images, is introduced so that a group of theorems of the same type, including Lanski's results, can be proved via a more or less unified argument. We show also that numerous radicals of $\bar{S}$ are merely the contractions from those of R. As a consequence, we see that R modulo its prime radical behaves much like $\bar{S}$ in many respects.

In §3 we establish a corresponding theory for $\bar{K}$, the subring generated by all skew elements. The only result hitherto known concerning $\bar{K}$ was as follows [4], [12]: If R is simple and $\dim_Z R > 4$, then $\bar{K} = R$. As a matter of fact, the subring $\overline{K^2}$ is more closely related to R than $\bar{K}$ is. We apply the technique developed in §2 to study the relationship between R and $\overline{K^2}$, and then derive some parallel theorems for $\bar{K}$.

II. Symmetric subrings. Our work depends heavily on the notion of *Lie ideals*. By a Lie ideal U of R we mean an additive subgroup which is invariant under all inner derivations of R. That is, $[u, x] = ux - xu \in U$ for all $u \in U$ and $x \in R$. The following lemma concerning Lie ideals will be referred to frequently in the sequel, and it is a combination of some results in [5].

LEMMA 1. *Let R be a semi-prime ring and U a subring and Lie ideal of R. Then U contains the ideal of R which is generated by $[U, U]$. If U is commutative, then $u^2 \in Z$ for all $u \in U$.*

Rings with involution abound with examples of Lie ideals. One can easily show that any subring, generated by symmetric elements and containing $T = \{x + x^* \mid x \in R\}$ the set of all traces, must be a Lie ideal. In particular, both $\bar{S}$ and $\bar{T}$ are Lie ideals.

Another essential property of $\bar{S}$ follows from the next lemma. We denote by N the set of all norms, i.e. $N = \{xx^* \mid x \in R\}$.

LEMMA 2. *Let U be an additive subgroup of R such that $T \subseteq U \subseteq S$ and $xUx^* \subseteq U$ for all $x \in R$. If $N \subseteq \bar{U}$, then $x\bar{U}x^* \subseteq \bar{U}$ for all $x \in R$.*

Proof. We prove by induction that $xu_1 \cdots u_n x^* \in \bar{U}$ for all $x \in R$ and $u_1, \cdots, u_n \in U$. The case $n = 1$ is clear. Assume the assertion holds for $n - 1$; then

$$xu_1u_2 \cdots u_n x^* = [x, u_1][u_2 \cdots u_n, x^*] + (xu_1x^*)u_2 \cdots u_n + u_1(xu_2 \cdots u_n x^*) - u_1xx^*u_2 \cdots u_n \in \bar{U}$$

because $\bar{U}$ is a Lie ideal.

DEFINITION. *A subring U of R is called a symmetric subring if:*
1. *U is generated by a set of symmetric elements.*
2. *$T \cup N \subseteq U$*
3. *$xUx^* \subseteq U$ for all $x \in R$.*

In light of Lemma 2, we know that $\bar{S}$ is a symmetric subring. From now on, U will always denote a symmetric subring of R. We call an ideal I of R a $*$-ideal if $I^* = I$.

LEMMA 3. *If R is semi-prime and I is a $*$-ideal of R such that $I \cap U = 0$, then $I = 0$.*

Proof. For any $a \in I$, $a^2 = a(a + a^*) - aa^* = 0$. Then I is nil of index 2 and hence $I = 0$.

Recall that a ring R is called a $*$-simple ring if $R^2 \neq 0$ and R has no $*$-ideal other than 0 and R. It is well-known that R is $*$-simple if and only if either R is simple or $R = A \oplus A^*$ for some simple ring A [**8**, p. 14]. Let $Z^+ = Z \cap S$. Then if R is $*$-simple, we have $Z^+ = 0$ or Z^+ is a field.

THEOREM 4. *If R is $*$-simple, then either $U = R$ or U is a field contained in Z^+.*

Proof. If U is not commutative, by Lemma 1 it contains a nonzero $*$-ideal of R so $U = R$. Assume that $[U, U] = 0$; then $U \subseteq S$. In this case, we need only to prove $U \subseteq Z$, for if $u \in U$ and $u \neq 0$ then $u^{-1} = u^{-1}u(u^{-1})^* \in U$.

If $R = A \oplus A^*$ for some simple ring A, then $T = U = S$. Thus $[U, U] = 0$ implies $[A, A] = 0$ and so R is commutative. If R is simple, then U, being a commutative subring and Lie ideal of R, must be central unless $2R = 0$ and $\dim_Z R = 4$ [5, Theorem 1.5]. So let us examine all possible 4-dimensional cases.

If R is a division ring, then $x^{-1}Ux = x^{-1}(xUx^*)x = Ux^*x \subseteq U$ for all $x \in R$ with $x \neq 0$. Hence $U \subseteq Z$ by the Brauer-Cartan-Hua theorem [7, Theorem 7.13.1, Cor.].

There remains the case $R = F_2$ where F is a field with char.$F = 2$. We claim that * must be of symplectic type. Assume the contrary,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & \alpha^{-1}\bar{c} \\ \alpha\bar{b} & \bar{d} \end{bmatrix}$$

for some $\alpha \in F$ with $\bar{\alpha} = \alpha$, where $-$ denotes the induced automorphism on F. Thus

$$U \subseteq S = \left\{ \begin{bmatrix} a & b \\ \alpha\bar{b} & c \end{bmatrix} \mid \bar{a} = a, \bar{c} = c \right\}.$$

For any $a \in F$, we have

$$\begin{bmatrix} 0 & a+\bar{a} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+\bar{a} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \in T^2 \subseteq U$$

so $\bar{a} = a$. Next, if $\begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix} \in U$ then

$$\begin{bmatrix} b & 0 \\ a+c & b \end{bmatrix} = \begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix} \in U$$

and hence $a = c$. But if $\begin{bmatrix} a & b \\ \alpha b & a \end{bmatrix} \in U$, then

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha b & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U$$

yields $a = 0$. So $U = T = \left\{ \begin{bmatrix} 0 & b \\ \alpha b & 0 \end{bmatrix} \middle| \, b \in F \right\}$ which is ridiculous because T is not a subring. Consequently, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$ and

$$U \subseteq S = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \mid a, b, c \in F \right\}.$$

For any $\begin{bmatrix} a & b \\ c & a \end{bmatrix}, \begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \in U$, we have $\begin{bmatrix} a & b \\ c & a \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \in U$ and hence $bc' = b'c$ by comparing the diagonal entries of the product. If there exists $\begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \in U$ with $b' \neq 0$, then

$$U \subseteq \left\{ \begin{bmatrix} a & b \\ \alpha b & a \end{bmatrix} \mid a, b \in F \right\},$$

where $\alpha = c'b'^{-1}$. However,

$$\begin{bmatrix} 0 & 0 \\ b' & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in U$$

forces $b' = 0$, a contradiction. Hence $U \subseteq \left\{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \mid a, c \in F \right\}$. On the other hand, if $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \in U$,

$$\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in U$$

implies $c = 0$. Therefore, $U \subseteq Z$.

Following **[11]**, we say R is **-prime* if the product of any two nonzero *-ideals is still not zero. It is easy to see that R is *-prime if and only if $aRb = a^*Rb = 0$ implies $a = 0$ or $b = 0$. As a consequence, any nonzero element in Z^+ is regular in a *-prime ring R.

We remind the reader of of a well-known fact that a nonzero Lie ideal of a semi-prime ring always contains elements with nonzero square.

THEOREM 5. *If R is *-prime, so is U.*

Proof. If $[U, U] \neq 0$, then U contains a nonzero *-ideal I of R. For any two *-ideals A, B of U with $AB = 0$, we have $IAIB \subseteq AB = 0$, so either $IAI = 0$ or $B = 0$, ending up with $A = 0$ or $B = 0$. Assume that $U \neq 0$ while $[U, U] = 0$. By Lemma 1, there exists $u_0 \in U$ such that $u_0^2 \in Z$ but $u_0^2 \neq 0$. So consider the ring Q of fractions a/α with $a \in R$ and $\alpha \in Z \cap U$, $\alpha \neq 0$. Q is also *-prime with respect to the involution given by $(a/\alpha)^* = a^*/\alpha$, and $U' = \{u/\alpha \in Q \mid u \in U\}$ is a symmetric subring of Q. As a matter of fact, Q is *-simple. For if J is any nonzero *-ideal of Q, $J \cap U' \neq 0$ and hence $(v/\beta)^2 \neq 0$ for some $v/\beta \in J \cap U'$. Since $v^2 \in Z$, v/β is invertible and so $J = Q$. By the previous theorem, $U' \subseteq Z^+(Q)$ and hence U is an integral domain contained in $Z^+(R)$.

Let $C_R(V) = \{x \in R \mid xv = vx \text{ for all } v \in V\}$ be the centralizer of a set V in R.

LEMMA 6. *Let $I \neq 0$ be an ideal (or *-ideal) of a prime* (resp. **-prime*) *ring R. Then $C_R(I) \subseteq Z$.*

Proof. For $a \in I$, $b \in C_R(I)$ and $x \in R$, we have $abx = bax = axb$, or equivalently, $a(bx - xb) = 0$. That is, $I[C_R(I), R] = 0$. Hence $[C_R(I), R] = 0$ and so $C_R(I) \subseteq Z$.

COROLLARY. *Let R be a prime (or *-prime) ring and I a nonzero ideal* (resp. **-ideal*) *of R such that $[I, I] = 0$. Then R is commutative.*

THEOREM 7. *If R is semi-prime, then $Z(U) \subseteq Z(R)$.*

Proof. Assume first that R is *-prime. If $[U, U] = 0$, then $Z(U) = U \subseteq Z(R)$ by Theorem 5. If $[U, U] \neq 0$, then U contains a nonzero *-ideal I of R, so $Z(U) \subseteq C_R(I) \subseteq Z(R)$ in view of Lemma 6. In either case, $[Z(U), R] = 0$. Now assume that R is semi-prime; then R is a subdirect sum of *-prime rings $\pi_\alpha(R)$. Since $\pi_\alpha(U)$ is a symmetric subring of $\pi_\alpha(R)$, we know $[\pi_\alpha(Z(U)), \pi_\alpha(R)] \subseteq [Z(\pi_\alpha(U)), \pi_\alpha(R)] = 0$ for all α. Hence, $[Z(U), R] = 0$.

The same reduction to *-prime rings together with Theorem 5 gives an alternate proof for Lanski's theorem:

THEOREM 8. *If R is semi-prime, so is U.*

With this established, we are able to consider the relationship between the prime radicals $\mathfrak{P}(R)$ and $\mathfrak{P}(U)$.

THEOREM 9. $\mathfrak{P}(U) = U \cap \mathfrak{P}(R)$.

Proof. Since $U/[U \cap \mathfrak{P}(R)] \simeq [U + \mathfrak{P}(R)]/\mathfrak{P}(R)$ which is a symmetric subring of the semi-prime ring $R/\mathfrak{P}(R)$, so $U/[U \cap \mathfrak{P}(R)]$ is semi-prime by Theorem 8 and hence $\mathfrak{P}(U) \subseteq U \cap \mathfrak{P}(R)$. On the other hand, if $a \in U \cap \mathfrak{P}(R)$, then $a \in U$ and any m-system in R containing a must contain 0. [**7**, Theorem 8.2.3]. Certainly, any m-system in U containing a contains 0. That is, $a \in \mathfrak{P}(U)$.

It is well-known that a ring without nonzero nil ideals is a subdirect sum of rings with the following property [**6**, p. 53]:

There exists a nonnilpotent element a such that $a^{n(I)} \in I$ for all nonzero ideal I.

One can impose this condition only on the *-ideals and show that it is a hereditary property. Then, making use of subdirect sum decomposition, we can prove that U inherits the freedom from nonzero nil ideals. Instead of doing this way, we prefer to present a direct proof by considering the nil radical $\mathfrak{N}(U)$ of U.

THEOREM 10. *If R has no nil ideal other than* 0, *neither does U.*

Proof. Let I be the ideal of R which is generated by $[U, U]$. Since R possesses no nonzero nil ideal, neither does I, considered as a ring. Hence $\mathfrak{N}(U) \cap I = 0$. For any $a \in \mathfrak{N}(U)$ and $u \in U$, we have $[a, u] \in \mathfrak{N}(U) \cap I = 0$. Thus $\mathfrak{N}(U) \subseteq Z(U)$. Since U is semi-prime by Theorem 8, $\mathfrak{N}(U) = 0$.

As an immediate consequence, we have

THEOREM 11. $\mathfrak{N}(U) = U \cap \mathfrak{N}(R)$.

Proceed as above with "locally nilpotent" in place of "nil" and with Levitzki radical $\mathfrak{L}$ in place of $\mathfrak{N}$, we get

THEOREM 12. *If R has no nonzero locally nilpotent ideal, neither does U.*

THEOREM 13. $\mathfrak{L}(U) = U \cap \mathfrak{L}(R)$.

In [2] the notion of **-primitive ring* was introduced as a ring admitting a **-faithful* irreducible module M (i.e. $Mr = Mr^* = 0$ implies $r = 0$). One can easily verify that a ring is *-primitive if and only if it is either primitive or a subdirect sum of a primitive ring and its opposite with the exchange involution.

We know that a nonzero ideal of a primitive ring is itself primitive. The proof is applicable to the following more general fact.

LEMMA 14. *Let R be a primitive* (*or *-primitive*) *ring. Suppose that I is a nonzero ideal* (resp. **-ideal*) *of R, and A is a subring* (resp. **-subring*, i.e. $A^* = A$) *containing I. Then A is also primitive* (resp. **-primitive*).

THEOREM 15. *If R is primitive or *-primitive, so is U.*

Proof. If $[U, U] \neq 0$, U contains a nonzero *-ideal of R, so it is primitive or *-primitive by Lemma 14. Assume that U is commutative. Then $U \subseteq Z^+$ and every element in R is quadratic over

Z^+. Hence R satisfies a polynomial identity. According to Kaplansky's theorem [**6**, Theorem 6.3.1], R is *-simple and hence U is a field by Theorem 4.

Using the fact that a semi-simple ring is a subdirect sum of *-primitive rings, we get immediately

THEOREM 16. *If R is semi-simple, so is U.*

In fact, the semi-simplicity of $\bar{S}$ was first proved by Herstein. His elegant proof was the inspiration of our next theorem which relates the Jacobson radicals of R and U.

THEOREM 17. $\mathfrak{J}(U) = U \cap \mathfrak{J}(R)$.

Proof. For $a \in \mathfrak{J}(U)$ and $x \in R$, we have

$$ax \circ ax^* = ax + ax^* + axax^* = a(x + x^* + xax^*) \in \mathfrak{J}(U)U \subseteq \mathfrak{J}(U).$$

Thus aR is quasi-regular and hence $a \in U \cap \mathfrak{J}(R)$. Conversely, if $a \in U \cap \mathfrak{J}(R)$, $a \circ b = 0$ for some $b \in R$, then $b = b \circ (a \circ b)^* = (b \circ b^*) \circ a^* \in U$. That is, $U \cap \mathfrak{J}(R)$ is a quasi-regular ideal of U, so $U \cap \mathfrak{J}(R) \subseteq \mathfrak{J}(U)$.

With Theorem 17 in hand, we are ready to study some non-semi-simple rings. Following [**7**], we say R is *semi-primary*, *primary*, or *completely primary* according as $R/\mathfrak{J}(R)$ is an artinian, simple artinian, or division ring respectively. Since $U/\mathfrak{J}(U)$ is isomorphic to a symmetric subring of $R/J(R)$, by Theorem 4 we have

THEOREM 18. *If R is primary or completely primary, so is U.*

As to semi-primary rings, we need some information about the descending chain condition. In a paper [**10**] which is to appear, Lanski proved that if R is artinian and $\frac{1}{2} \in R$, then so is $\bar{S}$. For our purpose, we prove

LEMMA 19. *If R is semi-prime artinian, so is U.*

Proof. By the Wedderburn-Artin theorem, we may write $R = R_1 \oplus \cdots \oplus R_n$ where each R_i is *-simple. Denote by e_i the identity of R_i, then $e_i \in Z^+$ and so e_iUe_i is a symmetric subring of R_i for each i. By Theorem 4, each e_iUe_i is artinian, so is $U = e_1Ue_1 \oplus \cdots \oplus e_nUe_n$.

THEOREM 20. *If R is semi-primary, so is U.*

We remark that the assertion corresponding to Lemma 19 for ascending chain condition is not true even if R is a commutative integral domain. A counter example can be found in **[13]**.

Let $\mathfrak{R}$ stand for any of the four radicals $\mathfrak{P}$, $\mathfrak{L}$, $\mathfrak{N}$ and $\mathfrak{J}$. We have shown $\mathfrak{R}(U) = U \cap \mathfrak{R}(R)$. If $\mathfrak{R}(U) = U$, then $U \subseteq \mathfrak{R}(R)$, so 0 is a symmetric subring of the semi-prime ring $R/\mathfrak{R}(R)$, and hence $\mathfrak{R}(R) = R$ by Lemma 3. That is, if U is locally nilpotent, nil or quasi-regular, so is R.

On the other hand, $\mathfrak{R}(U) = 0$ need not imply $\mathfrak{R}(R) = 0$. For example, let $R = F + A$ be the algebra obtained by adjunction of an identity to a trivial algebra A over a field F with char. $F \neq 2$. Define $(\alpha + a)^* = \alpha - a$ for $\alpha \in F$ and $a \in A$. Then $\bar{S} = F$ is a field, while $\mathfrak{R}(R) = A$ is a nilpotent ideal. In case A has infinite dimension, this example shows also that R is not artinian although $\bar{S}$ is.

However, we still have some results on $\mathfrak{R}(R)$. For if $\mathfrak{R}(U) = 0$, then the *-ideal $\mathfrak{R}(R)$ has trivial intersection with U, hence is nil of index 2. Thus we have $aRa = 0$ for any $a \in \mathfrak{R}(R)$ and consequently $\mathfrak{R}(R) = \mathfrak{P}(R)$. Besides, U is isomorphic to a symmetric subring of $R/\mathfrak{P}(R)$. Realizing this fact, one might not be surprised to see that $R/\mathfrak{P}(R)$, instead of R itself, satisfies the same properties as U does.

LEMMA 21. *Let R be a semi-prime ring and e the identity of U. Then e is also the identity of R.*

Proof. By Theorem 7, $e \in Z(U) \subseteq Z(R)$. Since $e \in S$, $I = \{x - ex \mid x \in R\}$ is a *-ideal of R. If $a - ea \in U$, then $a - ea = e(a - ea) = 0$. Thus $I \cap U = 0$ and so $I = 0$. In other words, e is the identity of R.

The case when R is semi-prime and $\bar{S}$ is simple was thoroughly studied by Lanski **[9]**. An example was given there that R is an integral domain but not simple while $\bar{S}$ is. In the presence of an identity, we have

THEOREM 22. *Let R be a semi-prime ring. If U is a *-simple ring with identity, so is R.*

Proof. Let I be any nonzero *-ideal of R. Then $I \cap U \neq 0$, and the *-simplicity of U implies $U \subseteq I$. By Lemna 21, U contains the identity of R, so $I = R$.

Even if U is a field, R can be semi-prime but not simple. The simplest example is the direct sum of two copies of a field with the exchange involution. This example illustrates why we deal with only *-primeness and *-primitivity in what follows.

THEOREM 23. (1) *If* U *is semi-prime,* $\mathfrak{P}(R)$ *is nil of index* 2. (2) *If* U *is *-prime, so is* $R/\mathfrak{P}(R)$.

Proof. We have proved (1) in the discussion before Lemma 21. As to (2), we may assume without loss of generality that R is semi-prime. Let I and J be *-ideals of R such that $IJ = 0$. Then $(I \cap U)(J \cap U) = 0$, so $I \cap U = 0$ or $J \cap U = 0$, ending up with $I = 0$ or $J = 0$.

Suppose that R is a *-prime ring and I a nonzero *-ideal of R. If I possesses a *-faithful irreducible module M, write $M = mI$ for some $m \in M$ and $m \neq 0$, and define a map from $M \times R$ into M by sending (ma, r) to $m(ar)$. One can easily check that such a map is well defined and that M becomes a *-faithful irreducible R-module. This is the content of

LEMMA 24. *Let* R *be a *-prime ring and* I *a nonzero ideal of* R. *If* I *is *-primitive, so is* R.

THEOREM 25. (1) *If* U *is semi-simple, then* $\mathfrak{J}(R) = \mathfrak{P}(R)$ *is nil of index* 2. (2) *If* U *is *-primitive, so is* $R/\mathfrak{P}(R)$.

Proof. We have seen the proof of (1) earlier. As to (2), we assume that R is semi-prime. By Theorem 23, R is *-prime. If $[U, U] \neq 0$, then U contains a nonzero *-ideal I of R. Lemma 14 shows that I is itself *-primitive and hence R is also *-primitive by the previous lemma. If U is commutative, it is *-simple with identity. It follows from Theorem 22 that R is *-primitive.

THEOREM 26. *If* U *is semi-primary, so is* R.

Proof. It suffices to show that if R is semi-prime and U is artinian, then R is also artinian. In this case, we have $U = U_1 \oplus \cdots \oplus U_n$, where each U_i is *-simple artinian. Let e_i be the identity of U_i; then $e_i \in Z(U) \subseteq Z(R)$. Since $1 = e_1 + \cdots + e_n$, $R = R_1 \oplus \cdots \oplus R_n$, with $R_i = e_i R$. Each R_i is then semi-prime and contains U_i as a symmetric subring. By Theorem 22 R_i is *-simple, so either $U_i = R_i$ or U_i is a field. If U_i is a field, then R_i satisfies a polynomial identity and hence is a finite dimensional algebra over a field contained in $Z(R_i)$. In either case, R_i is always artinian. Hence R must be also artinian.

III. Subrings generated by skew elements. In contrast to $\bar{S}$, $\bar{K}$ is not necessarily a Lie ideal of R. For instance, in F_2 with char. $F \neq 2$ and transpose as $*$, $\bar{K} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| \, a, b \in F \right\}$. Although $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in K$,

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

falls outside of $\bar{K}$. However, both $\overline{K^2}$ and $\overline{K_0^2}$, where $K_0 = \{x - x^* \mid x \in R\}$, are always Lie ideals.

DEFINITION. *By a skew subgroup V of R we mean a subgroup of R such that $K_0 \subseteq V \subseteq K$ and $xVx^* \subseteq V$ for all $x \in R$.*

Henceforth we shall use V to stand for a skew subgroup of R without further explanation.

LEMMA 27. *$\overline{V^2}$ is a Lie ideal of R.*

Proof. For $v_1, v_2 \in V$ and $x \in R$, we have

$$[v_1v_2, x] = v_1(v_2x + x^*v_2) - (v_1x^* + xv_1)v_2 \in V^2.$$

If $w_1, \cdots, w_n \in V^2$ and $x \in R$, then

$$[w_1 \cdots w_n, x] = w_1[w_2 \cdots w_n, x] + [w_1, x]w_2 \cdots w_n.$$

Hence, this lemma can be proved by induction.

LEMMA 28. *Let R be a semi-prime ring and n a natural number. If $v^{2^n} = 0$ for all $v \in V$, then $V = 0$.*

Proof. If $v^2 = 0$ for all $v \in V$, then for any $x \in R$ $(vx + x^*v)^2 = 0$ so $(vx)^3 = 0$. By Levitzki's lemma [**5**, Lemma 1.1], $v = 0$ for all $v \in V$. Assume that $n > 1$. For any $v \in V$ and $x \in R$, we have $(v^{2^{n-1}}x - x^*v^{2^{n-1}})^{2^n} = 0$ and hence $(v^{2^{n-1}}x)^{2^n+1} = 0$. Applying Levitzki's lemma again and using the induction hypothesis, we conclude that $V = 0$.

One might have noticed that the study of a symmetric subring U in R is based on the fact: *If R is semi-prime, either $U \subseteq Z^+$ or U contains a nonzero ideal of R.* For a skew subgroup V, we have a parallel result for $\overline{V^2}$.

LEMMA 29. *If R is $*$-prime and $[\overline{V^2}, \overline{V^2}] = 0$, then $\overline{V^2} \subseteq Z$ and $[V, V] = 0$. Further, R satisfies the standard identity $S[x_1, x_2, x_3, x_4]$ in 4 variables.*

Proof. Consider first the situation when R is *-simple. If $R = A \oplus A^*$ for some simple ring A, then $K_0 = V = K$, and so $[V^2, V^2] = 0$ implies $[A^2, A^2] = 0$. Since $A^2 = A$, R is also commutative, and the conclusions follow trivially. Assume that R is simple. Then $V^2 \subseteq Z$ unless possibly $2R = 0$ and $\dim_Z R = 4$. If R is a division ring, we have $xV^2x^{-1} = xVx^*(x^{-1})^* Vx^{-1} \subseteq V^2$, so $x\overline{V^2}x^{-1} \subseteq \overline{V^2}$ for all $x \in R$, $x \neq 0$. Hence $V^2 \subseteq Z$ by the Brauer-Cartan-Hua theorem. Suppose that $R = F_2$ for some field F with char.$F = 2$. If $Z \cap T \neq 0$, say, $\alpha = a + a^* \in Z$ for some $a \notin S$, then $1 = \alpha^{-1}a + (\alpha^{-1}a)^* \in T \subseteq V$ and hence $N \subseteq V$. By Lemma 2, $\bar{V}$ is a symmetric subring. Since $V = 1 \cdot V \subseteq V^2$, $[V, V] = 0$ so $V \subseteq Z$ by Theorem 4. If $Z \cap T = 0$, then $Z \subseteq S$ and * must be of transpose type, namely $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & \alpha^{-1}c \\ \alpha b & d \end{bmatrix}$ for some $\alpha \in F$. In this case, $V \subseteq S = \left\{\begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix} \middle| a, b, c \in F\right\}$. Since $\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \in T$, $\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix}$ commutes with $\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}\begin{bmatrix} a' & b' \\ \alpha b' & c' \end{bmatrix}$ for any $\begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix}, \begin{bmatrix} a' & b' \\ \alpha b' & c' \end{bmatrix} \in V$. Comparing the $(1,1)$-entries of the products, we get $ca' = ac'$. An argument like that in Theorem 4 shows $V = T = \left\{\begin{bmatrix} 0 & b \\ \alpha b & 0 \end{bmatrix} \middle| b \in F\right\}$. Hence $V^2 = Z$. Thus we have $V^2 \subseteq Z$ always. By Lemma 28, there exists $v \in V$ such that $v^2 \neq 0$ provided $V \neq 0$. Then v is invertible. Further, $v^{-1} = v^{-1}(-v)(v^{-1})^* \in V$, so $Vv^{-1} \subseteq Z$ and $V \subseteq Zv$. Consequently $[V, V] = 0$.

Now assume that R is *-prime and $V \neq 0$. By Lemmas 1 and 28, $Z^+ \neq 0$, so we may consider the quotient ring $Q = \{a/\alpha^2 \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$. Q can be equipped with * by defining $(a/\alpha^2)^* = a^*/\alpha^2$. Then Q is *-prime and $V' = \{v/\alpha^2 \in Q \mid v \in V\}$ is a skew subgroup of Q. If there is a nonzero *-ideal I of Q such that $I \cap V' \neq 0$, then $I \subseteq S(Q)$ and hence $[I, I] = 0$. By the corollary to Lemma 6, Q is commutative and we are done. Suppose that $J \cap V' \neq 0$ for any nonzero *-ideal J of Q. Since $J \cap V'$ contains an element a such that $a^4 \in Z$ and $a^8 \neq 0$ by Lemmas 1 and 28, and a^8 is invertible, we have $J = Q$. In other words, Q is *-simple and so $V'^2 \subseteq Z(Q)$ and $[V', V'] = 0$. Hence $V^2 \subseteq Z(R)$ and $[V, V] = 0$.

Since $K_0 \subseteq V$, we have $[K_0, K_0] = 0$ and hence R satisfies $S_4[x_1, x_2, x_3, x_4]$ by Amitsur's Theorem **[1]**.

We are now in a position to prove a series of theorems concerning $\overline{V^2}$. Since the proofs are parallel to those for U, we shall omit them unless some modification is needed.

THEOREM 30. *If R is *-simple and $V \neq 0$, then either $\overline{V^2} = R$ or $\overline{V^2}$ is a field contained in Z^+.*

Proof. By Lemmas 1, 27 and 29, we have either $\overline{V^2} = R$ or $\overline{V^2} \subseteq Z^+$. So it suffices to show that $\overline{V^2}$ contains with invertible elements their inverses. First $a^{-1}V = a^{-1}V(a^{-1})^* a^* \subseteq V\overline{V^2}$ if $a \in \overline{V^2}$. Similarly, $Va^{-1} \subseteq \overline{V^2}V$ and hence $a^{-1}\overline{V^2}a^{-1} \subseteq \overline{V^2}$ if $a \in \overline{V^2}$. Thus $a^{-1} = a^{-1}aa^{-1} \in \overline{V^2}$, if $a \in \overline{V^1}$ and is invertible.

THEOREM 31. *If R is prime or *-prime, so is $\overline{V^2}$.*

THEOREM 32. *If R is semi-prime, then $Z(\overline{V^2}) \subseteq Z(R)$.*

THEOREM 33. *If R is semi-prime, so is $\overline{V^2}$.*

THEOREM 34. $\mathfrak{P}(\overline{V^2}) = \overline{V^2} \cap \mathfrak{P}(R)$.

THEOREM 35. *If R has no nil ideal other than* 0, *neither does $\overline{V^2}$.*

THEOREM 36. $\mathfrak{N}(\overline{V^2}) = \overline{V^2} \cap \mathfrak{N}(R)$.

THEOREM 37. *If R has no nonzero locally nilpotent ideals, neither does $\overline{V^2}$.*

THEOREM 38. $\mathfrak{L}(\overline{V^2}) = \overline{V^2} \cap \mathfrak{L}(R)$.

THEOREM 39. *If R is primitive or *-primitive, so is $\overline{V^2}$ provided* $V \neq 0$.

THEOREM 40. *If R is semi-simple, so is $\overline{V^2}$.*

THEOREM 41. $\mathfrak{J}(\overline{V^2}) = \overline{V^2} \cap \mathfrak{J}(R)$.

Proof. It suffices to show that if $a \in \overline{V^2}$ and $a \circ b = 0 = b \circ a$ then $b \in \overline{V^2}$. The argument used in Theorem 30 shows that $(1+b)\overline{V^2}(1+b) \subseteq \overline{V^2}$. (The formal use of the symbol 1 is all right.) Then $b = -(1+b)(a+a^2)(1+b) \in \overline{V^2}$.

THEOREM 42. *If R is semi-primary, primary, or completely primary, so is $\overline{V^2}$ provided $V \neq J(R)$.*

In the example given in **[13]**, $2R = 0$ and $1 \in R$, so $\overline{K^2} = \bar{S}$. Hence $\overline{K^2}$ need not be noetherian even if R is a commutative noetherian domain. However, $\overline{K^2}$, as well as $\bar{S}$, inherits Goldie conditions when R is semi-prime. The proof of the next theorem is based on Lanski's argument **[10]** but is a little simpler.

THEOREM 43. *If R is a semi-prime Goldie ring, so is $\overline{V^2}$.*

Proof. Since the a.c.c. on right annihilators is inherited by subrings, it suffices to show that $\overline{V^2}$ has no infinite direct sum of nonzero right ideals. Let $\{\rho_\alpha\}$ be a set of right ideals of $\overline{V^2}$ such that $\Sigma_\alpha \rho_\alpha$ is direct. Denote by I the ideal of R generated by $[\overline{V^2}, \overline{V^2}]$. Then $\Sigma_\alpha \rho_\alpha I$ is a direct sum of right ideals of R, so $\rho_\alpha I = 0$ and hence $\rho_\alpha \subseteq \overline{V^2} \cap \mathrm{Ann}.I \subseteq Z(\overline{V^2})$ for almost all α. Being a commutative semi-prime subring of a Goldie ring, $Z(\overline{V^2})$ is itself a Goldie ring and hence $\rho_\alpha = 0$ for almost all α.

Let $R = F_2$, where F is a field with char.$F = 2$ and $*$ is given by transpose. In this case, $\bar{T} = K_0 = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \, a, b \in F \right\}$ possesses the nilpotent ideal $\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| \, a \in F \right\}$ even though R is simple. This example kills the hope for $\bar{T}$ or $\overline{K_0}$ to inherit those nice properties we have discussed so far. Fortunately, the behavior of $\bar{K}$ is not that bad.

THEOREM 44. *If R is $*$-simple, either $\bar{K} = R$ or $\bar{K}$ is a commutative $*$-simple ring provided $K \neq 0$.*

Proof. If char.$R = 2$, then $K = S$ and hence the assertion follows from Theorem 4. Assume that char.$R \neq 2$. If $[K^2, K^2] \neq 0$, then $\bar{K}$ also contains the nonzero $*$-ideal of R generated by $[\overline{K^2}, \overline{K^2}]$, so $\bar{K} = R$. If $\overline{K^2}$ is commutative, then $K^2 \subseteq Z^+$ by Theorem 30. Suppose that $Z \not\subseteq S$, then $\alpha^* \neq \alpha$ for some $\alpha \in Z$, so $\beta = \alpha - \alpha^* \neq 0$. Thus, $S\beta^{-1} \subseteq K$ and hence $S \subseteq K\beta$. Therefore, $R = S + K \subseteq \bar{K}$. Next, assume that $Z \subseteq S$. Then R must be simple. By Lemma 29, R satisfies an identity of degree 4 and hence $\dim_Z R \leqq 4$ by Kaplansky's Theorem. If R is a division ring, choose $a \in K$, $a \neq 0$, then $Ka^{-1} \subseteq K^2 \subseteq Z$. So $K \subseteq Za \subseteq K$, that is, $K = Za$. Hence $\bar{K} = Z(a)$ is a field. If $R = F_2$ for some field F, the commutativity of K forces $*$ to be of transpose type, say, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & \sigma^{-1}c \\ \sigma b & d \end{bmatrix}$ for some $\sigma \in F$. Then $\bar{K} = \left\{ \begin{bmatrix} a & b \\ -\sigma b & a \end{bmatrix} \middle| \, a, b \in F \right\}$. If $-\sigma$ is not a square in F, $\bar{K}$ is a field; while if $-\sigma = \pi^2$ for some $\pi \in F$, $\bar{K} = L_1 \oplus L_2$ where $L_1 = \left\{ \begin{bmatrix} a & \pi^{-1}a \\ \pi a & a \end{bmatrix} \middle| \, a \in F \right\}$ and $L_2 = \left\{ \begin{bmatrix} a & -\pi^{-1}a \\ -\pi a & a \end{bmatrix} \middle| \, a \in F \right\}$ are two fields which are isomorphic via the map induced by $*$.

THEOREM 45. *If R is $*$-prime, so is $\bar{K}$.*

Proof. If $\overline{K^2}$ is not commutative, then $\bar{K}$ also contains the ideal generated by $[\overline{K^2}, \overline{K^2}]$. An argument exactly like that in Theorem 5 proves the $*$-primeness of $\bar{K}$. Now we assume that $\overline{K^2}$ is a nonzero commutative ring. The quotient ring $Q = \{a/\alpha \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$

is either a *-simple ring or a commutative *-prime ring relative to the involution $(a/\alpha)^* = a^*/\alpha$. In the former case, $\overline{K(Q)}$ is a commutative *-simple ring by the previous theorem. So in either case $\overline{K(R)}$ is contained in a commutative *-prime ring and hence is *-prime.

LEMMA 46. *If R is semi-prime, then $C_{\bar{V}}(\overline{V^2}) = Z(\bar{V})$.*

Proof. Assume first that R is *-prime. If $\overline{V^2}$ is not commutative, then it contains a nonzero *-ideal I of R, so $C_{\bar{V}}(\overline{V^2}) \subseteq C_R(I) \subseteq Z(R)$ by Lemma 6 and hence $C_{\bar{V}}(\overline{V^2}) = Z(\bar{V})$. If $[V^2, V^2] = 0$, then $V^2 \subseteq Z(R)$ and $[V, V] = 0$ by Lemma 29 and hence $C_{\bar{V}}(\overline{V^2}) = \bar{V} = Z(\bar{V})$. The semi-prime case can be built up easily via subdirect sum.

The next lemma is crucial in the study of $\bar{K}$.

LEMMA 47. *Let R be a semi-prime ring and I a *-ideal of $\bar{K}$. If $I \cap K = 0$, then $I = 0$.*

Proof. If $I \cap K = 0$, then $I \subseteq S$. For any $a \in I$ and $k \in K$, $ak = (ak)^* = -ka$. Hence $I \subseteq C_{\bar{K}}(\overline{K^2}) = Z(\bar{K})$ by Lemma 46. Thus $IK \subseteq I \cap K = 0$, so $I\bar{K} = 0$, and in particular $I^2 = 0$. For any $a \in I$ and $x \in R$, we have $a(x - x^*) = 0$, that is, $ax = ax^*$ and hence $axa = a(xa)^* = a^2x^* = 0$. Since R is semi-prime, it follows that $I = 0$.

LEMMA 48. *If R is semi-prime, and $k \in K$ with $kKk = 0$, then $k = 0$.*

Proof. For any $x \in R$, $k(x - x^*)k = 0$ so $kxk = kx^*k$. Then $kxkxk = k(xkx^*)k = 0$ and hence kR is nil of index 3. So, $k = 0$ by Levitzki's lemma.

THEOREM 49. *If R is semi-prime, so is $\bar{K}$.*

Proof. Let I be a *-ideal of $\bar{K}$ such that $I^2 = 0$. For any $a \in I \cap K$, we have $aKa \subseteq I^2 = 0$ so $a = 0$ by Lemma 48. Lemma 47 shows $I = 0$, so $\bar{K}$ has no nonzero nilpotent *-ideal and hence is semi-prime.

THEOREM 50. *If R has no nil ideal other than 0, neither does $\bar{K}$.*

Proof. Let I be the ideal of R which is generated by $[\overline{K^2}, \overline{K^2}]$. Then $\mathfrak{N}(I) = 0$ and $I \subseteq \bar{K}$. If $a \in \mathfrak{N}(\bar{K}) \cap K$ and $b \in \overline{K^2}$, then $a^2b - ba^2 \in I \cap \mathfrak{N}(\bar{K}) = 0$. Thus $a^2 \in Z(\overline{K^2})$ and by Lemma 46 $a^2 \in Z(\bar{K})$. But $\bar{K}$ is semi-prime and a is nilpotent, so $a^2 = 0$ for all $a \in \mathfrak{N}(\bar{K}) \cap K$. In view of Lemma 28, $\mathfrak{N}(\bar{K}) \cap K = 0$ because $\mathfrak{N}(\bar{K})$ is itself a semi-prime ring. Hence, it follows from Lemma 47 that $\mathfrak{N}(\bar{K}) = 0$.

A similar argument proves the following

THEOREM 51. *If R has no nonzero locally nilpotent ideal, neither does $\bar{K}$.*

The proof of the next theorem is exactly like that of Theorem 39.

THEOREM 52. *If R is *-primitive, so is $\bar{K}$ provided $K \neq 0$.*

THEOREM 53. *If R is semi-simple, so is $\bar{K}$.*

Proof. Let $a \in \mathfrak{J}(\bar{K}) \cap K$. For any $x \in R$, we have

$$ax \circ (-ax^*) = a(x - x^* - xax^*) \in \mathfrak{J}(\bar{K})K \subseteq \mathfrak{J}(\bar{K}).$$

Hence aR is quasi-regular, so $a = 0$. By Lemma 47, $\mathfrak{J}(\bar{K}) = 0$.

THEOREM 54. *If R is semi-prime artinian, so is $\bar{K}$.*

Proof. Immediate from Theorem 44.

Unlike $\bar{S}$, the semi-prime assumption on R is not sufficient to get the converse theorems for $\bar{K}$ or $\overline{K^2}$. For example, let F be a field with char.$F \neq 2$, σ an automorphism on F with $\sigma^2 = 1$, and A a commutative semi-prime algebra over F. Put $R = F \oplus A$ and define $(\alpha, a)^* = (\alpha^\sigma, a)$. Then $\bar{K} = F$ and $\overline{K^2} = F^\sigma$ are fields provided $\sigma \neq 1$, while R is not even *-prime. Further, if A possesses an identity and $\dim_F A = \infty$, then R is neither artinian nor Goldie.

On the other hand, the *-primeness is sufficient for our purpose. To begin with, we prove a lemma which is analogous to Lemma 3.

LEMMA 55. *Let R be a *-prime ring and I a nonzero *-ideal of R such that $I \cap K_0^2 = 0$. If $K_0 \neq 0$, then $I = 0$.*

Proof. If $I \cap K_0^2 = 0$, then $(I \cap K_0)^2 = 0$. Since I is itself a semi-prime ring, and $I \cap K_0$ is a skew subgroup of I, so $I \cap K_0 = 0$ by Lemma 28. Hence $I \subseteq S$. For any $a \in I$ and $x \in R$, we have $ax = (ax)^* = x^*a$. So if $a, b \in I$ and $x \in R$, then $abx = ax^*b = xab = abx^*$. That is, $I^2K_0 = 0$. Since R is *-prime and $K_0 \neq 0$, it follows $I = 0$.

LEMMA 56. *Let R be a *-prime ring and e the identity of $\bar{K}$ or $\overline{V^2}$. If $e \neq 0$, then it is the identity of R.*

Proof. Since the only nonzero central symmetric idempotent in a $*$-prime ring is the identity, it suffices to show that $e \in Z(R)$. If e is the identity of $\overline{V^2}$, then $ex - xe \in \overline{V^2}$ for all $x \in R$ because $\overline{V^2}$ is a Lie ideal. If e works for $\bar{K}$, then $ex - xe = e(x - x^*) + (ex^* - xe) \in \bar{K}$ for all $x \in R$. Hence $e(ex - xe) = ex - xe = (ex - xe)e$ and this implies that $e \in Z(R)$.

On the basis of Lemma 55, we can prove the converse theorems by using an argument parallel to that for U.

THEOREM 57. *If R is $*$-prime, and $\bar{K}$ or $\overline{V^2}$ is a $*$-simple ring with identity, so is R.*

THEOREM 58. *If R is $*$-prime, and $\bar{K}$ or $\overline{V^2}$ is $*$-primitive, so is R.*

THEOREM 59. *Let R be a $*$-prime ring and $*$ not the identity map. If $\bar{K}$ or $\overline{V^2}$ is semi-simple, so is R.*

Proof. Since $\mathfrak{J}(\overline{V^2}) = \overline{V^2} \cap \mathfrak{J}(R)$, so $\mathfrak{J}(R) \cap K_0^2 = 0$ if $\overline{V^2}$ is semi-simple. By Lemma 55, R must be also semi-simple. In case $\bar{K}$ is semi-simple, so is $\overline{K^2}$ by Theorem 41, and hence R is also semi-simple.

THEOREM 60. *If R is $*$-prime, and $\bar{K}$ or $\overline{V^2}$ has no nil ideal other than 0, then neither does R.*

THEOREM 61. *If R is $*$-prime, and $\bar{K}$ or $\overline{V^2}$ has no nonzero locally nilpotent ideal, then neither does R.*

We close this paper with two theorems on chain conditions.

THEOREM 62. *Let R be a $*$-prime ring. If $*$ is not the identity map and either $\bar{K}$ or $\overline{V^2}$ is artinian, then so is R.*

Proof. By Theorems 31 and 45, both $\bar{K}$ and $\overline{V^2}$ are $*$-prime. Say, if $\bar{K}$ is artinian, then it is $*$-simple with identity, so R is also $*$-simple by Theorem 57 and hence $\bar{K} = R$ or $\bar{K}$ is commutative by Theorem 44. In the later case, R satisfies a polynomial identity, and is finite dimensional over a field contained in Z. Hence, R is artinian. The situation when $\overline{V^2}$ is artinian is the same.

For $a \in R$, let $r_R(a) = \{x \in R \mid ax = 0\}$ be the right annihilator of a in R. Denote by $\mathcal{Z}(R)$ the right singular ideal of R, that is, $\mathcal{Z}(R) = \{a \in R \mid r_R(a) \cap \rho \neq 0$ for any nonzero right ideal ρ of $R\}$.

THEOREM 63. *Let R be a $*$-prime ring. If $\overline{V^2}$ is a Goldie ring, so is R.*

Proof. If R is commutative, then $Q = \{a/\alpha \mid a \in R, \alpha \in S, \alpha \neq 0\}$ is a commutative *-simple ring, and hence R is a Goldie ring. Assume that R is not commutative, while $[V^2, V^2] = 0$. Then $V^2 \subseteq Z^+$ and $Q = \{a/\alpha \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$ is a *-simple ring. Since $[V, V] = 0$, it follows that Q satisfies a polynomial identity, and hence is artinian. So, R is a Goldie ring. Lastly, assume that $[V^2, V^2] \neq 0$ and let I be the ideal of R generated by $\{\overline{V^2}, \overline{V^2}]$. Suppose $\{\rho_\alpha\}$ is a set of right ideals of R which forms a direct sum. Then $\rho_\alpha I \subseteq \rho_\alpha \cap I \subseteq \overline{V^2}$ and $\rho_\alpha I = 0$ for almost all α. Consequently $\rho_\alpha = 0$ for almost all α. Consider $\mathfrak{Z}(R) \cap I$. If $a \in \mathfrak{Z}(R) \cap I$, then for any nonzero right ideal ρ of I, $\rho I \neq 0$, so $r_R(a) \cap \rho I \neq 0$ and hence $r_I(a) \cap \rho \neq 0$. In other words, $\mathfrak{Z}(R) \cap I \subseteq \mathfrak{Z}(I) = 0$ because I is itself a semi-prime Goldie ring. So $\mathfrak{Z}(R) = 0$.

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