ON TWO THEOREMS OF FROBENIUS

MARVIN MARCUS AND HENRYK MINC

This note contains simple proofs of two classical theorems of Frobenius, on nonnegative matrices. These concern powers of a primitive matrix and the maximal root of a principal submatrix of an irreducible matrix.

The purpose of this note is to give simple and straightforward proofs for two classical theorems of Frobenius [1].

A matrix \( A \) is said to be nonnegative (positive) if all its entries are nonnegative (positive); we write \( A \geq 0 \) (\( A > 0 \)). A nonnegative square matrix is called reducible if there exists a permutation matrix \( P \) such that

\[
P A P^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},
\]

where \( B \) and \( D \) are square; otherwise \( A \) is irreducible.

It was shown by Frobenius [1] that a nonnegative square matrix has a real maximal root \( r \) such that \( r \leq |\lambda_i| \) for every root \( \lambda_i \) of \( A \) and that to \( r \) corresponds a nonnegative characteristic vector. Moreover, if \( A \) is irreducible, then the maximal root \( r \) of \( A \) is simple and there is a positive characteristic vector corresponding to it. An irreducible matrix is said to be primitive if its maximal root is strictly greater than the moduli of the other roots.

We prove the following remarkable two results due to Frobenius (see [1]; also Theorem 8 and Proposition 4, p. 69, in [2]).

**Theorem 1.** If \( A \) is primitive then

\[
A^m > 0
\]

for some positive integer \( m \).

**Theorem 2.** The maximal root of an irreducible matrix is greater than the maximal root of any of its principal submatrices.

**Proof of Theorem 1.** Let \( A \) be a primitive matrix with maximal root \( r \). Then the matrix \( 1/r A \) is primitive as well, its maximal root is 1, and all its other roots have moduli less than 1. Let

\[
S^{-1} \left( \frac{1}{r} A \right) S = 1 + B,
\]

(1)
where \( 1 + B \) is, e.g., the Jordan normal form of \( 1/rA \). We can deduce immediately from (1) that:

(i) the moduli of all roots of \( B \) are less than 1 and therefore \( \lim_{t \to \infty} B^t = 0 \);
(ii) the first column of \( S \) is a character vector of \( A \) corresponding to the maximal root 1 and therefore has no zero coordinates;
(iii) the first row of \( S^{-1} \) is a characteristic vector of the transpose of \( 1/rA \) corresponding to its maximal root and thus cannot have zero coordinates.

Now,

\[
\lim_{t \to \infty} \left( \frac{1}{r} A \right)^t = \lim_{t \to \infty} (S(1 + B)S^{-1})^t = S(1 + (\lim_{t \to \infty} B^t))S^{-1} = S(1 + 0)S^{-1}
\]

is a nonnegative matrix. But the \((i, j)\) entry of \( S(1 + 0)S^{-1} \) is the nonzero product \( S_{1i}(S^{-1})_{ij} \). Hence \( S(1 + 0)S^{-1} \) must be strictly positive, i.e.,

\[
\lim_{t \to \infty} \left( \frac{1}{r} A \right)^t > 0.
\]

It follows that for sufficiently large integer \( m \),

\[
\left( \frac{1}{r} A \right)^m > 0,
\]

and therefore

\[
A^m > 0.
\]

In order to prove Theorem 2 we require the following lemma obtained in a more general form by Wielandt [3]:

*Let \( A \) be an \( n \times n \) irreducible matrix with maximal root \( r \). If \( x \) is a nonnegative \( n \)-tuple, \( x \neq 0 \), and \( k \) a nonnegative number satisfying

\[
Ax - kx \geq 0,
\]

then

(2) \( k \leq r \).

Equality can hold in (2) only if \( x > 0 \).
Proof of Theorem 2. We can assume without loss of generality that the principal submatrix in question lies in the first $t$ rows and first $t$ columns, i.e., that

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix},$$

where $B$ is the principal $t \times t$ submatrix. Let $r$ and $k$ be the maximal roots of $A$ and $B$, respectively. Let $y$ be a nonnegative characteristic vector of $B$ corresponding to $k$ and let

$$x = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

be the $n$-tuple whose first $t$ coordinates are those of $y$ and whose last $n - t$ coordinates are 0. Then

$$Ax = \begin{bmatrix} By \\ Dy \end{bmatrix} = k \begin{bmatrix} y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ Dy \end{bmatrix},$$

$$Ax - kx = \begin{bmatrix} 0 \\ Dy \end{bmatrix} \geq 0.$$

It follows immediately by the preceding lemma that

$$k < r.$$

References


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