ON SUBRINGS OF RINGS WITH INVOLUTION

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We give a systematic account on the relationship between a ring $R$ with involution and its subrings $\mathcal{S}$ and $\mathcal{K}$, which are generated by all its symmetric elements or skew elements respectively.

I. Introduction. Let $R$ be a ring with involution $*$ and $\mathcal{S}$ the subring generated by the set $\mathcal{S}$ of all symmetric elements in $R$. The relationship between $R$ and $\mathcal{S}$ has been studied by various authors. In [3] Dieudonné showed that if $R$ is a division ring of characteristic not 2, then either $\mathcal{S} = R$ or $\mathcal{S} \subseteq Z(R)$, the center of $R$. Later Herstein [4] extended this result by proving $\mathcal{S} = R$ for any simple ring $R$ with $\dim_2 R > 4$ and $\text{char.} R \neq 2$. The restriction on characteristic was removed by Montgomery [12]. Recently, Lanski [9] proved that if $R$ is prime or semi-prime, so is $\mathcal{S}$. In §2 of this paper, we show that $\mathcal{S}$ can inherit a number of ring-theoretic properties such as primitivity, semisimplicity, absence of nonzero nil ideals etc. In doing so, a notion called symmetric subring, which is a generalization of $\mathcal{S}$ and its $*$-homomorphic images, is introduced so that a group of theorems of the same type, including Lanski's results, can be proved via a more or less unified argument. We show also that numerous radicals of $\mathcal{S}$ are merely the contractions from those of $R$. As a consequence, we see that $R$ modulo its prime radical behaves much like $\mathcal{S}$ in many respects.

In §3 we establish a corresponding theory for $\mathcal{K}$, the subring generated by all skew elements. The only result hitherto known concerning $\mathcal{K}$ was as follows [4], [12]: If $R$ is simple and $\dim_2 R > 4$, then $\mathcal{K} = R$. As a matter of fact, the subring $\mathcal{K}^2$ is more closely related to $R$ than $\mathcal{K}$ is. We apply the technique developed in §2 to study the relationship between $R$ and $\mathcal{K}^2$, and then derive some parallel theorems for $\mathcal{K}$.

II. Symmetric subrings. Our work depends heavily on the notion of Lie ideals. By a Lie ideal $U$ of $R$ we mean an additive subgroup which is invariant under all inner derivations of $R$. That is, $[u, x] = ux - xu \in U$ for all $u \in U$ and $x \in R$. The following lemma concerning Lie ideals will be referred to frequently in the sequel, and it is a combination of some results in [5].

Lemma 1. Let $R$ be a semi-prime ring and $U$ a subring and Lie ideal of $R$. Then $U$ contains the ideal of $R$ which is generated by $[U, U]$. If $U$ is commutative, then $u^2 \in Z$ for all $u \in U$. 

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Rings with involution abound with examples of Lie ideals. One can easily show that any subring, generated by symmetric elements and containing \( T = \{ x + x^* \mid x \in R \} \) the set of all traces, must be a Lie ideal. In particular, both \( \bar{S} \) and \( \bar{T} \) are Lie ideals.

Another essential property of \( \bar{S} \) follows from the next lemma. We denote by \( N \) the set of all norms, i.e. \( N = \{ xx^* \mid x \in R \} \).

**Lemma 2.** Let \( U \) be an additive subgroup of \( R \) such that \( T \subseteq U \subseteq S \) and \( xUx^* \subseteq U \) for all \( x \in R \). If \( N \subseteq \bar{U} \), then \( xUx^* \subseteq \bar{U} \) for all \( x \in R \).

**Proof.** We prove by induction that \( xu_1 \cdots u_n x^* \in \bar{U} \) for all \( x \in R \) and \( u_1, \cdots, u_n \in U \). The case \( n = 1 \) is clear. Assume the assertion holds for \( n - 1 \); then

\[
xu_1u_2 \cdots u_n x^* = [x, u_1][u_2 \cdots u_n, x^*] + (xu_1 x^*)u_2 \cdots u_n + u_1(xu_2 \cdots u_n x^*) - u_1 xx^* u_2 \cdots u_n \in \bar{U}
\]

because \( \bar{U} \) is a Lie ideal.

**Definition.** A subring \( U \) of \( R \) is called a symmetric subring if:
1. \( U \) is generated by a set of symmetric elements.
2. \( T \cup N \subseteq U \)
3. \( xUx^* \subseteq U \) for all \( x \in R \).

In light of Lemma 2, we know that \( \bar{S} \) is a symmetric subring. From now on, \( U \) will always denote a symmetric subring of \( R \). We call an ideal \( I \) of \( R \) a \(*\)-ideal if \( I^* = I \).

**Lemma 3.** If \( R \) is semi-prime and \( I \) is a \(*\)-ideal of \( R \) such that \( I \cap U = 0 \), then \( I = 0 \).

**Proof.** For any \( a \in I \), \( a^2 = a(a + a^*) - aa^* = 0 \). Then \( I \) is nil of index 2 and hence \( I = 0 \).

Recall that a ring \( R \) is called a \(*\)-simple ring if \( R^2 \neq 0 \) and \( R \) has no \(*\)-ideal other than 0 and \( R \). It is well-known that \( R \) is \(*\)-simple if and only if either \( R \) is simple or \( R = A \oplus A^* \) for some simple ring \( A \) [8, p. 14]. Let \( Z^* = Z \cap S \). Then if \( R \) is \(*\)-simple, we have \( Z^* = 0 \) or \( Z^* \) is a field.

**Theorem 4.** If \( R \) is \(*\)-simple, then either \( U = R \) or \( U \) is a field contained in \( Z^* \).

**Proof.** If \( U \) is not commutative, by Lemma 1 it contains a nonzero \(*\)-ideal of \( R \) so \( U = R \). Assume that \( [U, U] = 0 \); then \( U \subseteq S \). In this...
case, we need only to prove \( \mathcal{U} \subseteq \mathcal{Z} \), for if \( u \in \mathcal{U} \) and \( u \neq 0 \) then \( u^{-1} = u^{-1}u(u^{-1})^* \in \mathcal{U} \).

If \( R = A \oplus A^* \) for some simple ring \( A \), then \( T = U = S \). Thus \([U, U] = 0 \) implies \([A, A] = 0 \) and so \( R \) is commutative. If \( R \) is simple, then \( U \), being a commutative subring and Lie ideal of \( R \), must be central unless \( 2R = 0 \) and \( \dim \mathcal{z} = 4 \) [5, Theorem 1.5]. So let us examine all possible 4-dimensional cases.

If \( R \) is a division ring, then \( x^{-1}Ux = x^{-1}(xUx^*)x = Ux^*x \subseteq \mathcal{U} \) for all \( x \in R \) with \( x \neq 0 \). Hence \( \mathcal{U} \subseteq \mathcal{Z} \) by the Brauer-Cartan-Hua theorem [7, Theorem 7.13.1, Cor.].

There remains the case \( R = \mathbb{F}_2 \) where \( \mathbb{F} \) is a field with \( \text{char} \mathbb{F} = 2 \). We claim that \( * \) must be of symplectic type. Assume the contrary,

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}^* = \begin{bmatrix}
  a & \alpha^{-1}c \\
  \bar{a} & \bar{d}
\end{bmatrix}
\]

for some \( \alpha \in \mathbb{F} \) with \( \bar{a} = \alpha \), where \(-\) denotes the \( * \)-induced automorphism on \( \mathbb{F} \). Thus

\[
\mathcal{U} \subseteq S = \left\{ \begin{bmatrix}
  a & b \\
  \bar{a} & \bar{c}
\end{bmatrix} \mid \bar{a} = a, \bar{c} = c \right\}.
\]

For any \( a \in \mathbb{F} \), we have

\[
\begin{bmatrix}
  0 & a + \bar{a} \\
  0 & 0
\end{bmatrix} = \begin{bmatrix}
  a + \bar{a} & 0 \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  0 & 1 \\
  \alpha & 0
\end{bmatrix} \in T^2 \subseteq \mathcal{U}
\]

so \( \bar{a} = a \). Next, if \( \begin{bmatrix}
  a & b \\
  \bar{a} & \bar{c}
\end{bmatrix} \in \mathcal{U} \) then

\[
\begin{bmatrix}
  b & 0 \\
  a + c & b
\end{bmatrix} = \begin{bmatrix}
  a & b \\
  \bar{a} & \bar{c}
\end{bmatrix} \begin{bmatrix}
  0 & 0 \\
  1 & 0
\end{bmatrix} + \begin{bmatrix}
  0 & 0 \\
  1 & 0
\end{bmatrix} \begin{bmatrix}
  a & b \\
  \bar{a} & \bar{c}
\end{bmatrix} \in \mathcal{U}
\]

and hence \( a = c \). But if \( \begin{bmatrix}
  a & b \\
  \bar{a} & a
\end{bmatrix} \in \mathcal{U} \), then

\[
\begin{bmatrix}
  a & 0 \\
  0 & 0
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  a & b \\
  \bar{a} & \bar{c}
\end{bmatrix} \begin{bmatrix}
  1 & 0 \\
  0 & 0
\end{bmatrix} \in \mathcal{U}
\]

yields \( a = 0 \). So \( U = T = \left\{ \begin{bmatrix}
  0 & b \\
  \alpha & 0
\end{bmatrix} \mid b \in \mathbb{F} \right\} \) which is ridiculous because \( T \) is not a subring. Consequently, \( \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}^* = \begin{bmatrix}
  d & b \\
  c & a
\end{bmatrix} \) and

\[
\mathcal{U} \subseteq S = \left\{ \begin{bmatrix}
  a & b \\
  c & a
\end{bmatrix} \mid a, b, c \in \mathbb{F} \right\}.
\]
For any \([a \ b] [c' \ a'] \in U\), we have \([a' \ b'] [c' \ a'] \in U\) and hence \(bc' = b'c\) by comparing the diagonal entries of the product. If there exists \([a' \ b'] \in U\) with \(b' \neq 0\), then

\[
U \subseteq \left\{ \left[ \begin{array}{cc} a & b \\ \alpha b & a \end{array} \right] | a, b \in F \right\},
\]

where \(\alpha = c'b'^{-1}\). However,

\[
\left[ \begin{array}{cc} 0 & 0 \\ b' & 0 \end{array} \right] = \left[ \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{cc} a' & b' \\ c' & a' \end{array} \right] \left[ \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right] \in U
\]

forces \(b' = 0\), a contradiction. Hence \(U \subseteq \left\{ \left[ \begin{array}{cc} a & 0 \\ c & a \end{array} \right] | a, c \in F \right\}\). On the other hand, if \(\left[ \begin{array}{cc} a & 0 \\ c & a \end{array} \right] \in U\),

\[
\left[ \begin{array}{cc} 0 & 0 \\ 0 & c \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \left[ \begin{array}{cc} a & 0 \\ c & a \end{array} \right] \left[ \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \in U
\]

implies \(c = 0\). Therefore, \(U \subseteq Z\).

Following [11], we say \(R\) is *-prime if the product of any two nonzero *-ideals is still not zero. It is easy to see that \(R\) is *-prime if and only if \(aRb = a^*Rb = 0\) implies \(a = 0\) or \(b = 0\). As a consequence, any nonzero element in \(Z^*\) is regular in a *-prime ring \(R\).

We remind the reader of of a well-known fact that a nonzero Lie ideal of a semi-prime ring always contains elements with nonzero square.

**Theorem 5.** If \(R\) is *-prime, so is \(U\).

**Proof.** If \([U, U] \neq 0\), then \(U\) contains a nonzero *-ideal \(I\) of \(R\). For any two *-ideals \(A, B\) of \(U\) with \(AB = 0\), we have \(IAIB \subseteq AB = 0\), so either \(IAI = 0\) or \(B = 0\), ending up with \(A = 0\) or \(B = 0\). Assume that \(U \neq 0\) while \([U, U] = 0\). By Lemma 1, there exists \(u_0 \in U\) such that \(u_0^2 \in Z\) but \(u_0^2 \neq 0\). So consider the ring \(Q\) of fractions \(a/\alpha\) with \(a \in R\) and \(\alpha \in Z \cap U, \alpha \neq 0\). \(Q\) is also *-prime with respect to the involution given by \((a/\alpha)^* = a^*/\alpha\), and \(U' = \{u/\alpha \in Q | u \in U\}\) is a symmetric subring of \(Q\). As a matter of fact, \(Q\) is *-simple. For if \(J\) is any nonzero *-ideal of \(Q\), \(J \cap U' \neq 0\) and hence \((v/\beta)^2 \neq 0\) for some \(v/\beta \in J \cap U'\). Since \(v^2 \in Z\), \(v/\beta\) is invertible and so \(J = Q\). By the
previous theorem, \( U' \subseteq Z^+(Q) \) and hence \( U \) is an integral domain contained in \( Z^+(R) \).

Let \( C_R(V) = \{x \in R \mid xv = vx \text{ for all } v \in V\} \) be the centralizer of a set \( V \) in \( R \).

**Lemma 6.** Let \( I \neq 0 \) be an ideal (or \(*\)-ideal) of a prime (resp. \(*\)-prime) ring \( R \). Then \( C_R(I) \subseteq Z \).

**Proof.** For \( a \in I, b \in C_R(I) \) and \( x \in R \), we have \( abx = bax = axb \), or equivalently, \( a(bx - xb) = 0 \). That is, \( I[C_R(I), R] = 0 \). Hence \( [C_R(I), R] = 0 \) and so \( C_R(I) \subseteq Z \).

**Corollary.** Let \( R \) be a prime (or \(*\)-prime) ring and \( I \) a nonzero ideal (resp. \(*\)-ideal) of \( R \) such that \([I, I] = 0\). Then \( R \) is commutative.

**Theorem 7.** If \( R \) is semi-prime, then \( Z(U) \subseteq Z(R) \).

**Proof.** Assume first that \( R \) is \(*\)-prime. If \([U, U] = 0\), then \( Z(U) = U \subseteq Z(R) \) by Theorem 5. If \([U, U] \neq 0\), then \( U \) contains a nonzero \(*\)-ideal \( I \) of \( R \), so \( Z(U) \subseteq C_R(I) \subseteq Z(R) \) in view of Lemma 6. In either case, \([Z(U), R] = 0\). Now assume that \( R \) is semi-prime; then \( R \) is a subdirect sum of \(*\)-prime rings \( \pi_a(R) \). Since \( \pi_a(U) \) is a symmetric subring of \( \pi_a(R) \), we know \([\pi_a(Z(U)), \pi_a(R)] \subseteq [Z(\pi_a(U)), \pi_a(R)] = 0\) for all \( a \). Hence, \([Z(U), R] = 0\).

The same reduction to \(*\)-prime rings together with Theorem 5 gives an alternate proof for Lanski's theorem:

**Theorem 8.** If \( R \) is semi-prime, so is \( U \).

With this established, we are able to consider the relationship between the prime radicals \( \Bbb{P}(R) \) and \( \Bbb{P}(U) \).

**Theorem 9.** \( \Bbb{P}(U) = U \cap \Bbb{P}(R) \).

**Proof.** Since \( U/[U \cap \Bbb{P}(R)] \approx [U + \Bbb{P}(R)]/\Bbb{P}(R) \) which is a symmetric subring of the semi-prime ring \( R/\Bbb{P}(R) \), so \( U/[U \cap \Bbb{P}(R)] \) is semi-prime by Theorem 8 and hence \( \Bbb{P}(U) \subseteq U \cap \Bbb{P}(R) \). On the other hand, if \( a \in U \cap \Bbb{P}(R) \), then \( a \in U \) and any \( m \)-system in \( R \) containing \( a \) must contain 0. \[7, \text{Theorem 8.2.3}\]. Certainly, any \( m \)-system in \( U \) containing \( a \) contains 0. That is, \( a \in \Bbb{P}(U) \).

It is well-known that a ring without nonzero nil ideals is a subdirect sum of rings with the following property [6, p. 53]:

There exists a non-nilpotent element \( a \) such that \( a^{n(I)} \in I \) for all nonzero ideal \( I \).
One can impose this condition only on the $*$-ideals and show that it is a hereditary property. Then, making use of subdirect sum decomposition, we can prove that $U$ inherits the freedom from nonzero nil ideals. Instead of doing this way, we prefer to present a direct proof by considering the nil radical $\mathcal{N}(U)$ of $U$.

Theorem 10. If $R$ has no nil ideal other than 0, neither does $U$.

Proof. Let $I$ be the ideal of $R$ which is generated by $[U, U]$. Since $R$ possesses no nonzero nil ideal, neither does $I$, considered as a ring. Hence $\mathcal{N}(U) \cap I = 0$. For any $a \in \mathcal{N}(U)$ and $u \in U$, we have $[a, u] \in \mathcal{N}(U) \cap I = 0$. Thus $\mathcal{N}(U) \subseteq Z(U)$. Since $U$ is semi-prime by Theorem 8, $\mathcal{N}(U) = 0$.

As an immediate consequence, we have

Theorem 11. $\mathcal{N}(U) = U \cap \mathcal{N}(R)$.

Proceed as above with "locally nilpotent" in place of "nil" and with Levitzki radical $\mathcal{L}$ in place of $\mathcal{N}$, we get

Theorem 12. If $R$ has no nonzero locally nilpotent ideal, neither does $U$.

Theorem 13. $\mathcal{L}(U) = U \cap \mathcal{L}(R)$.

In [2] the notion of $*$-primitive ring was introduced as a ring admitting a $*$-faithful irreducible module $M$ (i.e. $Mr = Mr^* = 0$ implies $r = 0$). One can easily verify that a ring is $*$-primitive if and only if it is either primitive or a subdirect sum of a primitive ring and its opposite with the exchange involution.

We know that a nonzero ideal of a primitive ring is itself primitive. The proof is applicable to the following more general fact.

Lemma 14. Let $R$ be a primitive (or $*$-primitive) ring. Suppose that $I$ is a nonzero ideal (resp. $*$-ideal) of $R$, and $A$ is a subring (resp. $*$-subring, i.e. $A^* = A$) containing $I$. Then $A$ is also primitive (resp. $*$-primitive).

Theorem 15. If $R$ is primitive or $*$-primitive, so is $U$.

Proof. If $[U, U] \neq 0$, $U$ contains a nonzero $*$-ideal of $R$, so it is primitive or $*$-primitive by Lemma 14. Assume that $U$ is commutative. Then $U \subseteq Z^*$ and every element in $R$ is quadratic over
Z+. Hence R satisfies a polynomial identity. According to Kaplansky’s theorem [6, Theorem 6.3.1], R is *-simple and hence U is a field by Theorem 4.

Using the fact that a semi-simple ring is a subdirect sum of *-primitive rings, we get immediately

**Theorem 16.** If R is semi-simple, so is U.

In fact, the semi-simplicity of S was first proved by Herstein. His elegant proof was the inspiration of our next theorem which relates the Jacobson radicals of R and U.

**Theorem 17.** \( \mathfrak{J}(U) = U \cap \mathfrak{J}(R) \).

**Proof.** For \( a \in \mathfrak{J}(U) \) and \( x \in R \), we have

\[
ax \circ ax^* = ax + ax^* + axax^* = a(x + x^* + xax^*) \in \mathfrak{J}(U)U \subseteq \mathfrak{J}(U).
\]

Thus \( aR \) is quasi-regular and hence \( a \in U \cap \mathfrak{J}(R) \). Conversely, if \( a \in U \cap \mathfrak{J}(R) \), \( a \circ b = 0 \) for some \( b \in R \), then \( b = b \circ (a \circ b)^* = (b \circ b^*) \circ a^* \in U \). That is, \( U \cap \mathfrak{J}(R) \) is a quasi-regular ideal of U, so \( U \cap \mathfrak{J}(R) \subseteq \mathfrak{J}(U) \).

With Theorem 17 in hand, we are ready to study some non-semi-simple rings. Following [7], we say R is semi-primary, primary, or completely primary according as \( R/\mathfrak{J}(R) \) is an artinian, simple artinian, or division ring respectively. Since \( U/\mathfrak{J}(U) \) is isomorphic to a symmetric subring of \( R/J(R) \), by Theorem 4 we have

**Theorem 18.** If R is primary or completely primary, so is U.

As to semi-primary rings, we need some information about the descending chain condition. In a paper [10] which is to appear, Lanski proved that if R is artinian and \( \frac{1}{2} \in R \), then so is S. For our purpose, we prove

**Lemma 19.** If R is semi-prime artinian, so is U.

**Proof.** By the Wedderburn-Artin theorem, we may write \( R = R_1 \oplus \cdots \oplus R_n \) where each \( R_i \) is *-simple. Denote by \( e_i \) the identity of \( R_i \), then \( e_i \in Z^+ \) and so \( e_i U e_i \) is a symmetric subring of \( R_i \) for each i. By Theorem 4, each \( e_i U e_i \) is artinian, so is \( U = e_1 U e_1 \oplus \cdots \oplus e_n U e_n \).

**Theorem 20.** If R is semi-primary, so is U.
We remark that the assertion corresponding to Lemma 19 for ascending chain condition is not true even if $R$ is a commutative integral domain. A counter example can be found in [13].

Let $\mathfrak{N}$ stand for any of the four radicals $\mathfrak{H}, \mathfrak{L}, \mathfrak{N}$ and $\mathfrak{F}$. We have shown $\mathfrak{N}(U) = U \cap \mathfrak{N}(R)$. If $\mathfrak{N}(U) = U$, then $U \subseteq \mathfrak{N}(R)$, so $0$ is a symmetric subring of the semi-prime ring $R/\mathfrak{N}(R)$, and hence $\mathfrak{N}(R) = R$ by Lemma 3. That is, if $U$ is locally nilpotent, nil or quasi-regular, so is $R$.

On the other hand, $\mathfrak{N}(U) = 0$ need not imply $\mathfrak{N}(R) = 0$. For example, let $R = F + A$ be the algebra obtained by adjunction of an identity to a trivial algebra $A$ over a field $F$ with char. $F \neq 2$. Define $(\alpha + a)^* = \alpha - a$ for $\alpha \in F$ and $a \in A$. Then $S = F$ is a field, while $\mathfrak{N}(R) = A$ is a nilpotent ideal. In case $A$ has infinite dimension, this example shows also that $R$ is not artinian although $S$ is.

However, we still have some results on $\mathfrak{N}(R)$. For if $\mathfrak{N}(U) = 0$, then the *-ideal $\mathfrak{N}(R)$ has trivial intersection with $U$, hence is nil of index 2. Thus we have $aRa = 0$ for any $a \in \mathfrak{N}(R)$ and consequently $\mathfrak{N}(R) = \mathfrak{H}(R)$. Besides, $U$ is isomorphic to a symmetric subring of $R/\mathfrak{H}(R)$. Realizing this fact, one might not be surprised to see that $R/\mathfrak{H}(R)$, instead of $R$ itself, satisfies the same properties as $U$ does.

**Lemma 21.** Let $R$ be a semi-prime ring and $e$ the identity of $U$. Then $e$ is also the identity of $R$.

**Proof.** By Theorem 7, $e \in Z(U) \subseteq Z(R)$. Since $e \in S$, $I = \{x - ex \mid x \in R\}$ is a *-ideal of $R$. If $a - ea \in U$, then $a - ea = e(a - ea) = 0$. Thus $I \cap U = 0$ and so $I = 0$. In other words, $e$ is the identity of $R$.

The case when $R$ is semi-prime and $\tilde{S}$ is simple was thoroughly studied by Lanski [9]. An example was given there that $R$ is an integral domain but not simple while $\tilde{S}$ is. In the presence of an identity, we have

**Theorem 22.** Let $R$ be a semi-prime ring. If $U$ is a *-simple ring with identity, so is $R$.

**Proof.** Let $I$ be any nonzero *-ideal of $R$. Then $I \cap U \neq 0$, and the *-simplicity of $U$ implies $U \subseteq I$. By Lemma 21, $U$ contains the identity of $R$, so $I = R$.

Even if $U$ is a field, $R$ can be semi-prime but not simple. The simplest example is the direct sum of two copies of a field with the exchange involution. This example illustrates why we deal with only *-primeness and *-primitivity in what follows.
THEOREM 23. (1) If U is semi-prime, \( \mathfrak{P}(R) \) is nil of index 2. (2) If U is \(*\)-prime, so is \( R/\mathfrak{P}(R) \).

Proof. We have proved (1) in the discussion before Lemma 21. As to (2), we may assume without loss of generality that \( R \) is semi-prime. Let \( I \) and \( J \) be \(*\)-ideals of \( R \) such that \( IJ = 0 \). Then \( (I \cap U)(J \cap U) = 0 \), so \( I \cap U = 0 \) or \( J \cap U = 0 \), ending up with \( I = 0 \) or \( J = 0 \).

Suppose that \( R \) is a \(*\)-prime ring and \( I \) a nonzero \(*\)-ideal of \( R \). If \( I \) possesses a \(*\)-faithful irreducible module \( M \), write \( M = ml \) for some \( m \in M \) and \( m \neq 0 \), and define a map from \( M \times R \) into \( M \) by sending \( (ma, r) \) to \( m(ar) \). One can easily check that such a map is well defined and that \( M \) becomes a \(*\)-faithful irreducible \( R \)-module. This is the content of

Lemma 24. Let \( R \) be a \(*\)-prime ring and \( I \) a nonzero ideal of \( R \). If \( I \) is \(*\)-primitive, so is \( R \).

THEOREM 25. (1) If \( U \) is semi-simple, then \( \mathfrak{F}(R) = \mathfrak{P}(R) \) is nil of index 2. (2) If \( U \) is \(*\)-primitive, so is \( R/\mathfrak{P}(R) \).

Proof. We have seen the proof of (1) earlier. As to (2), we assume that \( R \) is semi-prime. By Theorem 23, \( R \) is \(*\)-prime. If \( [U, U] \neq 0 \), then \( U \) contains a nonzero \(*\)-ideal \( I \) of \( R \). Lemma 14 shows that \( I \) is itself \(*\)-primitive and hence \( R \) is also \(*\)-primitive by the previous lemma. If \( U \) is commutative, it is \(*\)-simple with identity. It follows from Theorem 22 that \( R \) is \(*\)-primitive.

THEOREM 26. If \( U \) is semi-primary, so is \( R \).

Proof. It suffices to show that if \( R \) is semi-prime and \( U \) is artinian, then \( R \) is also artinian. In this case, we have \( U = U_1 \oplus \cdots \oplus U_n \), where each \( U_i \) is \(*\)-simple artinian. Let \( e_i \) be the identity of \( U_i \); then \( e_i \in Z(U) \subseteq Z(R) \). Since \( 1 = e_1 + \cdots + e_n \), \( R = R_1 \oplus \cdots \oplus R_n \), with \( R_i = e_iR \). Each \( R_i \) is then semi-prime and contains \( U_i \) as a symmetric subring. By Theorem 22 \( R_i \) is \(*\)-simple, so either \( U_i = R_i \) or \( U_i \) is a field. If \( U_i \) is a field, then \( R_i \) satisfies a polynomial identity and hence is a finite dimensional algebra over a field contained in \( Z(R_i) \). In either case, \( R_i \) is always artinian. Hence \( R \) must be also artinian.

III. Subrings generated by skew elements. In contrast to \( S \), \( \overline{K} \) is not necessarily a Lie ideal of \( R \). For instance, in \( F_2 \) with
char. \( F \neq 2 \) and transpose as \( * \), \( \bar{K} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\} \). Although \[
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in K,
\]

\[
\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

falls outside of \( \bar{K} \). However, both \( \bar{K}^2 \) and \( \bar{K}_0^2 \), where \( K_0 = \{ x - x^* | x \in R \} \), are always Lie ideals.

**Definition.** By a skew subgroup \( V \) of \( R \) we mean a subgroup of \( R \) such that \( K_0 \subseteq V \subseteq K \) and \( xVx^* \subseteq V \) for all \( x \in R \).

Henceforth we shall use \( V \) to stand for a skew subgroup of \( R \) without further explanation.

**Lemma 27.** \( \bar{V}^2 \) is a Lie ideal of \( R \).

**Proof.** For \( v_1, v_2 \in V \) and \( x \in R \), we have

\[
[v_1v_2, x] = v_1(v_2x + x^*v_2) - (v_1x^* + xv_1)v_2 \in V^2.
\]

If \( w_1, \ldots, w_n \in V^2 \) and \( x \in R \), then

\[
[w_1 \cdots w_n, x] = w_1[w_2 \cdots w_n, x] + [w_1, x]w_2 \cdots w_n.
\]

Hence, this lemma can be proved by induction.

**Lemma 28.** Let \( R \) be a semi-prime ring and \( n \) a natural number. If \( v^{2^n} = 0 \) for all \( v \in V \), then \( V = 0 \).

**Proof.** If \( v^2 = 0 \) for all \( v \in V \), then for any \( x \in R \) \((vx + x^*v)^2 = 0\) so \((vx)^3 = 0\). By Levitzki's lemma [5, Lemma 1.1], \( v = 0 \) for all \( v \in V \). Assume that \( n > 1 \). For any \( v \in V \) and \( x \in R \), we have \((v^{2^{n-1}}x - x^*v^{2^{n-1}})^{2^n} = 0\) and hence \((v^{2^{n-1}}x)^{2^n+1} = 0\). Applying Levitzki's lemma again and using the induction hypothesis, we conclude that \( V = 0 \).

One might have noticed that the study of a symmetric subring \( U \) in \( R \) is based on the fact: *If \( R \) is semi-prime, either \( U \subseteq Z^* \) or \( U \) contains a nonzero ideal of \( R \).* For a skew subgroup \( V \), we have a parallel result for \( \bar{V}^2 \).

**Lemma 29.** If \( R \) is *-prime and \( [V^2, V^2] = 0 \), then \( V^2 \subseteq Z \) and \( [V, V] = 0 \). Further, \( R \) satisfies the standard identity \( S[x_1, x_2, x_3, x_4] \) in 4 variables.
Proof. Consider first the situation when $R$ is $\ast$-simple. If $R = A \oplus A^\ast$ for some simple ring $A$, then $K_0 = V = K$, and so $[V^2, V^2] = 0$ implies $[A^2, A^2] = 0$. Since $A^2 = A$, $R$ is also commutative, and the conclusions follow trivially. Assume that $R$ is simple. Then $V^2 \subseteq Z$ unless possibly $2R = 0$ and $\dim_Z R = 4$. If $R$ is a division ring, we have $xV^2x^{-1} = xVx^*(x^{-1})Vx^{-1} \subseteq V^2$, so $xV^2x^{-1} \subseteq \overline{V^2}$ for all $x \in R$, $x \neq 0$. Hence $V^2 \subseteq Z$ by the Brauer-Cartan-Hua theorem. Suppose that $R = F_2$ for some field $F$ with $\text{char.} F = 2$. If $V = 0$, say, $\alpha = \alpha + \alpha^\ast \in Z$ for some $\alpha \notin S$, then $1 = \alpha^{-1}(\alpha - \alpha^\ast) \in T \subseteq V$ and hence $N \subseteq V$. By Lemma 2, $V$ is a symmetric subring. Since $V = V \cdot V \subseteq V^2$, $[V, V] = 0$ so $V \subseteq Z$ by Theorem 4. If $Z \cap T = 0$, then $Z \subseteq S$ and $\ast$ must be of transpose type, namely $[a \ b] = [a \ a^{-1}c]$
for some $\alpha \in F$. In this case, $V \subseteq S = \left\{ \begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix} \mid a, b, c \in F \right\}$. Since
$$
\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \in T, \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix} \text{ commutes with } \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} a' & b' \\ \alpha b' & c' \end{bmatrix}
$$
for any $\begin{bmatrix} a & b \\ \alpha b & c \end{bmatrix}, \begin{bmatrix} a' & b' \\ \alpha b' & c' \end{bmatrix} \in V$. Comparing the $(1,1)$-entries of the products, we get $ca' = ac'$. An argument like that in Theorem 4 shows $V = T = \left\{ \begin{bmatrix} 0 & b \\ \alpha b & 0 \end{bmatrix} \mid b \in F \right\}$. Hence $V^2 = Z$. Thus we have $V^2 \subseteq Z$ always. By Lemma 28, there exists $v \in V$ such that $v^2 \neq 0$ provided $V \neq 0$. Then $v$ is invertible. Further, $v^{-1} = v^{-1}(-v)(v^{-1})^\ast \in V$, so $Vv^{-1} \subseteq Z$ and $V \subseteq Zv$. Consequently $[V, V] = 0$.

Now assume that $R$ is $\ast$-prime and $V \neq 0$. By Lemmas 1 and 28, $Z^+ \neq 0$, so we may consider the quotient ring $Q = \{a/\alpha^2 \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$. $Q$ can be equipped with $\ast$ by defining $(a/\alpha^2)^\ast = a^\ast/\alpha^2$. Then $Q$ is $\ast$-prime and $V' = \{v/\alpha^2 \in Q \mid v \in V\}$ is a skew subgroup of $Q$. If there is a nonzero $\ast$-ideal $I$ of $Q$ such that $I \cap V' \neq 0$, then $I \subseteq S(Q)$ and hence $[I, I] = 0$. By the corollary to Lemma 6, $Q$ is commutative and we are done. Suppose that $J \cap V' \neq 0$ for any nonzero $\ast$-ideal $J$ of $Q$. Since $J \cap V'$ contains an element $a$ such that $a^4 \in Z$ and $a^8 \neq 0$ by Lemmas 1 and 28, and $a^8$ is invertible, we have $J = Q$. In other words, $Q$ is $\ast$-simple and so $V^2 \subseteq Z(Q)$ and $[V', V'] = 0$. Hence $V^2 \subseteq Z(R)$ and $[V, V] = 0$.

Since $K_0 \subseteq V$, we have $[K_0, K_0] = 0$ and hence $R$ satisfies $S_4[x_1, x_2, x_3, x_4]$ by Amitsur's Theorem [1].

We are now in a position to prove a series of theorems concerning $\overline{V^2}$. Since the proofs are parallel to those for $U$, we shall omit them unless some modification is needed.

**Theorem 30.** If $R$ is $\ast$-simple and $V \neq 0$, then either $\overline{V^2} = R$ or $\overline{V^2}$ is a field contained in $Z^+$. 

Proof. By Lemmas 1, 27 and 29, we have either $V^2 = R$ or $V^2 \subseteq Z^+$. So it suffices to show that $V^2$ contains with invertible elements their inverses. First $a^{-1}V = a^{-1}V(a^{-1})^*a^* \subseteq VV^2$ if $a \in V^2$. Similarly, $Va^{-1} \subseteq V^2V$ and hence $a^{-1}V^2a^{-1} \subseteq V^2$ if $a \in V^2$. Thus $a^{-1} = a^{-1}aa^{-1} \in V^2$, if $a \in V^1$ and is invertible.

**Theorem 31.** If $R$ is prime or $^*$-prime, so is $V^2$.

**Theorem 32.** If $R$ is semi-prime, then $Z(V^2) \subseteq Z(R)$.

**Theorem 33.** If $R$ is semi-prime, so is $V^2$.

**Theorem 34.** $\mathfrak{p}(V^2) = V^2 \cap \mathfrak{p}(R)$.

**Theorem 35.** If $R$ has no nil ideal other than 0, neither does $V^2$.

**Theorem 36.** $\mathfrak{n}(V^2) = V^2 \cap \mathfrak{n}(R)$.

**Theorem 37.** If $R$ has no nonzero locally nilpotent ideals, neither does $V^2$.

**Theorem 38.** $\mathfrak{q}(V^2) = V^2 \cap \mathfrak{q}(R)$.

**Theorem 39.** If $R$ is primitive or $^*$-primitive, so is $V^2$ provided $V \neq 0$.

**Theorem 40.** If $R$ is semi-simple, so is $V^2$.

**Theorem 41.** $\mathfrak{z}(V^2) = V^2 \cap \mathfrak{z}(R)$.

**Proof.** It suffices to show that if $a \in V^2$ and $a \cdot b = 0 = b \cdot a$ then $b \in V^2$. The argument used in Theorem 30 shows that $(1 + b)V^2(1 + b) \subseteq V^2$. (The formal use of the symbol 1 is all right.) Then $b = -(1 + b)(a + a^2)(1 + b) \in V^2$.

**Theorem 42.** If $R$ is semi-primary, primary, or completely primary, so is $V^2$ provided $V \neq J(R)$.

In the example given in [13], $2R = 0$ and $1 \in R$, so $K^2 = \bar{S}$. Hence $K^2$ need not be noetherian even if $R$ is a commutative noetherian domain. However, $K^2$, as well as $\bar{S}$, inherits Goldie conditions when $R$ is semi-prime. The proof of the next theorem is based on Lanski's argument [10] but is a little simpler.

**Theorem 43.** If $R$ is a semi-prime Goldie ring, so is $V^2$. 
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Proof. Since the a.c.c. on right annihilators is inherited by subrings, it suffices to show that $V^2$ has an infinite direct sum of nonzero right ideals. Let $\{\rho_a\}$ be a set of right ideals of $V^2$ such that $\Sigma_a\rho_a$ is direct. Denote by $I$ the ideal of $R$ generated by $[V^2, V^2]$. Then $\Sigma_a\rho_aI$ is a direct sum of right ideals of $R$, so $\rho_aI = 0$ and hence $\rho_a \subseteq V^2 \cap \text{Ann.}I \subseteq Z(V^2)$ for almost all $a$. Being a commutative semi-prime subring of a Goldie ring, $Z(V^2)$ is itself a Goldie ring and hence $\rho_a = 0$ for almost all $a$.

Let $R = F_2$, where $F$ is a field with $\text{char.} F = 2$ and $*$ is given by transpose. In this case, $\overline{T} = K_0 = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \bigg| a, b \in F \right\}$ possesses the nilpotent ideal $\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \bigg| a \in F \right\}$ even though $R$ is simple. This example kills the hope for $\overline{T}$ or $\overline{K}_0$ to inherit those nice properties we have discussed so far. Fortunately, the behavior of $\overline{K}$ is not that bad.

THEOREM 44. If $R$ is $*$-simple, either $\overline{K} = R$ or $\overline{K}$ is a commutative $*$-simple ring provided $K \neq 0$.

Proof. If $\text{char.} R = 2$, then $K = S$ and hence the assertion follows from Theorem 4. Assume that $\text{char.} R \neq 2$. If $[K^2, K^2] \neq 0$, then $\overline{K}$ also contains the nonzero $*$-ideal of $R$ generated by $[K^2, K^2]$, so $\overline{K} = R$. If $K^2$ is commutative, then $K^2 \subseteq Z^+$ by Theorem 30. Suppose that $Z \not\subseteq S$, then $\alpha^* \neq \alpha$ for some $\alpha \in Z$, so $\beta = \alpha - \alpha^* \neq 0$. Thus, $S\beta^{-1} \subseteq K$ and hence $S \subseteq K\beta$. Therefore, $R = S + K \subseteq \overline{K}$. Next, assume that $Z \subseteq S$. Then $R$ must be simple. By Lemma 29, $R$ satisfies an identity of degree 4 and hence dim$_Z R \leq 4$ by Kaplansky's Theorem. If $R$ is a division ring, choose $a \in K$, $a \neq 0$, then $Ka^{-1} \subseteq K^2 \subseteq Z$. So $K \subseteq Za \subseteq K$, that is, $K = Za$. Hence $\overline{K} = Z(a)$ is a field. If $R = F_2$ for some field $F$, the commutativity of $K$ forces $*$ to be of transpose type, say, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a & \sigma^{-1}c \\ \sigma b & d \end{bmatrix}$ for some $\sigma \in F$. Then $\overline{K} = \left\{ \begin{bmatrix} a & b \\ -\sigma b & a \end{bmatrix} \bigg| a, b \in F \right\}$. If $-\sigma$ is not a square in $F$, $\overline{K}$ is a field; while if $-\sigma = \pi^2$ for some $\pi \in F$, $\overline{K} = L_1 \oplus L_2$ where $L_1 = \left\{ \begin{bmatrix} a \pi^{-1}a \\ a \end{bmatrix} \bigg| a \in F \right\}$ and $L_2 = \left\{ \begin{bmatrix} a & -\pi^{-1}a \\ \pi a & a \end{bmatrix} \bigg| a \in F \right\}$ are two fields which are isomorphic via the map induced by $*$.

THEOREM 45. If $R$ is $*$-prime, so is $\overline{K}$.

Proof. If $\overline{K}^2$ is not commutative, then $\overline{K}$ also contains the ideal generated by $[\overline{K}^2, \overline{K}^2]$. An argument exactly like that in Theorem 5 proves the $*$-primeness of $\overline{K}$. Now we assume that $\overline{K}^2$ is a nonzero commutative ring. The quotient ring $Q = \{ a/\alpha \big| a \in R, \alpha \in Z^+, \alpha \neq 0 \}$
is either a *-simple ring or a commutative *-prime ring relative to the involution $(a/\alpha)^* = a*/\alpha$. In the former case, $K(Q)$ is a commutative *-simple ring by the previous theorem. So in either case $K(R)$ is contained in a commutative *-prime ring and hence is *-prime.

**Lemma 46.** If $R$ is semi-prime, then $C_V(V^2) = Z(V)$.

*Proof.* Assume first that $R$ is *-prime. If $V^2$ is not commutative, then it contains a nonzero *-ideal $I$ of $R$, so $C_V(V^2) \subseteq C_I(I) \subseteq Z(R)$ by Lemma 6 and hence $C_V(V^2) = Z(V)$. If $[V^2, V^2] = 0$, then $V^2 \subseteq Z(R)$ and $[V, V] = 0$ by Lemma 29 and hence $C_V(V^2) = \bar{V} = Z(V)$. The semi-prime case can be built up easily via subdirect sum.

The next lemma is crucial in the study of $\bar{K}$.

**Lemma 47.** Let $R$ be a semi-prime ring and $I$ a *-ideal of $\bar{K}$. If $I \cap K = 0$, then $I = 0$.

*Proof.* If $I \cap K = 0$, then $I \subseteq S$. For any $a \in I$ and $k \in K$, $ak = (ak)^* = -ka$. Hence $I \subseteq C_K(K^2) = Z(\bar{K})$ by Lemma 46. Thus $IK \subseteq I \cap K = 0$, so $I\bar{K} = 0$, and in particular $I^2 = 0$. For any $a \in I$ and $x \in R$, we have $a(x - x*) = 0$, that is, $ax = ax^*$ and hence $axa = a(xa)^* = a^2x^* = 0$. Since $R$ is semi-prime, it follows that $I = 0$.

**Lemma 48.** If $R$ is semi-prime, and $k \in K$ with $kKk = 0$, then $k = 0$.

*Proof.* For any $x \in R$, $k(x - x^*)k = 0$ so $kxk = kx^*k$. Then $kxkxk = k(xkx^*)k = 0$ and hence $kR$ is nil of index 3. So, $k = 0$ by Levitzki's lemma.

**Theorem 49.** If $R$ is semi-prime, so is $\bar{K}$.

*Proof.* Let $I$ be a *-ideal of $\bar{K}$ such that $I^2 = 0$. For any $a \in I \cap K$, we have $aKa \subseteq I^2 = 0$ so $a = 0$ by Lemma 48. Lemma 47 shows $I = 0$, so $\bar{K}$ has no nonzero nilpotent *-ideal and hence is semi-prime.

**Theorem 50.** If $R$ has no nil ideal other than $0$, neither does $\bar{K}$.

*Proof.* Let $I$ be the ideal of $R$ which is generated by $[K^2, K^2]$. Then $\mathcal{N}(I) = 0$ and $I \subseteq \bar{K}$. If $a \in \mathcal{N}(\bar{K}) \cap K$ and $b \in K^2$, then $a^2b - ba^2 \in I \cap \mathcal{N}(\bar{K}) = 0$. Thus $a^2 \in Z(K^2)$ and by Lemma 46 $a^2 \in Z(\bar{K})$. But $\bar{K}$ is semi-prime and $a$ is nilpotent, so $a^2 = 0$ for all $a \in \mathcal{N}(\bar{K}) \cap K$. In view of Lemma 28, $\mathcal{N}(\bar{K}) \cap K = 0$ because $\mathcal{N}(\bar{K})$ is itself a semi-prime ring. Hence, it follows from Lemma 47 that $\mathcal{N}(\bar{K}) = 0$. 


A similar argument proves the following

**Theorem 51.** If $R$ has no nonzero locally nilpotent ideal, neither does $\overline{K}$.

The proof of the next theorem is exactly like that of Theorem 39.

**Theorem 52.** If $R$ is *-primitive, so is $\overline{K}$ provided $K \neq 0$.

**Theorem 53.** If $R$ is semi-simple, so is $\overline{K}$.

**Proof.** Let $a \in \mathcal{J}(\overline{K}) \cap K$. For any $x \in R$, we have

$$ax \circ (-ax^*) = a(x - x^* - xax^*) \in \mathcal{J}(\overline{K})K \subseteq \mathcal{J}(\overline{K}).$$

Hence $aR$ is quasi-regular, so $a = 0$. By Lemma 47, $\mathcal{J}(\overline{K}) = 0$.

**Theorem 54.** If $R$ is semi-prime artinian, so is $\overline{K}$.

**Proof.** Immediate from Theorem 44.

Unlike $\overline{S}$, the semi-prime assumption on $R$ is not sufficient to get the converse theorems for $\overline{K}$ or $\overline{K^2}$. For example, let $F$ be a field with $\text{char. } F \neq 2$, $\sigma$ an automorphism on $F$ with $\sigma^2 = 1$, and $A$ a commutative semi-prime algebra over $F$. Put $R = F \oplus A$ and define $(\alpha, a)^* = (\alpha^*, a)$. Then $\overline{K} = F$ and $\overline{K^2} = F^\sigma$ are fields provided $\sigma \neq 1$, while $R$ is not even *-prime. Further, if $A$ possesses an identity and $\dim FA = \infty$, then $R$ is neither artinian nor Goldie.

On the other hand, the *-primeness is sufficient for our purpose. To begin with, we prove a lemma which is analogous to Lemma 3.

**Lemma 55.** Let $R$ be a *-prime ring and $I$ a nonzero *-ideal of $R$ such that $I \cap K_0^2 = 0$. If $K_0 \neq 0$, then $I = 0$.

**Proof.** If $I \cap K_0^2 = 0$, then $(I \cap K_0)^2 = 0$. Since $I$ is itself a semi-prime ring, and $I \cap K_0$ is a skew subgroup of $I$, so $I \cap K_0 = 0$ by Lemma 28. Hence $I \subseteq S$. For any $a \in I$ and $x \in R$, we have $ax = (ax)^* = x^*a$. So if $a, b \in I$ and $x \in R$, then $abx = ax^*b = xab = abx^*$. That is, $I^2K_0 = 0$. Since $R$ is *-prime and $K_0 \neq 0$, it follows $I = 0$.

**Lemma 56.** Let $R$ be a *-prime ring and $e$ the identity of $\overline{K}$ or $\overline{V^2}$. If $e \neq 0$, then it is the identity of $R$. 

Proof. Since the only nonzero central symmetric idempotent in a *-prime ring is the identity, it suffices to show that \( e \in Z(R) \). If \( e \) is the identity of \( V^2 \), then \( ex - xe \in V^2 \) for all \( x \in R \) because \( V^2 \) is a Lie ideal. If \( e \) works for \( K \), then \( ex - xe = e(x - x^*) + (ex^* - xe) \in K \) for all \( x \in R \). Hence \( e(ex - xe) = ex - xe = (ex - xe)e \) and this implies that \( e \in Z(R) \).

On the basis of Lemma 55, we can prove the converse theorems by using an argument parallel to that for \( U \).

**Theorem 57.** If \( R \) is *-prime, and \( K \) or \( V^2 \) is a *-simple ring with identity, so is \( R \).

**Theorem 58.** If \( R \) is *-prime, and \( K \) or \( V^2 \) is *-primitive, so is \( R \).

**Theorem 59.** Let \( R \) be a *-prime ring and * not the identity map. If \( K \) or \( V^2 \) is semi-simple, so is \( R \).

Proof. Since \( \mathcal{N}(V^2) = V^2 \cap \mathcal{N}(R) \), so \( \mathcal{N}(R) \cap K_0^2 = 0 \) if \( V^2 \) is semi-simple. By Lemma 55, \( R \) must be also semi-simple. In case \( K \) is semi-simple, so is \( K^2 \) by Theorem 41, and hence \( R \) is also semi-simple.

**Theorem 60.** If \( R \) is *-prime, and \( K \) or \( V^2 \) has no nil ideal other than 0, then neither does \( R \).

**Theorem 61.** If \( R \) is *-prime, and \( K \) or \( V^2 \) has no nonzero locally nilpotent ideal, then neither does \( R \).

We close this paper with two theorems on chain conditions.

**Theorem 62.** Let \( R \) be a *-prime ring. If * is not the identity map and either \( K \) or \( V^2 \) is artinian, then so is \( R \).

Proof. By Theorems 31 and 45, both \( K \) and \( V^2 \) are *-prime. Say, if \( K \) is artinian, then it is *-simple with identity, so \( R \) is also *-simple by Theorem 57 and hence \( K = R \) or \( K \) is commutative by Theorem 44. In the later case, \( R \) satisfies a polynomial identity, and is finite dimensional over a field contained in \( Z \). Hence, \( R \) is artinian. The situation when \( V^2 \) is artinian is the same.

For \( a \in R \), let \( r_R(a) = \{ x \in R \mid ax = 0 \} \) be the right annihilator of \( a \) in \( R \). Denote by \( \mathcal{J}(R) \) the right singular ideal of \( R \), that is, \( \mathcal{J}(R) = \{ a \in R \mid r_R(a) \cap \rho \neq 0 \text{ for any nonzero right ideal } \rho \text{ of } R \} \).

**Theorem 63.** Let \( R \) be a *-prime ring. If \( V^2 \) is a Goldie ring, so is \( R \).
Proof. If $R$ is commutative, then $Q = \{a/\alpha \mid a \in R, \alpha \in S, \alpha \neq 0\}$ is a commutative $*$-simple ring, and hence $R$ is a Goldie ring. Assume that $R$ is not commutative, while $[V^2, V^2] = 0$. Then $V^2 \subseteq Z^+$ and $Q = \{a/\alpha \mid a \in R, \alpha \in Z^+, \alpha \neq 0\}$ is a $*$-simple ring. Since $[V, V] = 0$, it follows that $Q$ satisfies a polynomial identity, and hence is artinian. So, $R$ is a Goldie ring. Lastly, assume that $[V^2, V^2] \neq 0$ and let $I$ be the ideal of $R$ generated by $\{V^2, V^2\}$. Suppose $\{\rho_\alpha\}$ is a set of right ideals of $R$ which forms a direct sum. Then $\rho_\alpha I \subseteq \rho_\alpha \cap I \subseteq V^2$ and $\rho_\alpha I = 0$ for almost all $\alpha$. Consequently $\rho_\alpha = 0$ for almost all $\alpha$. Consider $\overline{\mathfrak{Z}}(R) \cap I$. If $a \in \overline{\mathfrak{Z}}(R) \cap I$, then for any nonzero right ideal $\rho$ of $I$, $\rho I \neq 0$, so $r_\rho(a) \cap \rho I \neq 0$ and hence $r_\rho(a) \cap \rho \neq 0$. In other words, $\overline{\mathfrak{Z}}(R) \cap I \subseteq \overline{\mathfrak{Z}}(I) = 0$ because $I$ is itself a semi-prime Goldie ring. So $\overline{\mathfrak{Z}}(R) = 0$.

REFERENCES


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