ON SUBRINGS OF RINGS WITH INVOLUTION

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We give a systematic account on the relationship between a ring $R$ with involution and its subrings $\hat{S}$ and $\hat{K}$, which are generated by all its symmetric elements or skew elements respectively.

I. Introduction. Let $R$ be a ring with involution $*$ and $\hat{S}$ the subring generated by the set $S$ of all symmetric elements in $R$. The relationship between $R$ and $\hat{S}$ has been studied by various authors. In [3] Dieudonné showed that if $R$ is a division ring of characteristic not 2, then either $\hat{S} = R$ or $\hat{S} \subseteq Z(R)$, the center of $R$. Later Herstein [4] extended this result by proving $\hat{S} = R$ for any simple ring $R$ with $\dim_{\mathbb{F}}R > 4$ and char.$R \neq 2$. The restriction on characteristic was removed by Montgomery [12]. Recently, Lanski [9] proved that if $R$ is prime or semi-prime, so is $\hat{S}$. In §2 of this paper, we show that $\hat{S}$ can inherit a number of ring-theoretic properties such as primitivity, semisimplicity, absence of nonzero nil ideals etc.. In doing so, a notion called symmetric subring, which is a generalization of $\hat{S}$ and its $*$-homomorphic images, is introduced so that a group of theorems of the same type, including Lanski's results, can be proved via a more or less unified argument. We show also that numerous radicals of $\hat{S}$ are merely the contractions from those of $R$. As a consequence, we see that $R$ modulo its prime radical behaves much like $\hat{S}$ in many respects.

In §3 we establish a corresponding theory for $\hat{K}$, the subring generated by all skew elements. The only result hitherto known concerning $\hat{K}$ was as follows [4], [12]: If $R$ is simple and $\dim_{\mathbb{F}}R > 4$, then $\hat{K} = R$. As a matter of fact, the subring $K^2$ is more closely related to $R$ than $\hat{K}$ is. We apply the technique developed in §2 to study the relationship between $R$ and $K^2$, and then derive some parallel theorems for $\hat{K}$.

II. Symmetric subrings. Our work depends heavily on the notion of Lie ideals. By a Lie ideal $U$ of $R$ we mean an additive subgroup which is invariant under all inner derivations of $R$. That is, $\left[u, x\right] = ux - xu \in U$ for all $u \in U$ and $x \in R$. The following lemma concerning Lie ideals will be referred to frequently in the sequel, and it is a combination of some results in [5].

Lemma 1. Let $R$ be a semi-prime ring and $U$ a subring and Lie ideal of $R$. Then $U$ contains the ideal of $R$ which is generated by $\left[U, U\right]$. If $U$ is commutative, then $u^2 \in Z$ for all $u \in U$.
Rings with involution abound with examples of Lie ideals. One can easily show that any subring, generated by symmetric elements and containing \( T = \{x + x^* | x \in R\} \) the set of all traces, must be a Lie ideal. In particular, both \( \tilde{S} \) and \( \tilde{T} \) are Lie ideals.

Another essential property of \( \tilde{S} \) follows from the next lemma. We denote by \( N \) the set of all norms, i.e. \( N = \{xx^* | x \in R\} \).

**Lemma 2.** Let \( U \) be an additive subgroup of \( R \) such that \( T \subseteq U \subseteq S \) and \( xU x^* \subseteq U \) for all \( x \in R \). If \( N \subseteq \widetilde{U} \), then \( x\widetilde{U} x^* \subseteq \widetilde{U} \) for all \( x \in R \).

**Proof.** We prove by induction that \( xu_1 \cdots u_n x^* \in \widetilde{U} \) for all \( x \in R \) and \( u_1, \ldots, u_n \in U \). The case \( n = 1 \) is clear. Assume the assertion holds for \( n - 1 \); then
\[
xu_1 u_2 \cdots u_n x^* = [x, u_1] [u_2 \cdots u_n, x^*] + (xu_1 x^*) u_2 \cdots u_n + u_1 (xu_2 \cdots u_n x^*)
\]
\[
- u_1 xx^* u_2 \cdots u_n \in \widetilde{U}
\]
because \( \widetilde{U} \) is a Lie ideal.

**Definition.** A subring \( U \) of \( R \) is called a symmetric subring if:
1. \( U \) is generated by a set of symmetric elements.
2. \( T \cup N \subseteq U \)
3. \( xU x^* \subseteq U \) for all \( x \in R \).

In light of Lemma 2, we know that \( \tilde{S} \) is a symmetric subring. From now on, \( U \) will always denote a symmetric subring of \( R \). We call an ideal \( I \) of \( R \) a \(*\)-ideal if \( I^* = I \).

**Lemma 3.** If \( R \) is semi-prime and \( I \) is a \(*\)-ideal of \( R \) such that \( I \cap U = 0 \), then \( I = 0 \).

**Proof.** For any \( a \in I \), \( a^2 = a(a + a^*) - aa^* = 0 \). Then \( I \) is nil of index 2 and hence \( I = 0 \).

Recall that a ring \( R \) is called a \(*\)-simple ring if \( R^2 \neq 0 \) and \( R \) has no \(*\)-ideal other than 0 and \( R \). It is well-known that \( R \) is \(*\)-simple if and only if either \( R \) is simple or \( R = A \oplus A^* \) for some simple ring \( A \) [8, p. 14]. Let \( Z^* = Z \cap S \). Then if \( R \) is \(*\)-simple, we have \( Z^* = 0 \) or \( Z^* \) is a field.

**Theorem 4.** If \( R \) is \(*\)-simple, then either \( U = R \) or \( U \) is a field contained in \( Z^* \).

**Proof.** If \( U \) is not commutative, by Lemma 1 it contains a nonzero \(*\)-ideal of \( R \) so \( U = R \). Assume that \([U, U] = 0\); then \( U \subseteq S \). In this
case, we need only to prove $U \subseteq Z$, for if $u \in U$ and $u \neq 0$ then $u^{-1} = u^{-1}u(u^{-1})^* \in U$.

If $R = A \oplus A^*$ for some simple ring $A$, then $T = U = S$. Thus $[U, U] = 0$ implies $[A, A] = 0$ and so $R$ is commutative. If $R$ is simple, then $U$, being a commutative subring and Lie ideal of $R$, must be central unless $2R = 0$ and $\dim_{Z} R = 4$ [5, Theorem 1.5]. So let us examine all possible 4-dimensional cases.

If $R$ is a division ring, then $x^{-1}Ux = x^{-1}(xUx^*)x = Ux^*x \subseteq U$ for all $x \in R$ with $x \neq 0$. Hence $U \subseteq Z$ by the Brauer-Cartan-Hua theorem [7, Theorem 7.13.1; Cor.].

There remains the case $R = F_2$ where $F$ is a field with char.$F = 2$. We claim that $*$ must be of symplectic type. Assume the contrary,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & \alpha^{-1}c \\ ab & \bar{d} \end{bmatrix}$$

for some $\alpha \in F$ with $\bar{\alpha} = \alpha$, where $-$ denotes the induced automorphism on $F$. Thus

$$U \subseteq S = \left\{ \begin{bmatrix} a & b \\ ab & c \end{bmatrix} | \bar{a} = a, \bar{c} = c \right\}.$$ 

For any $a \in F$, we have

$$\begin{bmatrix} 0 & a + \bar{a} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a + \bar{a} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \in T^2 \subseteq U$$

so $\bar{a} = a$. Next, if $\begin{bmatrix} a & b \\ ab & c \end{bmatrix} \in U$ then

$$\begin{bmatrix} b & 0 \\ a + c & b \end{bmatrix} = \begin{bmatrix} a & b \\ ab & c \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ ab & c \end{bmatrix} \in U$$

and hence $a = c$. But if $\begin{bmatrix} a & b \\ ab & a \end{bmatrix} \in U$, then

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ ab & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U$$

yields $a = 0$. So $U = T = \left\{ \begin{bmatrix} 0 & b \\ \alpha b & 0 \end{bmatrix} \right\}$ which is ridiculous because $T$ is not a subring. Consequently, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$ and

$$U \subseteq S = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} | a, b, c \in F \right\}.$$
For any $\begin{bmatrix} a & b \\ c & a \end{bmatrix}$, $\begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \in U$, we have $\begin{bmatrix} a & b \\ c & a \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \in U$ and hence $bc' = b'c$ by comparing the diagonal entries of the product. If there exists $\begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \in U$ with $b' \neq 0$, then

$$U \subseteq \left\{ \begin{bmatrix} a & b \\ ab & a \end{bmatrix} \mid a, b \in F \right\},$$

where $\alpha = c'b^{-1}$. However,

$$\begin{bmatrix} 0 & 0 \\ b' & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & a' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in U$$

forces $b' = 0$, a contradiction. Hence $U \subseteq \left\{ \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \mid a, c \in F \right\}$. On the other hand, if $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \in U$,

$$\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in U$$

implies $c = 0$. Therefore, $U \subseteq Z$.

Following [11], we say $R$ is *-prime if the product of any two nonzero *-ideals is still not zero. It is easy to see that $R$ is *-prime if and only if $aRb = a^*Rb = 0$ implies $a = 0$ or $b = 0$. As a consequence, any nonzero element in $Z^+$ is regular in a *-prime ring $R$.

We remind the reader of a well-known fact that a nonzero Lie ideal of a semi-prime ring always contains elements with nonzero square.

**Theorem 5.** If $R$ is *-prime, so is $U$.

**Proof.** If $[U, U] \neq 0$, then $U$ contains a nonzero *-ideal $I$ of $R$. For any two *-ideals $A, B$ of $U$ with $AB = 0$, we have $IAIB \subseteq AB = 0$, so either $IAI = 0$ or $B = 0$, ending up with $A = 0$ or $B = 0$. Assume that $U \neq 0$ while $[U, U] = 0$. By Lemma 1, there exists $u_0 \in U$ such that $u_0^3 \in Z$ but $u_0^2 \neq 0$. So consider the ring $Q$ of fractions $a/\alpha$ with $a \in R$ and $\alpha \in Z \cap U$, $\alpha \neq 0$. $Q$ is also *-prime with respect to the involution given by $(a/\alpha)^* = a^*/\alpha$, and $U' = \{u/\alpha \in Q \mid u \in U\}$ is a symmetric subring of $Q$. As a matter of fact, $Q$ is *-simple. For if $J$ is any nonzero *-ideal of $Q$, $J \cap U' \neq 0$ and hence $(v/\beta)^2 \neq 0$ for some $v/\beta \in J \cap U'$. Since $v^2 \in Z$, $v/\beta$ is invertible and so $J = Q$. By the
previous theorem, \( U' \subseteq Z'(Q) \) and hence \( U \) is an integral domain contained in \( Z'(R) \).

Let \( C_R(V) = \{ x \in R \mid xv = vx \text{ for all } v \in V \} \) be the centralizer of a set \( V \) in \( R \).

**Lemma 6.** Let \( I \neq 0 \) be an ideal (or \(*\)-ideal) of a prime (resp. \(*\)-prime) ring \( R \). Then \( C_R(I) \subseteq Z \).

**Proof.** For \( a \in I, b \in C_R(I) \) and \( x \in R \), we have \( abx = bax = axb \), or equivalently, \( a(bx - xb) = 0 \). That is, \( I[C_R(I), R] = 0 \). Hence \([C_R(I), R] = 0 \) and so \( C_R(I) \subseteq Z \).

**Corollary.** Let \( R \) be a prime (or \(*\)-prime) ring and \( I \) a nonzero ideal (resp. \(*\)-ideal) of \( R \) such that \([I, I] = 0 \). Then \( R \) is commutative.

**Theorem 7.** If \( R \) is semi-prime, then \( Z(U) \subseteq Z(R) \).

**Proof.** Assume first that \( R \) is \(*\)-prime. If \([U, U] = 0 \), then \( Z(U) = U \subseteq Z(R) \) by Theorem 5. If \([U, U] \neq 0 \), then \( U \) contains a nonzero \(*\)-ideal \( I \) of \( R \), so \( Z(U) \subseteq C_R(I) \subseteq Z(R) \) in view of Lemma 6. In either case, \([Z(U), R] = 0 \). Now assume that \( R \) is semi-prime; then \( R \) is a subdirect sum of \(*\)-prime rings \( \pi_\alpha(R) \). Since \( \pi_\alpha(U) \) is a symmetric subring of \( \pi_\alpha(R) \), we know \([\pi_\alpha(Z(U)), \pi_\alpha(R)] \subseteq [Z(\pi_\alpha(U)), \pi_\alpha(R)] = 0 \) for all \( \alpha \). Hence, \([Z(U), R] = 0 \).

The same reduction to \(*\)-prime rings together with Theorem 5 gives an alternate proof for Lanski's theorem:

**Theorem 8.** If \( R \) is semi-prime, so is \( U \).

With this established, we are able to consider the relationship between the prime radicals \( \mathfrak{p}(R) \) and \( \mathfrak{p}(U) \).

**Theorem 9.** \( \mathfrak{p}(U) = U \cap \mathfrak{p}(R) \).

**Proof.** Since \( U/[U \cap \mathfrak{p}(R)] = [U + \mathfrak{p}(R)]/\mathfrak{p}(R) \) which is a symmetric subring of the semi-prime ring \( R/\mathfrak{p}(R) \), so \( U/[U \cap \mathfrak{p}(R)] \) is semi-prime by Theorem 8 and hence \( \mathfrak{p}(U) \subseteq U \cap \mathfrak{p}(R) \). On the other hand, if \( a \in U \cap \mathfrak{p}(R) \), then \( a \in U \) and any \( m \)-system in \( R \) containing \( a \) must contain 0. [7, Theorem 8.2.3]. Certainly, any \( m \)-system in \( U \) containing \( a \) contains 0. That is, \( a \in \mathfrak{p}(U) \).

It is well-known that a ring without nonzero nil ideals is a subdirect sum of rings with the following property [6, p. 53]:

**There exists a nonnilpotent element \( a \) such that \( a^{n(i)} \in I \) for all nonzero ideal \( I \).**
One can impose this condition only on the *-ideals and show that it is a hereditary property. Then, making use of subdirect sum decomposition, we can prove that $U$ inherits the freedom from nonzero nil ideals. Instead of doing this way, we prefer to present a direct proof by considering the nil radical $\mathfrak{N}(U)$ of $U$.

**Theorem 10.** If $R$ has no nil ideal other than 0, neither does $U$.

*Proof.* Let $I$ be the ideal of $R$ which is generated by $[U, U]$. Since $R$ possesses no nonzero nil ideal, neither does $I$, considered as a ring. Hence $\mathfrak{N}(U) \cap I = 0$. For any $a \in \mathfrak{N}(U)$ and $u \in U$, we have $[a, u] \in \mathfrak{N}(U) \cap I = 0$. Thus $\mathfrak{N}(U) \subseteq Z(U)$. Since $U$ is semi-prime by Theorem 8, $\mathfrak{N}(U) = 0$.

As an immediate consequence, we have

**Theorem 11.** $\mathfrak{N}(U) = U \cap \mathfrak{N}(R)$.

Proceed as above with “locally nilpotent” in place of “nil” and with Levitzki radical $\mathfrak{L}$ in place of $\mathfrak{N}$, we get

**Theorem 12.** If $R$ has no nonzero locally nilpotent ideal, neither does $U$.

**Theorem 13.** $\mathfrak{L}(U) = U \cap \mathfrak{L}(R)$.

In [2] the notion of *-primitive ring was introduced as a ring admitting a *-faithful irreducible module $M$ (i.e. $Mr = Mr^* = 0$ implies $r = 0$). One can easily verify that a ring is *-primitive if and only if it is either primitive or a subdirect sum of a primitive ring and its opposite with the exchange involution.

We know that a nonzero ideal of a primitive ring is itself primitive. The proof is applicable to the following more general fact.

**Lemma 14.** Let $R$ be a primitive (or *-primitive) ring. Suppose that $I$ is a nonzero ideal (resp. *-ideal) of $R$, and $A$ is a subring (resp. *-subring, i.e. $A^* = A$) containing $I$. Then $A$ is also primitive (resp. *-primitive).

**Theorem 15.** If $R$ is primitive or *-primitive, so is $U$.

*Proof.* If $[U, U] \neq 0$, $U$ contains a nonzero *-ideal of $R$, so it is primitive or *-primitive by Lemma 14. Assume that $U$ is commutative. Then $U \subseteq Z^+$ and every element in $R$ is quadratic over
Z\. Hence \( R \) satisfies a polynomial identity. According to Kaplansky's theorem [6, Theorem 6.3.1], \( R \) is \(*\)-simple and hence \( U \) is a field by Theorem 4.

Using the fact that a semi-simple ring is a subdirect sum of \(*\)-primitive rings, we get immediately

**Theorem 16.** If \( R \) is semi-simple, so is \( U \).

In fact, the semi-simplicity of \( \tilde{S} \) was first proved by Herstein. His elegant proof was the inspiration of our next theorem which relates the Jacobson radicals of \( R \) and \( U \).

**Theorem 17.** \( \mathfrak{J}(U) = U \cap \mathfrak{J}(R) \).

**Proof.** For \( a \in \mathfrak{J}(U) \) and \( x \in R \), we have

\[
ax \circ ax^* = ax + ax^* + axax^* = a(x + x^* + xa^*) \in \mathfrak{J}(U)U \subseteq \mathfrak{J}(U).
\]

Thus \( aR \) is quasi-regular and hence \( a \in U \cap \mathfrak{J}(R) \). Conversely, if \( a \in U \cap \mathfrak{J}(R) \), \( a \circ b = 0 \) for some \( b \in R \), then \( b = b \circ (a \circ b)^* = (b \circ b^*) \circ a^* \in U \). That is, \( U \cap \mathfrak{J}(R) \) is a quasi-regular ideal of \( U \), so \( U \cap \mathfrak{J}(R) \subseteq \mathfrak{J}(U) \).

With Theorem 17 in hand, we are ready to study some non-semi-simple rings. Following [7], we say \( R \) is semi-primary, primary, or completely primary according as \( R/\mathfrak{J}(R) \) is an artinian, simple artinian, or division ring respectively. Since \( U/\mathfrak{J}(U) \) is isomorphic to a symmetric subring of \( R/J(R) \), by Theorem 4 we have

**Theorem 18.** If \( R \) is primary or completely primary, so is \( U \).

As to semi-primary rings, we need some information about the descending chain condition. In a paper [10] which is to appear, Lanski proved that if \( R \) is artinian and \( \frac{1}{2} \in R \), then so is \( \tilde{S} \). For our purpose, we prove

**Lemma 19.** If \( R \) is semi-prime artinian, so is \( U \).

**Proof.** By the Wedderburn-Artin theorem, we may write \( R = R_1 \oplus \cdots \oplus R_n \) where each \( R_i \) is \(*\)-simple. Denote by \( e_i \) the identity of \( R_i \), then \( e_i \in Z^+ \) and so \( e_iUe_i \) is a symmetric subring of \( R_i \) for each \( i \). By Theorem 4, each \( e_iUe_i \) is artinian, so is \( U = e_1Ue_1 \oplus \cdots \oplus e_nUe_n \).

**Theorem 20.** If \( R \) is semi-primary, so is \( U \).
We remark that the assertion corresponding to Lemma 19 for ascending chain condition is not true even if $R$ is a commutative integral domain. A counter example can be found in [13].

Let $\mathfrak{R}$ stand for any of the four radicals $\mathfrak{P}$, $\mathfrak{L}$, $\mathfrak{R}$ and $\mathfrak{S}$. We have shown $\mathfrak{R}(U) = U \cap \mathfrak{R}(R)$. If $\mathfrak{R}(U) = U$, then $U \subseteq \mathfrak{R}(R)$, so $0$ is a symmetric subring of the semi-prime ring $R/\mathfrak{R}(R)$, and hence $\mathfrak{R}(R) = R$ by Lemma 3. That is, if $U$ is locally nilpotent, nil or quasi-regular, so is $R$.

On the other hand, $\mathfrak{R}(U) = 0$ need not imply $\mathfrak{R}(R) = 0$. For example, let $R = F + A$ be the algebra obtained by adjunction of an identity to a trivial algebra $A$ over a field $F$ with char. $F \neq 2$. Define $(\alpha + a)^* = \alpha - a$ for $\alpha \in F$ and $a \in A$. Then $\overline{S} = F$ is a field, while $\mathfrak{R}(R) = A$ is a nilpotent ideal. In case $A$ has infinite dimension, this example shows also that $R$ is not artinian although $\overline{S}$ is.

However, we still have some results on $\mathfrak{R}(R)$. For if $\mathfrak{R}(U) = 0$, then the $*$-ideal $\mathfrak{R}(R)$ has trivial intersection with $U$, hence is nil of index 2. Thus we have $aRa = 0$ for any $a \in \mathfrak{R}(R)$ and consequently $\mathfrak{R}(R) = \mathfrak{P}(R)$. Besides, $U$ is isomorphic to a symmetric subring of $R/\mathfrak{P}(R)$. Realizing this fact, one might not be surprised to see that $R/\mathfrak{P}(R)$, instead of $R$ itself, satisfies the same properties as $U$ does.

Lemma 21. Let $R$ be a semi-prime ring and $e$ the identity of $U$. Then $e$ is also the identity of $R$.

Proof. By Theorem 7, $e \in Z(U) \subseteq Z(R)$. Since $e \in S$, $I = \{x - ex \mid x \in R\}$ is a $*$-ideal of $R$. If $a - ea \in U$, then $a - ea = e(a - ea) = 0$. Thus $I \cap U = 0$ and so $I = 0$. In other words, $e$ is the identity of $R$.

The case when $R$ is semi-prime and $\overline{S}$ is simple was thoroughly studied by Lanski [9]. An example was given there that $R$ is an integral domain but not simple while $\overline{S}$ is. In the presence of an identity, we have

Theorem 22. Let $R$ be a semi-prime ring. If $U$ is a $*$-simple ring with identity, so is $R$.

Proof. Let $I$ be any nonzero $*$-ideal of $R$. Then $I \cap U \neq 0$, and the $*$-simplicity of $U$ implies $U \subseteq I$. By Lemma 21, $U$ contains the identity of $R$, so $I = R$.

Even if $U$ is a field, $R$ can be semi-prime but not simple. The simplest example is the direct sum of two copies of a field with the exchange involution. This example illustrates why we deal with only $*$-primeness and $*$-primitivity in what follows.
THEOREM 23. (1) If $U$ is semi-prime, $\mathfrak{P}(R)$ is nil of index 2. (2) If $U$ is $\ast$-prime, so is $R/\mathfrak{P}(R)$.

Proof. We have proved (1) in the discussion before Lemma 21. As to (2), we may assume without loss of generality that $R$ is semi-prime. Let $I$ and $J$ be $\ast$-ideals of $R$ such that $IJ = 0$. Then $(I \cap U)(J \cap U) = 0$, so $I \cap U = 0$ or $J \cap U = 0$, ending up with $I = 0$ or $J = 0$.

Suppose that $R$ is a $\ast$-prime ring and $I$ a nonzero $\ast$-ideal of $R$. If $I$ possesses a $\ast$-faithful irreducible module $M$, write $M = mI$ for some $m \in M$ and $m \neq 0$, and define a map from $M \times R$ into $M$ by sending $(ma, r)$ to $m(ar)$. One can easily check that such a map is well defined and that $M$ becomes a $\ast$-faithful irreducible $R$-module. This is the content of

LEMMA 24. Let $R$ be a $\ast$-prime ring and $I$ a nonzero ideal of $R$. If $I$ is $\ast$-primitive, so is $R$.

THEOREM 25. (1) If $U$ is semi-simple, then $\mathfrak{S}(R) = \mathfrak{P}(R)$ is nil of index 2. (2) If $U$ is $\ast$-primitive, so is $R/\mathfrak{P}(R)$.

Proof. We have seen the proof of (1) earlier. As to (2), we assume that $R$ is semi-prime. By Theorem 23, $R$ is $\ast$-prime. If $[U, U] \neq 0$, then $U$ contains a nonzero $\ast$-ideal $I$ of $R$. Lemma 14 shows that $I$ is itself $\ast$-primitive and hence $R$ is also $\ast$-primitive by the previous lemma. If $U$ is commutative, it is $\ast$-simple with identity. It follows from Theorem 22 that $R$ is $\ast$-primitive.

THEOREM 26. If $U$ is semi-primary, so is $R$.

Proof. It suffices to show that if $R$ is semi-prime and $U$ is artinian, then $R$ is also artinian. In this case, we have $U = U_1 \oplus \cdots \oplus U_n$, where each $U_i$ is $\ast$-simple artinian. Let $e_i$ be the identity of $U_i$; then $e_i \in Z(U) \subseteq Z(R)$. Since $1 = e_1 + \cdots + e_n$, $R = R_1 \oplus \cdots \oplus R_n$ with $R_i = e_i R$. Each $R_i$ is then semi-prime and contains $U_i$ as a symmetric subring. By Theorem 22, $R_i$ is $\ast$-simple, so either $U_i = R_i$ or $U_i$ is a field. If $U_i$ is a field, then $R_i$ satisfies a polynomial identity and hence is a finite dimensional algebra over a field contained in $Z(R_i)$. In either case, $R_i$ is always artinian. Hence $R$ must be also artinian.

III. Subrings generated by skew elements. In contrast to $\bar{S}$, $\bar{K}$ is not necessarily a Lie ideal of $R$. For instance, in $F_2$ with
char. $F \neq 2$ and transpose as $\ast$, $\bar{K} = \{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in F \}$. Although 
\[
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in K,
\]
\[
\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]
falls outside of $\bar{K}$. However, both $K^2$ and $K_5$, where $K_0 = \{ x - x^\ast \mid x \in R \}$, are always Lie ideals.

**Definition.** By a skew subgroup $V$ of $R$ we mean a subgroup of $R$ such that $K_0 \subseteq V \subseteq K$ and $xVx^\ast \subseteq V$ for all $x \in R$.

Henceforth we shall use $V$ to stand for a skew subgroup of $R$ without further explanation.

**Lemma 27.** $\overline{V}^2$ is a Lie ideal of $R$.

**Proof.** For $v_1, v_2 \in V$ and $x \in R$, we have 
\[
[v_1v_2, x] = v_1(v_2x + x^\ast v_2) - (v_1x^\ast + xv_1)v_2 \in V^2.
\]

If $w_1, \ldots, w_n \in V^2$ and $x \in R$, then 
\[
[w_1 \cdots w_n, x] = w_1[w_2 \cdots w_n, x] + [w_1, x]w_2 \cdots w_n.
\]

Hence, this lemma can be proved by induction.

**Lemma 28.** Let $R$ be a semi-prime ring and $n$ a natural number. If $v^{2n} = 0$ for all $v \in V$, then $V = 0$.

**Proof.** If $v^2 = 0$ for all $v \in V$, then for any $x \in R$ $(vx + x^\ast v)^2 = 0$ so $(vx)^3 = 0$. By Levitzki's lemma [5, Lemma 1.1], $v = 0$ for all $v \in V$. Assume that $n > 1$. For any $v \in V$ and $x \in R$, we have 
\[
(v^{2n-1}x - x^\ast v^{2n-1})^2 = 0
\]
and hence $(v^{2n-1}x)^2 = 0$. Applying Levitzki's lemma again and using the induction hypothesis, we conclude that $V = 0$.

One might have noticed that the study of a symmetric subring $U$ in $R$ is based on the fact: If $R$ is semi-prime, either $U \subseteq Z^+$ or $U$ contains a nonzero ideal of $R$. For a skew subgroup $V$, we have a parallel result for $\overline{V}^2$.

**Lemma 29.** If $R$ is $\ast$-prime and $[V^2, V^2] = 0$, then $V^2 \subseteq Z$ and 
$[V, V] = 0$. Further, $R$ satisfies the standard identity $S[x_1, x_2, x_3, x_4]$ in 4 variables.
Proof. Consider first the situation when $R$ is $*$-simple. If $R = A \oplus A^*$ for some simple ring $A$, then $K_0 = V = K$, and so $[V^2, V^2] = 0$ implies $[A^2, A^3] = 0$. Since $A^2 = A$, $R$ is also commutative, and the conclusions follow trivially. Assume that $R$ is simple. Then $V^2 \subseteq Z$ unless possibly $2R = 0$ and $\dim_2 R = 4$. If $R$ is a division ring, we have $xV^2x^{-1} = xVx^*(x^{-1})^*xV^2$ for all $x \in R$, $x \neq 0$. Hence $V^2 \subseteq Z$ by the Brauer-Cartan-Hua theorem. Suppose that $R = F_2$ for some field $F$ with $\text{char.} F = 2$. If $Z \nsubseteq \Gamma \subseteq V$ for some $a \in S$, then $1 = a^{-1}a + (a^{-1}a)^* \in \Gamma \subseteq V$ and hence $N \subseteq V$. By Lemma 2, $\Gamma$ is a symmetric subring. Since $V = 1 \cdot V \subseteq V^2$, $[V, V] = 0$ so $V \subseteq Z$ by Theorem 4. If $Z \cap T = 0$, then $Z \subseteq S$ and $*$ must be of transpose type, namely $\left[\begin{array}{cc} a & b^* \\ c & d \end{array}\right] = \left[\begin{array}{cc} a & \alpha^{-1}c \\ \alpha b & d \end{array}\right]$ for some $\alpha \in F$. In this case, $V \subseteq S = \left\{ \left[\begin{array}{cc} a & b \\ c & d \end{array}\right] \mid a, b, c \in F \right\}$. Since $\left[\begin{array}{cc} 0 & 1 \\ \alpha & 0 \end{array}\right] \in T$, $\left[\begin{array}{cc} 0 & 1 \\ \alpha & 0 \end{array}\right] \left[\begin{array}{cc} a & b \\ \alpha c & d \end{array}\right]$ commutes with $\left[\begin{array}{cc} 0 & 1 \\ \alpha & 0 \end{array}\right] \left[\begin{array}{cc} a' & b' \\ \alpha b' & c' \end{array}\right]$ for any $\left[\begin{array}{cc} a & b \\ \alpha c & d \end{array}\right]$, $\left[\begin{array}{cc} a' & b' \\ \alpha b' & c' \end{array}\right] \in V$. Comparing the (1,1)-entries of the products, we get $ca' = ac$. An argument like that in Theorem 4 shows $V = T = \left\{ \left[\begin{array}{c} b \\ \alpha b \end{array}\right] \mid b \in F \right\}$. Hence $V^2 = Z$. Thus we have $V^2 \subseteq Z$ always. By Lemma 28, there exists $v \in V$ such that $v^2 \neq 0$ provided $V \neq 0$. Then $v$ is invertible. Further, $v^{-1} = v^{-1}(-v)(v^{-1})^* \in V$, so $Vv^{-1} \subseteq Z$ and $V \subseteq Zv$. Consequently $[V, V] = 0$.

Now assume that $R$ is $*$-prime and $V \neq 0$. By Lemmas 1 and 28, $Z^+ \neq 0$, so we may consider the quotient ring $Q = \{a/\alpha^2 \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$. $Q$ can be equipped with $*$ by defining $(a/\alpha^2)^* = a^*/\alpha^2$. Then $Q$ is $*$-prime and $Q' = \{v/\alpha^2 \in Q \mid v \in V\}$ is a skew subgroup of $Q$. If there is a nonzero $*$-ideal $I$ of $Q$ such that $I \cap V' \neq 0$, then $I \subseteq S(Q)$ and hence $[I, I] = 0$. By the corollary to Lemma 6, $Q$ is commutative and we are done. Suppose that $J \cap V' \neq 0$ for any nonzero $*$-ideal $J$ of $Q$. Since $J \cap V'$ contains an element $a$ such that $a^4 \in Z$ and $a^4 \neq 0$ by Lemmas 1 and 28, and $a^4$ is invertible, we have $J = Q$. In other words, $Q$ is $*$-simple and so $V'^2 \subseteq Z(Q)$ and $[V', V'] = 0$. Hence $V^2 \subseteq Z(Q)$ and $[V, V] = 0$.

Since $K_0 \subseteq V$, we have $[K_0, K_0] = 0$ and hence $R$ satisfies $S_3[x_1, x_2, x_3, x_4]$ by Amitsur's Theorem [1].

We are now in a position to prove a series of theorems concerning $V^2$. Since the proofs are parallel to those for $U$, we shall omit them unless some modification is needed.

**Theorem 30.** If $R$ is $*$-simple and $V \neq 0$, then either $V^2 = R$ or $V^2$ is a field contained in $Z^+$. 
Proof. By Lemmas 1, 27 and 29, we have either \( V^2 = R \) or \( V^2 \subseteq Z^+ \). So it suffices to show that \( V^2 \) contains with invertible elements their inverses. First \( a^{-1}V = a^{-1}V(a^{-1})^*a^* \subseteq VV^2 \) if \( a \in V^2 \). Similarly, \( Va^{-1} \subseteq VV^2 \) and hence \( a^{-1}V^2a^{-1} \subseteq V^2 \) if \( a \in V^2 \). Thus \( a^{-1} = a^{-1}aa^{-1} \in V^2 \), if \( a \in V^1 \) and is invertible.

**Theorem 31.** If \( R \) is prime or *-prime, so is \( V^2 \).

**Theorem 32.** If \( R \) is semi-prime, then \( Z(V^2) \subseteq Z(R) \).

**Theorem 33.** If \( R \) is semi-prime, so is \( V^2 \).

**Theorem 34.** \( \mathfrak{P}(V^2) = V^2 \cap \mathfrak{P}(R) \).

**Theorem 35.** If \( R \) has no nil ideal other than 0, neither does \( V^2 \).

**Theorem 36.** \( \mathfrak{N}(V^2) = V^2 \cap \mathfrak{N}(R) \).

**Theorem 37.** If \( R \) has no nonzero locally nilpotent ideals, neither does \( V^2 \).

**Theorem 38.** \( \mathfrak{L}(V^2) = V^2 \cap \mathfrak{L}(R) \).

**Theorem 39.** If \( R \) is primitive or *-primitive, so is \( V^2 \) provided \( V \neq 0 \).

**Theorem 40.** If \( R \) is semi-simple, so is \( V^2 \).

**Theorem 41.** \( \mathfrak{G}(V^2) = V^2 \cap \mathfrak{G}(R) \).

**Theorem 42.** If \( R \) is semi-primary, primary, or completely primary, so is \( V^2 \) provided \( V \neq J(R) \).

Proof. It suffices to show that if \( a \in V^2 \) and \( a \circ b = 0 = b \circ a \) then \( b \in V^2 \). The argument used in Theorem 30 shows that \( (1 + b)V^2(1 + b) \subseteq V^2 \). (The formal use of the symbol 1 is all right.) Then \( b = -(1 + b)(a + a^2)(1 + b) \in V^2 \).

**Theorem 43.** If \( R \) is a semi-prime Goldie ring, so is \( V^2 \).

Proof. By Lemmas 1, 27 and 29, we have either \( V^2 = R \) or \( V^2 \subseteq Z^+ \). So it suffices to show that \( V^2 \) contains with invertible elements their inverses. First \( a^{-1}V = a^{-1}V(a^{-1})^*a^* \subseteq VV^2 \) if \( a \in V^2 \). Similarly, \( Va^{-1} \subseteq VV^2 \) and hence \( a^{-1}V^2a^{-1} \subseteq V^2 \) if \( a \in V^2 \). Thus \( a^{-1} = a^{-1}aa^{-1} \in V^2 \), if \( a \in V^1 \) and is invertible.
Proof. Since the a.c.c. on right annihilators is inherited by sub-
rings, it suffices to show that $V^2$ has infinite direct sum of nonzero
right ideals. Let $\{\rho_\alpha\}$ be a set of right ideals of $V^2$ such that $\Sigma_\alpha \rho_\alpha$ is direct. Denote by $I$ the ideal of $R$ generated by $[V^2, V^2]$. Then $\Sigma_\alpha \rho_\alpha I$ is a direct sum of right ideals of $R$, so $\rho_\alpha I = 0$ and hence $\rho_\alpha \subseteq V^2 \cap \text{Ann}.I \subseteq Z(V^2)$ for almost all $\alpha$. Being a commutative semi-prime
subring of a Goldie ring, $Z(V^2)$ is itself a Goldie ring and hence $\rho_\alpha = 0$ for almost all $\alpha$.

Let $R = F_2$, where $F$ is a field with char.$F = 2$ and $*$ is given by
transpose. In this case, $\bar{T} = K_0 = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in F \right\}$ possesses the nil-
potent ideal $\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| a \in F \right\}$ even though $R$ is simple. This example
kills the hope for $\bar{T}$ or $K_0$ to inherit those nice properties we have
discussed so far. Fortunately, the behavior of $\bar{K}$ is not that bad.

Theorem 44. If $R$ is *-simple, either $\bar{K} = R$ or $\bar{K}$ is a commutative
*-simple ring provided $K \neq 0$.

Proof. If char.$R = 2$, then $K = S$ and hence the assertion follows
from Theorem 4. Assume that char.$R \neq 2$. If $[K^2, K^2] \neq 0$, then $\bar{K}$
also contains the nonzero *-ideal of $R$ generated by $[K^2, K^2]$, so $\bar{K} = R$. If $K^2$ is commutative, then $K^2 \subseteq Z^*$ by Theorem 30. Suppose
that $S \subseteq S$, then $\alpha^* \neq \alpha$ for some $\alpha \in Z$, so $\beta = \alpha - \alpha^* \neq 0$. Thus,
$S\beta^{-1} \subseteq K$ and hence $S \subseteq K\beta$. Therefore, $R = S + K \subseteq \bar{K}$. Next, assume
that $Z \subseteq S$. Then $R$ must be simple. By Lemma 29, $R$ satisfies
an identity of degree 4 and hence dim$_z R \leq 4$ by Kaplansky's
Theorem. If $R$ is a division ring, choose $a \in K, a \neq 0$, then $Ka^{-1} \subseteq K^2 \subseteq Z$.
So $K \subseteq Z a \subseteq K$, that is, $K = Z a$. Hence $\bar{K} = Z(a)$ is a field. If
$R = F_2$ for some field $F$, the commutativity of $K$ forces * to be of
transpose type, say, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & \sigma^{-1} c \\ \sigma b & d \end{bmatrix}$ for some $\sigma \in F$. Then
$\bar{K} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b \in F \right\}$. If $- \sigma$ is not a square in $F$, $\bar{K}$ is a field;
while if $- \sigma = \pi^2$ for some $\pi \in F$, $\bar{K} = L_1 \oplus L_2$ where $L_1 = \left\{ \begin{bmatrix} a & \pi^{-1} a \\ \pi a & a \end{bmatrix} \middle| a \in F \right\}$ and $L_2 = \left\{ \begin{bmatrix} a & - \pi^{-1} a \\ - \pi a & a \end{bmatrix} \middle| a \in F \right\}$ are two
fields which are isomorphic via the map induced by $*$.

Theorem 45. If $R$ is *-prime, so is $\bar{K}$.

Proof. If $K^2$ is not commutative, then $\bar{K}$ also contains the ideal
generated by $[K^2, K^2]$. An argument exactly like that in Theorem 5
proves the *-primeness of $\bar{K}$. Now we assume that $K^2$ is a nonzero
commutative ring. The quotient ring $Q = \{a/\alpha \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$
is either a *-simple ring or a commutative *-prime ring relative to the involution \((a/a)^* = a^*/a\). In the former case, \(K(Q)\) is a commutative *-simple ring by the previous theorem. So in either case \(K(R)\) is contained in a commutative *-prime ring and hence is *-prime.

Lemma 46. If \(R\) is semi-prime, then \(C_V(V^2) = Z(\bar{V})\).

Proof. Assume first that \(R\) is *-prime. If \(V^2\) is not commutative, then it contains a nonzero *-ideal \(I\) of \(R\), so \(C_V(V^2) \subseteq C_R(I) \subseteq Z(R)\) by Lemma 6 and hence \(C_V(V^2) = Z(\bar{V})\). If \([V^2, V^2] = 0\), then \(V^2 \subseteq Z(R)\) and \([V, V] = 0\) by Lemma 29 and hence \(C_V(V^2) = V = Z(\bar{V})\). The semi-prime case can be built up easily via subdirect sum.

The next lemma is crucial in the study of \(\bar{K}\).

Lemma 47. Let \(R\) be a semi-prime ring and \(I\) a *-ideal of \(\bar{K}\). If \(I \cap K = 0\), then \(I = 0\).

Proof. If \(I \cap K = 0\), then \(I \subseteq S\). For any \(a \in I\) and \(k \in K\), \(ak = (ak)^* = -ka\). Hence \(I \subseteq C_K(K^2) = Z(\bar{K})\) by Lemma 46. Thus \(IK \subseteq I \cap K = 0\), so \(I \bar{K} = 0\), and in particular \(I^2 = 0\). For any \(a \in I\) and \(x \in R\), we have \(a(x - x^*) = 0\), that is, \(ax = ax^*\) and hence \(axa = a(xa)^* = a^2x^* = 0\). Since \(R\) is semi-prime, it follows that \(I = 0\).

Lemma 48. If \(R\) is semi-prime, and \(k \in K\) with \(kKk = 0\), then \(k = 0\).

Proof. For any \(x \in R\), \(k(x - x^*)k = 0\) so \(kxk = kx^*k\). Then \(kxk xk = k(xk^*) k = 0\) and hence \(kR\) is nil of index 3. So, \(k = 0\) by Levitzki's lemma.

Theorem 49. If \(R\) is semi-prime, so is \(\bar{K}\).

Proof. Let \(I\) be a *-ideal of \(\bar{K}\) such that \(I^2 = 0\). For any \(a \in I \cap K\), we have \(aKa \subseteq I^2 = 0\) so \(a = 0\) by Lemma 48. Lemma 47 shows \(I = 0\), so \(\bar{K}\) has no nonzero nilpotent *-ideal and hence is semi-prime.

Theorem 50. If \(R\) has no nil ideal other than 0, neither does \(\bar{K}\).

Proof. Let \(I\) be the ideal of \(R\) which is generated by \([K^2, \bar{K}^2]\). Then \(\mathfrak{N}(I) = 0\) and \(I \subseteq \bar{K}\). If \(a \in \mathfrak{N}(\bar{K}) \cap K\) and \(b \in \bar{K}^2\), then \(a^2b - ba^2 \in I \cap \mathfrak{N}(\bar{K}) = 0\). Thus \(a^2 \in Z(\bar{K}^2)\) and by Lemma 46 \(a^2 \in Z(\bar{K})\). But \(\bar{K}\) is semi-prime and \(a\) is nilpotent, so \(a^2 = 0\) for all \(a \in \mathfrak{N}(\bar{K}) \cap K\). In view of Lemma 28, \(\mathfrak{N}(\bar{K}) \cap K = 0\) because \(\mathfrak{N}(\bar{K})\) is itself a semi-prime ring. Hence, it follows from Lemma 47 that \(\mathfrak{N}(\bar{K}) = 0\).
A similar argument proves the following

**Theorem 51.** If $R$ has no nonzero locally nilpotent ideal, neither does $\bar{K}$.

The proof of the next theorem is exactly like that of Theorem 39.

**Theorem 52.** If $R$ is $*$-primitive, so is $\bar{K}$ provided $K \neq 0$.

**Theorem 53.** If $R$ is semi-simple, so is $\bar{K}$.

**Proof.** Let $a \in \mathfrak{J}(\bar{K}) \cap K$. For any $x \in R$, we have

$$ax \cdot (-ax^*) = a(x - x^* - xax^*) \in \mathfrak{J}(\bar{K})K \subseteq \mathfrak{J}(\bar{K}).$$

Hence $aR$ is quasi-regular, so $a = 0$. By Lemma 47, $\mathfrak{J}(\bar{K}) = 0$.

**Theorem 54.** If $R$ is semi-prime artinian, so is $\bar{K}$.

**Proof.** Immediate from Theorem 44.

Unlike $\tilde{S}$, the semi-prime assumption on $R$ is not sufficient to get the converse theorems for $\bar{K}$ or $\bar{K}^2$. For example, let $F$ be a field with $\text{char.} F \neq 2$, $\sigma$ an automorphism on $F$ with $\sigma^2 = 1$, and $A$ a commutative semi-prime algebra over $F$. Put $R = F \oplus A$ and define $(a, a)^* = (\sigma^* a, a)$. Then $\bar{K} = F$ and $\bar{K}^2 = F^\sigma$ are fields provided $\sigma \neq 1$, while $R$ is not even $*$-prime. Further, if $A$ possesses an identity and $\dim_F A = \infty$, then $R$ is neither artinian nor Goldie.

On the other hand, the $*$-primeness is sufficient for our purpose. To begin with, we prove a lemma which is analogous to Lemma 3.

**Lemma 55.** Let $R$ be a $*$-prime ring and $I$ a nonzero $*-$ideal of $R$ such that $I \cap K_0^2 = 0$. If $K_0 \neq 0$, then $I = 0$.

**Proof.** If $I \cap K_0^2 = 0$, then $(I \cap K_0)^2 = 0$. Since $I$ is itself a semi-prime ring, and $I \cap K_0$ is a skew subgroup of $I$, so $I \cap K_0 = 0$ by Lemma 28. Hence $I \subseteq S$. For any $a \in I$ and $x \in R$, we have $ax = (ax)^* = x^* a$. So if $a, b \in I$ and $x \in R$, then $abx = ax^* b = xab = abx^*$. That is, $I^2 K_0 = 0$. Since $R$ is $*$-prime and $K_0 \neq 0$, it follows $I = 0$.

**Lemma 56.** Let $R$ be a $*$-prime ring and $e$ the identity of $\bar{K}$ or $\bar{V}$. If $e \neq 0$, then it is the identity of $R$. 
Proof. Since the only nonzero central symmetric idempotent in a \( * \)-prime ring is the identity, it suffices to show that \( e \in Z(R) \). If \( e \) is the identity of \( \overline{V^2} \), then \( ex - xe \in \overline{V^2} \) for all \( x \in R \) because \( \overline{V^2} \) is a Lie ideal. If \( e \) works for \( K \), then \( ex - xe = e(x - x^*) + (ex^* - xe) \in K \) for all \( x \in R \). Hence \( e(ex - xe) = ex - xe = (ex - xe)e \) and this implies that \( e \in Z(R) \).

On the basis of Lemma 55, we can prove the converse theorems by using an argument parallel to that for \( U \).

**Theorem 57.** If \( R \) is \( * \)-prime, and \( \overline{K} \) or \( \overline{V^2} \) is a \( * \)-simple ring with identity, so is \( R \).

**Theorem 58.** If \( R \) is \( * \)-prime, and \( \overline{K} \) or \( \overline{V^2} \) is \( * \)-primitive, so is \( R \).

**Theorem 59.** Let \( R \) be a \( * \)-prime ring and \( * \) not the identity map. If \( \overline{K} \) or \( \overline{V^2} \) is semi-simple, so is \( R \).

Proof. Since \( \overline{\mathcal{G}(V^2)} = \overline{V^2} \cap \mathcal{G}(R) \), so \( \mathcal{G}(R) \cap K^0 = 0 \) if \( \overline{V^2} \) is semi-simple. By Lemma 55, \( R \) must be also semi-simple. In case \( \overline{K} \) is semi-simple, so is \( \overline{K^2} \) by Theorem 41, and hence \( R \) is also semi-simple.

**Theorem 60.** If \( R \) is \( * \)-prime, and \( \overline{K} \) or \( \overline{V^2} \) has no nil ideal other than 0, then neither does \( R \).

**Theorem 61.** If \( R \) is \( * \)-prime, and \( \overline{K} \) or \( \overline{V^2} \) has no nonzero locally nilpotent ideal, then neither does \( R \).

We close this paper with two theorems on chain conditions.

**Theorem 62.** Let \( R \) be a \( * \)-prime ring. If \( * \) is not the identity map and either \( \overline{K} \) or \( \overline{V^2} \) is artinian, then so is \( R \).

Proof. By Theorems 31 and 45, both \( \overline{K} \) and \( \overline{V^2} \) are \( * \)-prime. Say, if \( \overline{K} \) is artinian, then it is \( * \)-simple with identity, so \( R \) is also \( * \)-simple by Theorem 57 and hence \( \overline{K} = R \) or \( \overline{K} \) is commutative by Theorem 44. In the later case, \( R \) satisfies a polynomial identity, and is finite dimensional over a field contained in \( Z \). Hence, \( R \) is artinian. The situation when \( \overline{V^2} \) is artinian is the same.

For \( a \in R \), let \( r_K(a) = \{ x \in R | ax = 0 \} \) be the right annihilator of \( a \) in \( R \). Denote by \( \mathcal{B}(R) \) the right singular ideal of \( R \), that is, \( \mathcal{B}(R) = \{ a \in R | r_K(a) \cap \rho \neq 0 \text{ for any nonzero right ideal } \rho \text{ of } R \} \).

**Theorem 63.** Let \( R \) be a \( * \)-prime ring. If \( \overline{V^2} \) is a Goldie ring, so is \( R \).
Proof. If $R$ is commutative, then $Q = \{a/\alpha \mid a \in R, \alpha \in S, \alpha \neq 0\}$ is a commutative *-simple ring, and hence $R$ is a Goldie ring. Assume that $R$ is not commutative, while $[V^2, V^2] = 0$. Then $V^2 \subseteq Z^+$ and $Q = \{a/\alpha \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$ is a *-simple ring. Since $[V, V] = 0$, it follows that $Q$ satisfies a polynomial identity, and hence is artinian. So, $R$ is a Goldie ring. Lastly, assume that $[V^2, V^2] \neq 0$ and let $I$ be the ideal of $R$ generated by $\{V^2, V^2\}$. Suppose $\{\rho_a\}$ is a set of right ideals of $R$ which forms a direct sum. Then $\rho_a I \subseteq \rho_a \cap I \subseteq V^2$ and $\rho_a I = 0$ for almost all $\alpha$. Consequently $\rho_a = 0$ for almost all $\alpha$. Consider $\mathfrak{J}(R) \cap I$. If $a \in \mathfrak{J}(R) \cap I$, then for any nonzero right ideal $\rho$ of $I$, $\rho I \neq 0$, so $r_\rho(a) \cap \rho I \neq 0$ and hence $r_\rho(a) \cap \rho I \neq 0$. In other words, $\mathfrak{J}(R) \cap I \subseteq \mathfrak{J}(I) = 0$ because $I$ is itself a semi-prime Goldie ring. So $\mathfrak{J}(R) = 0$.

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