

# Pacific Journal of Mathematics

## **A REPRESENTATION THEOREM FOR REAL CONVEX FUNCTIONS**

ROY MARTIN RAKESTRAW

## A REPRESENTATION THEOREM FOR REAL CONVEX FUNCTIONS

ROY M. RAKESTRAW

**The Krein-Milman theorem is used to prove the following result. A nonnegative function  $f$  on  $[0, 1]$  is convex if, and only if, there exist nonnegative Borel measures  $\mu_1$  and  $\mu_2$  on  $[0, 1]$  such that**

$$f(x) = \int_0^x (1 - \xi)^{-1}(x - \xi)d\mu_1(\xi) + \int_x^1 [1 - (x/\xi)]d\mu_2(\xi),$$

**for every  $x \in [0, 1]$ . An example is given for which the representation is not unique.**

**1. Extremal elements.** Let  $C$  be the set of all nonnegative convex real-valued functions on  $[0, 1]$ . Since the sum of two nonnegative convex functions is in  $C$  and since a nonnegative real multiple of a convex function is a convex function, the set  $C$  is a convex cone. It is the purpose of this paper to determine the extremal elements of this cone and to show that for the convex functions an integral representation in terms of extremal elements is possible (see [1] for terminology). We prove the following theorem which characterizes the extremal elements of  $C$ .

**THEOREM 1.** *The set of extremal elements of  $C$  consists of the following functions, where  $m > 0$ :*

$$\begin{aligned} e_+(m, \xi; x) &= 0, \quad x \in [0, \xi] \text{ and } m(x - \xi) \text{ for } x \in [\xi, 1], \text{ where } 0 \leq \xi < 1; \\ e_+(m, 1; x) &= 0, \quad x \in [0, 1) \text{ and } m \text{ for } x = 1; \\ e_-(m, 0; x) &= 0, \quad x \in (0, 1] \text{ and } m \text{ for } x = 0; \\ e_-(m, \xi; x) &= m(\xi - x), \quad x \in [0, \xi] \text{ and } 0 \text{ for } x \in [\xi, 1] \text{ where } 0 < \xi \leq 1. \end{aligned}$$

*Proof.* Let  $f$  be a function in  $C$  which assumes exactly one positive value in  $[0, 1]$ . If  $f = c > 0$ , then  $f(x) = cx + c(1 - x)$  for  $x \in [0, 1]$  and hence,  $f$  is not an extremal element of  $C$ . If  $f$  is not constant, then  $f$  must be positive at one end point of  $[0, 1]$ , since  $f$  is continuous on  $(0, 1)$  [5, p. 109]. It is evident that the two functions which are positive only at 0 and 1, respectively, are extremal elements of  $C$ . If  $f \neq 0$  on  $[0, 1)$  and

$$f(1) \neq f_-(1) = \lim_{x \rightarrow 1^-} f(x),$$

then  $f = f_1 + f_2$ , where  $f_1 = 0$  on  $[0, 1)$ ,  $f_1(1) = f(1) - f_-(1) > 0$  and  $f_2 = f - f_1$ , and hence  $f$  is not an extremal element. Similarly, if  $f$  is an extremal element and  $f \neq 0$  on  $(0, 1]$ , then  $f$  must be continuous at 0. Thus, all of the remaining extremal elements of  $C$  must be continuous on  $[0, 1]$ .

Let  $f \in C$  such that  $f$  is not constant and is continuous on  $[0, 1]$ , and let  $f(x_0) = \inf\{f(x) : 0 \leq x \leq 1\}$ . If  $f(x_0) > 0$ , then  $f = f(x_0) + [f - f(x_0)]$  and so  $f$  is not an extremal element of  $C$ . If  $f$  is not monotonic, then a nonproportional decomposition of  $f$  can be given by  $f = f_1 + f_2$ , where  $f_1(x) = f(x)$  for  $x \in [0, x_0]$ ,  $f_1(x) = f(x_0)$  for  $x \in [x_0, 1]$  and  $f_2 = f - f_1$ . Hence, all continuous extremal elements of  $C$  must assume the value 0 and must be monotonic.

Let  $f \in C$  such that  $f$  is continuous and monotonic and  $\inf\{f(x) : x \in [0, 1]\} = 0$ . Suppose, without loss of generality, that  $f'$  assumes at least two positive values in  $(0, 1)$ ; we know that  $f'$  exists and is left-continuous and nondecreasing on  $(0, 1)$ , since  $f$  is convex [5, p. 109]. Let  $0 < x_1 < x_2 < 1$  be such that  $0 < f'_-(x_1) < f'_-(x_2)$  and define  $f_1(x) = f(x)$ ,  $x \in [0, x_1]$ , and  $f_1(x) = f(x_1) + f'_-(x_1)(x - x_1)$ , for  $x \in [x_1, 1]$ . Then  $f_1 \in C$  and  $f_2 = f - f_1 \in C$ ; that is,  $f = f_1 + f_2$  and hence,  $f$  is not an extremal element of  $C$ . Thus, if  $f$  is a continuous extremal element of  $C$ , then  $f'$  (and  $f'_+$ ) assumes exactly one nonzero value in  $(0, 1)$ .

For  $m > 0$ , define the functions  $e_+(m, \xi; \cdot)$  and  $e_-(m, \xi; \cdot)$  as in the statement of the theorem. It is easily seen that  $e_+(m, \xi; \cdot)$ , where  $0 \leq \xi < 1$ , and  $e_-(m, \xi; \cdot)$ , where  $0 < \xi \leq 1$ , are extremal elements of  $C$  and moreover, they are the only continuous extremal elements of  $C$ . Thus, for  $m > 0$  and  $0 \leq \xi \leq 1$ ,

$$\text{extr } C = \{e_+(m, \xi; \cdot)\} \cup \{e_-(m, \xi; \cdot)\},$$

where  $\text{extr } C$  denotes the set of extremal elements of  $C$ . This completes the proof of Theorem 1.

**2. Integral representations.** The set of functions  $C - C = C + (-C)$  is the smallest linear space containing the convex cone  $C$ . With the topology of simple convergence,  $C - C$  is a Hausdorff locally convex space such that for each  $x \in [0, 1]$ , the linear functional  $L_x$  defined by  $L_x(f) = f(x)$  is continuous.

**THEOREM 2.** *In  $C - C$ , the cone  $C$  has a compact base  $C_0$ . Moreover, the extreme points of  $C_0$  form a compact set.*

*Proof.* The linear functional  $F$  defined on  $C - C$  by  $F(f) = f(x + 2h) - 2f(x + h) + f(x)$ , for  $[x, x + 2h] \subset [0, 1]$ , is continuous in the

topology of simple convergence. By definition,  $C$  is the intersection of a collection of closed half-spaces corresponding to such functional. Hence,  $C$  is closed in  $C - C$ .

If  $f \in C$ , then  $f$  is bounded by  $f(0) + f(1)$ , and it follows from the Tychonoff theorem that the normalized convex functions, namely

$$C_0 = \{f \in C: f(0) + f(1) = 1\},$$

form a compact base for  $C$ . If we let

$$E_+ = \{e_+((1 - \xi)^{-1}, \xi; \cdot): 0 \leq \xi < 1\} \cup \{e_+(1, 1; \cdot)\} \text{ and}$$

$$E_- = \{e_-(\xi^{-1}, \xi; \cdot): 0 < \xi \leq 1\} \cup \{e_-(1, 0; \cdot)\},$$

then  $\text{ext } C_0 = E_+ \cup E_-$ , where  $\text{ext } C_0$  denotes the set of extreme points of  $C_0$ .

Let  $U_0 = \{f \in C - C: |f(0)| < 1/2\}$  and  $U_1 = \{f \in C - C: |f(1)| < 1/2\}$ . Then  $U_0$  and  $U_1$  are open sets,  $E_+ \subset U_0$ ,  $E_- \subset U_1$  and  $U_0 \cap E_- = U_1 \cap E_+ = \emptyset$ . Hence,  $E_+ \cup E_-$  is a separation of  $\text{ext } C_0$ . If we define  $\alpha_+: [0, 1] \rightarrow E_+$  by  $\alpha_+(\xi) = e_+((1 - \xi)^{-1}, \xi; \cdot)$ , for  $0 \leq \xi < 1$ , and  $\alpha_+(1) = e_+(1, 1; \cdot)$ , then  $\alpha_+$  is a continuous bijection. Since  $[0, 1]$  is a compact space and  $E_+$  is a Hausdorff space, then  $\alpha_+$  is a homeomorphism. Likewise,  $\alpha_-: [0, 1] \rightarrow E_-$ , defined by  $\alpha_-(\xi) = e_-(\xi^{-1}, \xi; \cdot)$ , for  $0 < \xi \leq 1$ , and  $\alpha_-(0) = e_-(1, 0; \cdot)$ , is a homeomorphism. Hence,  $\text{ext } C_0 = E_+ \cup E_-$  is a compact set, and the proof is complete.

The mappings  $\alpha_+$  and  $\alpha_-$  introduced in the proof of Theorem 2 will now be used to prove the representation theorem.

**THEOREM 3.** *For each  $f \in C$ , there exist nonnegative Borel measures  $\mu_1$  and  $\mu_2$  on  $[0, 1]$  such that*

$$f(x) = \int_0^x (1 - \xi)^{-1}(x - \xi)d\mu_1(\xi) + \int_x^1 [1 - (x/\xi)]d\mu_2(\xi)$$

for every  $x \in [0, 1]$ .

*Proof.* Let  $f \in C_0$ . (Since each nonzero function in  $C$  is a positive scalar multiple of some function in  $C_0$ , we need only consider those functions in  $C_0$ .) Then, since  $C_0$  and  $\text{ext } C_0$  are compact subsets of the locally convex space  $C - C$ , by the Krein-Milman representation theorem there exists a probability measure  $\mu$  on  $\text{ext } C_0$  such that

$$L(f) = \int_{\text{ext } C_0} L d\mu,$$

for every continuous linear functional  $L$  on  $C - C$  [3, p. 6]. Thus,

$$f(x) = L_x(f) = \int_{\text{ext}C_0} L_x d\mu = \int_{E_+} L_x d\mu + \int_{E_-} L_x d\mu,$$

for all  $x \in [0, 1]$ . Define  $\mu_1$  on each Borel subset  $B$  of  $[0, 1]$  by  $\mu_1(B) = \mu[\alpha_+(B)]$ ; that is,  $\mu_1 = \mu\alpha_+$ . Since  $L_x[\alpha_+(\xi)] = 0$ , for  $x \in [0, \xi]$  and  $(1 - \xi)^{-1}(x - \xi)$ , for  $x \in [\xi, 1]$ , then

$$\begin{aligned} \int_{E_+} L_x d\mu &= \int_{(\alpha_+)^{-1}(E_+)} L_x \alpha_+ d(\mu\alpha_+) = \int_0^1 L_x[\alpha_+(\xi)] d\mu_1(\xi) \\ &= \int_0^x (1 - \xi)^{-1}(x - \xi) d\mu_1(\xi) \end{aligned}$$

[2, p. 163]. Similarly,

$$\int_{E_-} L_x d\mu = \int_x^1 \xi^{-1}(\xi - x) d\mu_2(\xi),$$

where  $\mu_2 = \mu\alpha_-$ , and the theorem is proved.

**3. Remarks.** If  $\mu_1$  and  $\mu_2$  are nonnegative Borel measures on  $[0, 1]$  and

$$f(x) = \int_0^x (1 - \xi)^{-1}(x - \xi) d\mu_1(\xi) + \int_x^1 \xi^{-1}(\xi - x) d\mu_2(\xi),$$

for every  $x \in [0, 1]$ , then it is easily seen that  $f$  is in  $C$ . The measures  $\mu_1$  and  $\mu_2$  which appear in the statement of Theorem 3 are not necessarily unique because the probability measure  $\mu$  in the proof of Theorem 3 will not always be unique. This follows from the fact that

$$(1/4)(f_1 + f_2 + f_3 + f_4) = (1/8)(f_3 + f_4 + 3f_5 + 3f_6),$$

where  $f_1 = e_+(1, 0; \cdot)$ ,  $f_2 = e_-(1, 1; \cdot)$ ,  $f_3 = e_+(4, (3/4); \cdot)$ ,  $f_4 = e_-(4, (1/4); \cdot)$ ,  $f_5 = e_+((4/3), (1/4); \cdot)$  and  $f_6 = e_-((4/3), (3/4); \cdot)$ . We also note that  $C - C$  properly contains the functions of bounded convexity on  $[0, 1]$  [4].

REFERENCES

1. N. Bourbaki, *Espaces vectoriels topologiques*, Act. Sci. Ind. No. 1189, Paris, 1953.
2. P. R. Halmos, *Measure Theory*, Van Nostrand, New York, N.Y., 1959.
3. R. R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand, New York, N.Y., 1966.
4. A. W. Roberts and D. E. Varberg, *Functions of bounded convexity*, Bull. Amer. Math. Soc., 75 (1969), 568-572.
5. H. L. Royden, *Real Analysis* (2nd ed.), Macmillan, New York, N.Y., 1968.

Received January 31, 1974.

UNIVERSITY OF MISSOURI

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION  
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics  
All Rights Reserved

# Pacific Journal of Mathematics

Vol. 60, No. 2

October, 1975

Waleed A. Al-Salam and A. Verma, <i>A fractional Leibniz <math>q</math>-formula</i> . . . . .	1
Robert A. Bekes, <i>Algebraically irreducible representations of <math>L_1(G)</math></i> . . . . .	11
Thomas Theodore Bowman, <i>Construction functors for topological semigroups</i> . . . . .	27
Stephen LaVern Campbell, <i>Operator-valued inner functions analytic on the closed disc. II</i> . . . . .	37
Leonard Eliezer Dor and Edward Wilfred Odell, Jr., <i>Monotone bases in <math>L_p</math></i> . . . . .	51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, <i>On the existence of global classical solution of initial-boundary value problem for <math>cmu - u^3 = f</math></i> . . . . .	63
Y. Gordon, <i>Unconditional Schauder decompositions of normed ideals of operators between some <math>l_p</math>-spaces</i> . . . . .	71
Gary Grefsrud, <i>Oscillatory properties of solutions of certain <math>n</math>th order functional differential equations</i> . . . . .	83
Irvin Roy Hentzel, <i>Generalized right alternative rings</i> . . . . .	95
Zensiro Goseki and Thomas Benny Rushing, <i>Embeddings of shape classes of compacta in the trivial range</i> . . . . .	103
Emil Grosswald, <i>Brownian motion and sets of multiplicity</i> . . . . .	111
Donald LaTorre, <i>A construction of the idempotent-separating congruences on a bisimple orthodox semigroup</i> . . . . .	115
Pjek-Hwee Lee, <i>On subrings of rings with involution</i> . . . . .	131
Marvin David Marcus and H. Minc, <i>On two theorems of Frobenius</i> . . . . .	149
Michael Douglas Miller, <i>On the lattice of normal subgroups of a direct product</i> . . . . .	153
Grattan Patrick Murphy, <i>A metric basis characterization of Euclidean space</i> . . . . .	159
Roy Martin Rakestraw, <i>A representation theorem for real convex functions</i> . . . . .	165
Louis Jackson Ratliff, Jr., <i>On Rees localities and <math>H_i</math>-local rings</i> . . . . .	169
Simeon Reich, <i>Fixed point iterations of nonexpansive mappings</i> . . . . .	195
Domenico Rosa, <i><math>B</math>-complete and <math>B_r</math>-complete topological algebras</i> . . . . .	199
Walter Roth, <i>Uniform approximation by elements of a cone of real-valued functions</i> . . . . .	209
Helmut R. Salzmann, <i>Homogene kompakte projektive Ebenen</i> . . . . .	217
Jerrold Norman Siegel, <i>On a space between <math>BH</math> and <math>B_\infty</math></i> . . . . .	235
Robert C. Sine, <i>On local uniform mean convergence for Markov operators</i> . . . . .	247
James D. Stafney, <i>Set approximation by lemniscates and the spectrum of an operator on an interpolation space</i> . . . . .	253
Árpád Szász, <i>Convolution multipliers and distributions</i> . . . . .	267
Kalathoor Varadarajan, <i>Span and stably trivial bundles</i> . . . . .	277
Robert Breckenridge Warfield, Jr., <i>Countably generated modules over commutative Artinian rings</i> . . . . .	289
John Yuan, <i>On the groups of units in semigroups of probability measures</i> . . . . .	303