Pacific Journal of Mathematics

FIXED POINT ITERATIONS OF NONEXPANSIVE MAPPINGS

SIMEON REICH

Vol. 60, No. 2 October 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 60, No. 2, 1975

FIXED POINT ITERATIONS OF NONEXPANSIVE MAPPINGS

SIMEON REICH

Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the fixed point property for nonexpansive mappings, and let $T: C \to C$ be nonexpansive. In this note it is shown (by a very simple argument) that if a sequence of iterates of T (generated with the aid of an infinite, lower triangular, regular row-stochastic matrix) is bounded, then T has a fixed point.

Dotson and Mann [3] proved this theorem under the additional assumption that E was uniformly convex. (Their complicated proof relied heavily on the uniform convexity of E.) We use our method also to establish a similar result (essentially due to Browder) for nonlinear nonexpansive semigroups.

Let C be a closed convex subset of a Banach space $(E, |\ \ |)$, and let $T: C \to C$ be nonexpansive (that is, $|Tx - Ty| \leqq |x - y|$ for all x and y in C). Let N denote the set of nonnegative integers, and suppose $A = \{a_{nk}: n, k \in N\}$ is an infinite matrix satisfying

$$a_{nk} \geqq 0 \quad \text{for all} \quad n, k \in N,$$

$$a_{nk} = 0 \quad \text{if} \quad k > n,$$

$$\sum_{k=0}^{n} a_{nk} = 1 \quad \text{for all} \quad n \in N,$$

$$\lim_{n \to \infty} a_{nk} = 0 \quad \text{for all} \quad k \in N.$$

If x_0 belongs to C, then a sequence $S = \{x_n: n \in N\} \subset C$ can be defined inductively by

$$x_n = a_{n0} x_0 + \sum_{k=1}^{n} a_{nk} T x_{k-1}, \qquad n \in N.$$

This iteration scheme is due to Mann [8].

It is not difficult to see that if T has a fixed point, then S is bounded. Dotson and Mann [3, Theorem 1] have proved that if E is uniformly convex, and if S is bounded for some x_0 in C, then T has a fixed point. Their proof is rather complicated and relies heavily on the uniform convexity of E. In this note we establish a far-reaching extension of the Dotson-Mann theorem in a very simple manner. We remark in passing that a special case of the Dotson-Mann result was independently established by Reinermann [**10**, p. 10]. He assumed that A is column-finite.

THEOREM 1. *Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the fixed point property for nonexpansive mappings, and that $T: C \to C$ is nonexpansive. If the sequence S defined above is bounded for some x_0 in C, then T has a fixed point.*

Proof. Pick a point y in C, and set $R = \limsup_{n\to\infty} |y - x_n|$. R is finite because S is bounded. Let $K = \{z \in C\colon \limsup_{n\to\infty} |z - x_n| \leqq R\}$. K is a non-empty bounded closed convex (hence weakly compact) subset of C. Now let z be in K. Then

$$\begin{aligned} |Tz - x_n| &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{n} a_{nk}\,|Tz - Tx_{k-1}| \\ &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{n} a_{nk}\,|z - x_{k-1}|. \end{aligned}$$

For each positive ϵ, there is $m(\epsilon) \in N$ such that $|z - x_k| < R + \epsilon$ for all $k > m$. Therefore we obtain for $n > m + 1$,

$$\begin{aligned} |Tz - x_n| &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{m+1} a_{nk}\,|z - x_{k-1}| + \sum_{k=m+2}^{n} a_{nk}\,(R + \epsilon) \\ &\leqq a_{n0}|Tz - x_0| + \sum_{k=1}^{m+1} a_{nk}\,|z - x_{k-1}| + R + \epsilon \\ &= h(n) + R + \epsilon \end{aligned}$$

where $\lim_{n\to\infty} h(n) = 0$. Thus Tz belongs to K, and the result follows.

REMARK. S need not converge, even if E is a Hilbert space [**5**].

In the setting of Theorem 1, let $r_m = \inf\{r\colon$ there exists $y \in C$ such that $|y - x_n| \leqq r$ for all $n \geqq m\}$, and $R = \lim_{n\to\infty} r_m$. Since C is convex and boundedly weakly compact, there is at least one point z in C such that $\limsup_{n\to\infty} |z - x_n| = R$. Such a point is called an asymptotic center of S with respect to C (cf. [**4**]). The proof of Theorem 1 shows that the set of asymptotic centers of S with respect to C is invariant under T (cf. [**9**, p. 253]). Consequently, it contains a fixed point of T. In particular, if the asymptotic center is unique (this indeed happens when E is uniformly convex, or more generally, uniformly convex in every direction [2]), then it is a fixed point of T. Note that a weakly compact convex subset of a Banach space which is uniformly convex in every direction has normal structure and therefore possesses the fixed point property for nonexpansive mappings [6].

The idea of the proof of Theorem 1 can be also applied to a result on nonlinear nonexpansive semigroups which is essentially due to Browder [1].

Recall that a nonexpansive semigroup on a subset D of a Banach space E is a function U: $[0, \infty) \times D \to D$ satisfying the following conditions:

$$U(t_1 + t_2, x) = U(t_1, U(t_2, x)) \quad \text{for} \quad t_1, t_2 \geqq 0$$

and $x \in D$,

$$|U(t, x) - U(t, y)| \leqq |x - y| \quad \text{for} \quad t \geqq 0$$

and $x, y \in D$,

$$U(0, x) = x \quad \text{for} \quad x \in D.$$

A semigroup U is called bounded if for each x in D there is $M(x)$ such that $|U(t, x)| \leqq M(x)$ for all $t \geqq 0$. It is said to have a fixed point x_0 if $U(t, x_0) = x_0$ for all $t \geqq 0$. If U has a fixed point, then it is clearly bounded. In order to prove the converse statement, we shall assume that D has the common fixed point property for nonexpansive mappings. This means that every commuting family of nonexpansive self-mappings of D has a common fixed point.

THEOREM 2. *Let C be a boundedly weakly compact convex subset of a Banach space E. Suppose that each weakly compact convex subset of C possesses the common fixed point property for nonexpansive mappings, and that U: $[0, \infty) \times C \to C$ is a nonexpansive semigroup. If U is bounded, then it has a fixed point.*

Proof. Fix a point x_0 in C, and let y be another point in C. $R = \limsup_{t \to \infty} |y - U(t, x_0)|$ is finite because the orbit $\{U(t, x_0): t \geqq 0\}$ is bounded. Let $K = \{z \in C: \limsup_{t \to \infty} |z - U(t, x_0)| \leqq R\}$. K is a nonempty bounded closed convex (hence weakly compact) subset of C. If $z \in K$, $t_0 \geqq 0$, $\epsilon > 0$, and t is large enough, then

$$\begin{aligned} |U(t_0, z) - U(t, x_0)| &= |U(t_0, z) - U(t_0, U(t - t_0, x_0))| \\ &\leqq |z - U(t - t_0, x_0)| < R + \epsilon. \end{aligned}$$

Consequently, $U(t_0, z)$ also belongs to K. Thus K is invariant under the commuting family of nonexpansive mappings $\{U(t, \cdot): t \geqq 0\}$. Hence the result.

In the setting of Theorem 2 we can also define an asymptotic center, this time for $\{U(t, x_0): t \geqq 0\}$. If E is uniformly convex in every direction, this asymptotic center is unique. Moreover, a weakly compact convex subset of E has normal structure and therefore possesses the common fixed point property for nonexpansive mappings **[7]**. The proof of Theorem 2 shows that in this case the asymptotic center of $\{U(t, x_0): t \geqq 0\}$ is a fixed point of U.

REMARK. A version of Theorem 2 is true for arbitrary commutative semigroups of nonexpansive mappings.

REFERENCES

1. F. E. Browder, *Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces*, Bull. Amer. Math. Soc., **73** (1967), 867–874.

2. M. M. Day, R. C. James and S. Swaminathan, *Normed linear spaces that are uniformly convex in every direction*, Canad. J. Math., **23** (1971), 1051–1059.

3. W. G. Dotson, Jr. and W. R. Mann, *A generalized corollary of the Browder-Kirk fixed point theorem*, Pacific J. Math., **26** (1968), 455–459.

4. M. Edelstein, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math. Soc., **78** (1972), 206–208.

5. A. Genel and J. Lindenstrauss, *An example concerning fixed points*, to appear.

6. W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

7. T.-C. Lim, *A fixed point theorem for families of nonexpansive mappings*, Pacific J. Math., **53** (1974), 487–493.

8. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

9. S. Reich, *Remarks on fixed points II*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **53** (1972), 250–254.

10. J. Reinermann, *Approximation von Fixpunkten*, Studia Math., **39** (1971), 1–15.

Received April 14, 1975.

TEL AVIV UNIVERSITY
Current address: The University of Chicago

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $ 72.00 a year (6 Vols., 12 issues). Special rate: $ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Pacific Journal of Mathematics
Vol. 60, No. 2 October, 1975

Waleed A. Al-Salam and A. Verma, *A fractional Leibniz q-formula* 1
Robert A. Bekes, *Algebraically irreducible representations of* $L_1(G)$ 11
Thomas Theodore Bowman, *Construction functors for topological semigroups* 27
Stephen LaVern Campbell, *Operator-valued inner functions analytic on the closed disc. II* 37
Leonard Eliezer Dor and Edward Wilfred Odell, Jr., *Monotone bases in* L_p 51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, *On the existence of global classical solution of initial-boundary value problem for* $cmu - u^3 = f$ 63
Y. Gordon, *Unconditional Schauder decompositions of normed ideals of operators between some* l_p*-spaces* 71
Gary Grefsrud, *Oscillatory properties of solutions of certain nth order functional differential equations* 83
Irvin Roy Hentzel, *Generalized right alternative rings* 95
Zensiro Goseki and Thomas Benny Rushing, *Embeddings of shape classes of compacta in the trivial range* 103
Emil Grosswald, *Brownian motion and sets of multiplicity* 111
Donald LaTorre, *A construction of the idempotent-separating congruences on a bisimple orthodox semigroup* 115
Pjek-Hwee Lee, *On subrings of rings with involution* 131
Marvin David Marcus and H. Minc, *On two theorems of Frobenius* 149
Michael Douglas Miller, *On the lattice of normal subgroups of a direct product* 153
Grattan Patrick Murphy, *A metric basis characterization of Euclidean space* 159
Roy Martin Rakestraw, *A representation theorem for real convex functions* 165
Louis Jackson Ratliff, Jr., *On Rees localities and* H_i*-local rings* 169
Simeon Reich, *Fixed point iterations of nonexpansive mappings* 195
Domenico Rosa, B*-complete and* B_r*-complete topological algebras* 199
Walter Roth, *Uniform approximation by elements of a cone of real-valued functions* 209
Helmut R. Salzmann, *Homogene kompakte projektive Ebenen* 217
Jerrold Norman Siegel, *On a space between* BH *and* B_∞ 235
Robert C. Sine, *On local uniform mean convergence for Markov operators* 247
James D. Stafney, *Set approximation by lemniscates and the spectrum of an operator on an interpolation space* 253
Árpád Száz, *Convolution multipliers and distributions* 267
Kalathoor Varadarajan, *Span and stably trivial bundles* 277
Robert Breckenridge Warfield, Jr., *Countably generated modules over commutative Artinian rings* 289
John Yuan, *On the groups of units in semigroups of probability measures* 303