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## **$B$ -COMPLETE AND $B_r$ -COMPLETE TOPOLOGICAL ALGEBRAS**

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**In this note, a topological algebra  $A$  is an algebra over the field of complex numbers together with a Hausdorff locally convex topology which makes multiplication jointly continuous.  $A$  is called  $B$ -complete ( $B_*$ -complete) if every continuous (continuous, one-to-one) and almost open algebra homomorphism from  $A$  onto any topological algebra is open. For a completely regular space  $X$ ,  $C(X)$  is the algebra of all continuous complex-valued functions with the usual pointwise operations and the compact-open topology. The main theorem states that  $C(X)$  is a  $B$ -complete algebra iff  $X$  is a  $k$ -space.**

Recall that a linear map between two locally convex spaces is called *almost open* if the closure of the image of every neighborhood of zero is a neighborhood of zero. Pták [7] called a locally convex space  $E$   *$B$ -complete* if every continuous and almost open linear map from  $E$  onto any locally convex space is open. A general treatment of  $B$ -complete spaces may be found in [2]. Our study of  $B$ -complete algebras is motivated by a problem raised in [7]; namely, what conditions on  $X$  are necessary and sufficient for  $C(X)$  to be a  $B$ -complete space. Necessary conditions are that  $X$  be a normal  $k$ -space, a result which is implicit in [7]. W. H. Summers [11] has obtained some partial results for weighted spaces of bounded continuous functions.

We shall investigate this problem within the framework of topological algebras rather than locally convex spaces. The definition of a  $B$ -complete algebra is motivated by T. Husain's extension of the notion of  $B$ -completeness for suitable classes of locally convex spaces [2] and for topological groups [3]. In Section 2 we obtain a characterization of a  $B$ -complete algebra  $A$  in terms of its topological dual  $A'$ . In Section 3 we prove that  $C(X)$  is a  $B$ -complete algebra iff  $X$  is a  $k$ -space. In Section 4 several counterexamples are provided, and in Section 5 a closed graph theorem is stated.

A convex subset  $U$  of an algebra  $A$  is called  *$m$ -convex* if  $U \cdot U \subset U$ . A topological algebra  $A$  is called *locally  $m$ -convex* (LMC) if there exists a basis for the neighborhoods of zero consisting of closed  $m$ -convex and circled sets. Equivalently,  $A$  is LMC iff its topology is generated by a set  $\{p_i; i \in I\}$  of sub-multiplicative seminorms ( $p_i(xy) \leq p_i(x)p_i(y)$  for  $i \in I$ ,  $x, y \in A$ ). Clearly  $C(X)$  is LMC; simply take the seminorms  $\{p_K; K \text{ is a compact subset of } X\}$  defined by

$p_K(f) = \sup\{|f(x)| : x \in K\}$ , for  $f \in C(X)$ . For  $b > 0$ , we will use the notation  $N(K, b) = N(p_K, b) = \{f \in C(X) : |p_K(f)| \leq b\}$  to denote basic closed neighborhoods of zero in  $C(X)$ .

**2. A characterization of  $B$ -complete algebras.** It follows directly from the definitions that every  $B$ -complete algebra is  $B_r$ -complete, and that the quotient of a  $B$ -complete algebra modulo a closed ideal is  $B$ -complete. Also, every topological algebra which is a  $B$ -complete space is a  $B$ -complete algebra, in particular, Banach algebras and complete metrizable topological algebras are  $B$ -complete algebras. For later reference we now state the following two lemmas whose proofs are immediate. At times subscripts will be used to denote topologies.

LEMMA 2.1. *Let  $A_u$  be a topological algebra. Then the following statements are equivalent.*

- (a)  $A_u$  is a  $B_r$ -complete algebra.
- (b) Whenever the identity map  $A_u \rightarrow A_v$ ,  $A_v$  a topological algebra, is continuous and almost open, it follows that  $u = v$ .

LEMMA 2.2. *A topological algebra  $A$  is a  $B$ -complete algebra iff  $A/I$  is a  $B_r$ -complete algebra for every closed ideal  $I$  of  $A$ .*

Let  $A$  be a topological algebra. A linear subspace  $S$  of  $A'$  is called *almost closed* if  $S \cap U^0$  is  $w^*$ -closed for every neighborhood  $U$  of zero in  $A$ , where  $U^0$  is the polar of  $U$ .  $S$  is said to be an  *$m$ -subspace* if whenever  $h \in S$  then  $h \circ l_a$  and  $h \circ r_a$  are in  $S$ , where  $l_a, r_a : A \rightarrow A$  are left and right multiplication by  $a$ , respectively. It can be readily checked that the polar of an ideal is an  $m$ -subspace and the polar of an  $m$ -subspace is an ideal.

The following theorem extends theorem IV, 8.1 of [9] to algebras. The author would like to thank Seth Warner for suggesting it.

THEOREM 2.3. *Let  $A$  be a topological algebra.*

- (a)  *$A$  is a  $B_r$ -complete algebra iff every  $w^*$ -dense and almost closed  $m$ -subspace of  $A'$  coincides with  $A'$ .*
- (b)  *$A$  is a  $B$ -complete algebra iff every almost closed  $m$ -subspace of  $A'$  is  $w^*$ -closed.*

*Proof.* (a) Suppose  $A$  is a  $B_r$ -complete algebra. Let  $S$  be a dense and almost closed  $m$ -subspace of  $A'$ , and let  $u$  be the Hausdorff locally convex topology on  $A$  defined by the family  $\{(U^0 \cap S)^0\}$  where  $U$  varies over all neighborhoods of zero in  $A$ . As in [2, Theorem 5, p. 54], the identity map  $id : A \rightarrow A_u$  is continuous and almost open, and  $(A_u)' = S$ . It remains to show that  $A_u$  is a topological algebra. Since  $S$  is an

$m$ -subspace it follows that  $l_a, r_a: A_v \rightarrow A_v$  are continuous for every  $a \in A$ , where  $v$  is the weak topology determined by  $S$ . Using the additional facts that  $id: A \rightarrow A_u$  is almost open and that the convex circled sets which are  $u$ -closed are also  $v$ -closed, we can easily conclude that  $l_a, r_a: A_u \rightarrow A_u$  are continuous for every  $a \in A$ . To show that multiplication is jointly continuous, let  $V$  be a closed neighborhood of zero in  $A_u$ . Then there exists a neighborhood  $U$  of zero in  $A$  such that  $U \cdot U \subset V$ . Since multiplication is separately continuous in  $A_u$ , it follows as in [4, Lemma 1.4(b)] that  $cl_u(U) \cdot cl_u(U) \subset V$ . Since the identity map is almost open, the factors on the left are  $u$ -neighborhoods of zero, hence  $A_u$  is a topological algebra. By our hypothesis  $id: A \rightarrow A_u$  is open, consequently  $S = (A_u)' = A'$ .

In view of Lemma 2.1, for the converse we may suppose that  $id: A \rightarrow A_u$ ,  $A_u$  a topological algebra, is continuous and almost open. By [2, Proposition 14, p. 54],  $(A_u)'$  is an almost closed subspace of  $A'$ , and clearly it is also a  $w^*$ -dense subspace. Since multiplication is jointly continuous in  $A_u$ , it follows that  $(A_u)'$  is an  $m$ -subspace of  $A'$ . By hypothesis  $(A_u)' = A'$  which implies that  $A$  and  $A_u$  have the same closed, convex and circled sets. This implies that  $id: A \rightarrow A_u$  is actually open, hence  $A$  is a  $B_r$ -complete algebra by Lemma 2.1.

(b) Suppose  $A$  is a  $B$ -complete algebra. Let  $S$  be an almost closed  $m$ -subspace of  $A'$ . Then  $A/S^0$  is a  $B$ -complete algebra, and  $S$  is a  $w^*$ -dense and almost closed  $m$ -subspace of  $(A/S^0)'$ . By (a)  $S = (A/S^0)' = S^{00}$  is  $w^*$ -closed in  $A'$ .

In view of Lemma 2.2, to prove the converse it suffices to show that  $A/I$  is  $B_r$ -complete for every closed ideal  $I$  of  $A$ . So let  $S$  be a  $w^*$ -dense and almost closed  $m$ -subspace of  $(A/I)'$ , and let  $q: A \rightarrow A/I$  be the quotient map. As in [2, Theorem 1, p. 46]  $q'(S)$  is an almost closed subspace of  $A'$ , where  $q'$  is the transpose of  $q$ . Since  $h \circ q \circ r_a = (h \circ r_{q(a)}) \circ q$  and  $h \circ q \circ l_a = (h \circ l_{q(a)}) \circ q$  for every  $a \in A$ , it follows that  $q'(S)$  is also an  $m$ -subspace of  $A'$ . By hypothesis  $q'(S)$  is closed. Thus  $S \cong q'(S) = I^0 \cong (A/I)'$  and consequently  $A/I$  is a  $B_r$ -complete algebra. By Lemma 2.2,  $A$  is a  $B$ -complete algebra.

Sulley [10] proved a very useful criterion for determining when dense subgroups of  $B_r$ -complete and  $B$ -complete topological abelian groups inherit the respective properties. The same criterion can be adapted for topological algebras. The proof is essentially the same and is therefore omitted. It should be pointed out that the proof of Theorem 1 in [10] uses the existence of completions. The same proof holds for topological algebras since the completion (of the additive group) of a topological algebra is again a topological algebra.

**THEOREM 2.4.** *Let  $B$  be a dense subalgebra of a topological algebra  $A$ .*

(a)  $B$  is a  $B_r$ -complete algebra iff  $A$  is a  $B_r$ -complete algebra and  $B$  has nonzero intersection with every nonzero closed ideal of  $A$ .

(b)  $B$  is a  $B$ -complete algebra iff  $A$  is a  $B$ -complete algebra and  $B \cap I$  is dense in  $I$  for every closed ideal  $I$  of  $A$ .

**COROLLARY 2.5.** *Let  $C^*(X)$  be the subalgebra of  $C(X)$  consisting of all bounded functions.*

(a)  $C^*(X)$  is a  $B_r$ -complete algebra iff  $C(X)$  is such.

(b)  $C^*(X)$  is a  $B$ -complete algebra iff  $C(X)$  is such.

*Proof.* Since  $C^*(X)$  is a dense subalgebra of  $C(X)$ , (a) and (b) will follow if we show that  $I \cap C^*(X)$  is dense in  $I$  for every closed nonzero ideal  $I$  of  $C(X)$ . So let  $I$  be a closed nonzero ideal of  $C(X)$ . By [6, Theorem 2.1]  $I = I_F = \{f \in C(X) : f(F) = \{0\}\}$  for some proper closed  $F \subset X$ . Let  $f \in I_F$  and let  $N(K, f, b)$  be a basic neighborhood of  $f$ . Let  $r = \sup\{|f(x)| : x \in K\}$  and let  $G = \{x \in X : |f(x)| \geq r + 1\}$ . Then  $G$  is a closed subset of  $X$  and  $G \cap K = \emptyset$ . Since  $X$  is completely regular, there exists  $g \in C^*(X)$ ,  $0 \leq g \leq 1$ , such that  $g(K) = \{1\}$  and  $g(G) = \{0\}$ . Consequently,  $gf \in C^*(X) \cap N(K, f, b)$ ; hence,  $I \cap C^*(X)$  is dense in  $I$ . The corollary follows from the theorem.

**COROLLARY 2.6.** *The subalgebra  $P$  of  $C([0, 1])$  consisting of all polynomials is not a  $B_r$ -complete algebra.*

*Proof.* Since  $C([0, 1])$  is a Banach algebra it is a  $B_r$ -complete algebra. Let  $I$  be the closed nonzero ideal of all functions which are zero on  $[0, 1/2]$ . Since  $P$  is dense in  $C([0, 1])$  and since  $I \cap P = \{0\}$  it follows by Theorem 2.4(a) that  $P$  is not a  $B_r$ -complete algebra.

As a consequence of Corollary 2.5 it follows that a  $B$ -complete algebra need not be complete. Let  $R$  be the real line with the usual topology. Then  $C(R)$  is complete and metrizable, hence a  $B$ -complete algebra. Thus  $C^*(R)$  is a  $B$ -complete algebra which is not complete. Corollary 2.6 shows that a normed algebra need not be  $B_r$ -complete.

**3. Necessary and sufficient conditions for  $C(X)$  to be a  $B$ -complete algebra.** Recall that a subset  $S$  of a topological space  $X$  is  $k$ -closed if  $S$  has closed intersection with every compact subset of  $X$ , and  $X$  is called a  $k$ -space if every  $k$ -closed subset of  $X$  is closed. (For example, locally compact spaces and first countable spaces are  $k$ -spaces.)

**THEOREM 3.1.**  *$C(X)$  is a  $B_*$ -complete algebra iff every dense and  $k$ -closed subset of  $X$  coincides with  $X$ .*

*Proof.* That the above condition is necessary was essentially proved by Pták [7, Theorem 6.4]. For suppose that  $C(X)$  is a  $B_*$ -complete algebra, and let  $S$  be a dense and  $k$ -closed subset of  $X$ . Let  $u$  be the topology of uniform convergence on the compact subsets of  $S$ . Clearly  $C(X)_u$  is LMC, hence a topological algebra. By [7, Theorem 6.4] the identity map  $C(X) \rightarrow C(X)_u$  is continuous and almost open, hence open by our hypothesis. It follows directly that  $S = X$ .

In view of Lemma 2.1, to prove the converse we may suppose that the identity map  $id: C(X) \rightarrow C(X)_u$  is continuous and almost open, where  $C(X)_u$  is a topological algebra. Let  $S = \{x \in X: h_x \text{ is } u\text{-continuous}\}$ , where  $h_x$  is defined by  $h_x(f) = f(x)$  for  $f \in C(X)$ . We will show that  $S$  is a dense and  $k$ -closed subset of  $X$ .

The first step is to show that the topology  $u$  is generated by a subset of the seminorms  $\{p_K: K \text{ is a compact subset of } X\}$ . Since  $C(X)_u$  is a topological algebra, by [4, Lemma 1.4(b)] it follows that  $cl_u N(K, b)$  is  $m$ -convex for every compact  $K \subset X$  and every  $0 \leq b \leq 1$ . Since the identity map is almost open, these sets form a basis for the topology  $u$ ; hence  $C(X)_u$  is LMC and  $u$  is generated by a set of submultiplicative seminorms  $\{p_j\}$ ,  $j \in J$ . Let  $p \in \{p_j\}$ ,  $j \in J$ . Since  $p$  is submultiplicative,  $\ker(p)$  is a closed ideal of  $C(X)_u$ , hence also closed in  $C(X)$ . By [6, Theorem 2.1]  $\ker(p) = I_H$  for some closed  $H \subset X$ . Since  $p$  is continuous on  $C(X)$ ,  $N(K, b) \subset N(p, 1)$  for some compact subset  $K$  of  $X$  and some  $b > 0$ . (Note that  $N(K, b)$  need not be a  $u$ -neighborhood of zero.) This implies  $I_K = \ker(p_K) \subset \ker(p) = I_H$ , which in turn implies that  $H \subset K$ . Thus  $H$  is a compact subset of  $X$ . Let  $h: C(X) \rightarrow C(X)/I_H = C(H)$  and  $k: C(X)_u \rightarrow C(X)_u/I_H = C(H)_u$  be the respective quotient homomorphisms, and let  $id^*$  be the unique one-to-one and onto homomorphism such that  $id^* \circ h = k \circ id$ . Since  $id$  is continuous and almost open, so is  $id^*$ . Since  $I_H$  is  $u$ -closed,  $C(H)_u$  is a (Hausdorff) topological algebra, hence  $id^*$  is open since  $C(H)$  is a Banach algebra. It follows that the seminorm  $p_H$  is continuous on  $C(X)_u$  and that  $N(p_H, b) \subset N(p, 1)$  for some  $b > 0$ . Since  $p$  was chosen arbitrarily we can conclude that the topology  $u$  is generated by a subset of  $\{p_K: K \text{ is a compact subset of } X\}$ .

A simple argument shows that:

$$\begin{aligned} S &= \cup \{H: H \subset X \text{ is compact and } p_H \text{ is continuous on } C(X)_u\}. \\ (1) \quad &= \cup \{H: H \subset X \text{ is compact and } N(H, 1) \text{ is a nbd. of zero in } C(X)_u\}. \end{aligned}$$

Since  $X$  is completely regular and since  $u$  is a Hausdorff topology

generated by seminorms of the form  $p_H$ , it follows directly that  $S$  is a dense subset of  $X$ . To show that  $X$  is  $k$ -closed, let  $K$  be a compact subset of  $X$ . Since the identity map is almost open,  $\text{cl}_u N(K, 1)$  is a neighborhood of zero in  $C(X)_u$ . Since  $N(K \cap S, 1)$  is closed in  $C(X)_u$ , it follows that  $-\text{cl}_u N(K, 1) \subset N(K \cap S, 1) = N(\text{cl}(K \cap S), 1)$  and consequently the latter is also a neighborhood of zero in  $C(X)_u$ . By (1) it follows that  $\text{cl}(K \cap S) \subset S$ . Thus  $\text{cl}(K \cap S) \subset K \cap S$ , which implies that  $S$  is  $k$ -closed. By our hypothesis  $S = X$ , hence  $h_x$  is  $u$ -continuous for each  $x \in X$ . Consequently,  $N(K, 1)$  is  $u$ -closed for each compact  $K \subset X$ , hence a  $u$ -neighborhood of zero since  $\text{id}: C(X) \rightarrow C(X)_u$  is almost open. Thus  $C(X) = C(X)_u$  and in view of Lemma 2.1 it follows that  $C(X)$  is a  $B_r$ -complete algebra.

From the above theorem it follows that  $C(X)$  is a  $B_r$ -complete algebra whenever  $X$  is a  $k$ -space. Example 4.2 shows that the converse is false.

For a closed subset  $F$  of  $X$ ,  $C_F(X)$  is the topological subalgebra of  $C(F)$  consisting of all functions which are restrictions of members of  $C(X)$ . Clearly,  $C_F(X)$  is a dense subalgebra of  $C(F)$  and the restriction map  $r: C(X) \rightarrow C_F(X)$  is continuous. Pták [7, Theorem 6.7] showed that  $r$  is open, hence  $C_F(X) \cong C(X)/I_F$ , the latter having the quotient topology from  $C(X)$ .

**THEOREM 3.2.**  *$C(X)$  is a  $B$ -complete algebra iff  $X$  is a  $k$ -space.*

*Proof.* Suppose  $C(X)$  is a  $B$ -complete algebra. Let  $S$  be a  $k$ -closed subset of  $X$  and let  $F = \text{cl}(S)$ . Then  $S$  is a dense and  $k$ -closed subset of  $F$ . Since  $C_F(X) \cong C(X)/I_F$ , it follows that  $C_F(X)$  is also a  $B$ -complete algebra. Since  $C_F(X)$  is dense in  $C(F)$ , by Theorem 2.4(b) it follows that  $C(F)$  is also a  $B$ -complete algebra. By Theorem 3.1,  $S = F$  implying that  $X$  is a  $k$ -space.

Conversely, suppose that  $X$  is a  $k$ -space. In view of Lemma 2.2, to show that  $C(X)$  is a  $B$ -complete algebra, it suffices to show that the quotient of  $C(X)$  modulo any closed ideal is a  $B_r$ -complete algebra. So let  $I$  be a closed ideal of  $C(X)$ . By [6, Theorem 2.1]  $I = I_F$  for some closed  $F \subset X$ . Since  $k$ -spaces are closed hereditary,  $F$  is also a  $k$ -space, hence by Theorem 3.1  $C(F)$  is a  $B_r$ -complete algebra. Since  $C_F(X)$  is dense in  $C(X)$  and has nonzero intersection with every nonzero closed ideal of  $C(F)$ , it follows from Theorem 2.4(a) that  $C_F(X)$  is also a  $B_r$ -complete algebra. Thus  $C_F(X) \cong C(X)/I_F = C(X)/I$  is a  $B_r$ -complete algebra. By Lemma 2.2 it follows that  $C(X)$  is a  $B$ -complete algebra.

**COROLLARY 3.3.** *If  $C(X)$  is a  $B$ -complete space, then  $X$  is a normal  $k$ -space.*

*Proof.* That  $X$  is a  $k$ -space follows from Theorem 3.2. To show that  $X$  is normal, let  $F$  be a closed subset of  $X$ . Since the restriction map  $r: C(X) \rightarrow C_F(X)$  is continuous and open it follows that  $C_F(X)$  is also a  $B$ -complete space. Since  $B$ -complete spaces are complete [9, 8.1 p. 162],  $C_F(X) = C(F)$ . By the Urysohn-Tychonoff theorem  $X$  is normal.

Theorem 3.2 gives rise to a large class of  $B$ -complete algebras which are not  $B$ -complete spaces. If  $X$  is a  $k$ -space which is not normal, in particular the Tychonoff plank, then  $C(X)$  is a  $B$ -complete algebra but not a  $B$ -complete space.

**4. Counterexamples.** In this section we will provide examples to show that a complete algebra need not be  $B_r$ -complete and that a  $B_r$ -complete algebra need not be  $B$ -complete.

Let  $R$  and  $R^2$  be the real line and the plane with their usual topologies. If  $X \subset R^2$  and  $x \in X$ , then a function  $f: X \rightarrow R$  is called *separately continuous* at  $x$  provided  $f|_{(L \cap X)}$  is continuous at  $x$  if  $L$  is either the horizontal or the vertical line through  $x$  in  $R^2$ . A completely regular space  $X$  is called a  $k_R$ -space if every  $f: X \rightarrow R$ , whose restriction to every compact  $K \subset X$  is continuous, is continuous on  $X$ . It is well known that  $C(X)$  is complete iff  $X$  is a  $k_R$ -space. Every completely regular  $k$ -space is a  $k_R$ -space. That the converse is false was first shown by an example of M. Katětov which appeared in [7, Theorem 6.17].

EXAMPLE 4.1. Let  $X_u$  be  $R^2$  with the usual topology and let  $F$  be the set of all  $f: X_u \rightarrow R$  which are separately continuous. Let  $v$  be the coarsest topology on  $X$  making every  $f \in F$  continuous. Clearly  $X_v$  is completely regular. It was shown in [7, Theorem 6.16] that  $X_v$  is a  $k_R$ -space which has a proper dense and  $k$ -closed subset. By Theorem 3.1  $C(X_v)$  is not a  $B_r$ -complete algebra; however, it is complete since  $X$  is a  $k_R$ -space.

EXAMPLE 4.2. Let  $N$  be the natural numbers and let  $p \in \beta N \setminus N$ , where  $\beta N$  is the Stone-Čech compactification of  $N$ , and let  $X = \beta N \amalg (N \cup \{p\})$  be the disjoint topological union. Let  $Y$  be the quotient of  $X$  which is obtained by identifying the element  $p \in \beta N$  with the element  $p \in N \cup \{p\}$ , and let  $f: X \rightarrow Y$  be the quotient map. It can be checked easily that  $Y$  is completely regular and that  $N \cup \{p\} \cong f(N \cup \{p\})$ . Now  $f(N)$  (the subset of  $f(N \cup \{p\})$ ) is not closed, however, from the counterexample following Proposition 2.2 of [1] it follows that  $f(N) \cap K$  is finite for every compact  $K \subset Y$ . Thus  $f(N)$  is  $k$ -closed but not closed, hence  $Y$  is not a  $k$ -space. Let  $D$  be a dense and  $k$ -closed subset of  $Y$ . Since  $f(N)$  is a discrete subset of both  $f(N \cup \{p\})$  and  $f(\beta N)$ , it follows that  $f(N) \subset f(N \cup \{p\}) \cap D$  and  $f(N) \subset f(\beta N) \cap$



$D$ . Since  $D$  is  $k$ -closed  $f(\beta N) = f(\beta N) \cap D$ , and thus  $D = Y$ . By Theorem 3.1  $C(Y)$  is a  $B_r$ -complete algebra, but  $C(Y)$  is not a  $B$ -complete algebra since  $Y$  is not a  $k$ -space. It should be pointed out that the completion of  $C(Y)$  is isomorphic to  $C(\beta N \amalg N)$  which is a  $B$ -complete algebra.

EXAMPLE 4.3. To show that a complete and  $B_r$ -complete algebra need not be  $B$ -complete, we will use the following topological space constructed in [5, Section 3]. Let  $X_u$  be as in Example 4.1, and let  $A$  be the  $x$ -axis. Let  $F$  be the set of all  $f: X_u \rightarrow \mathbb{R}$  which are continuous on  $X_u \setminus A$  and separately continuous at every  $x \in A$ , and let  $w$  be the coarsest topology making every  $f \in F$  continuous. As in [5, Section 3],  $X_w$  is a  $k_R$ -space which is not a  $k$ -space; also,  $u = w$  on  $X \setminus A$  and on every vertical and every horizontal line.

Let  $D$  be a dense and  $k$ -closed subset of  $X_w$ . Since  $X_w \setminus A$  is open,  $D \cap (X_w \setminus A)$  is dense and  $k$ -closed in  $X_w \setminus A$ ; hence,  $D \cap (X_w \setminus A) = X_w \setminus A$  since the latter is a  $k$ -space. For  $(a, 0) \in A$ , let  $B_a = \{(a, b) : -1 \leq b \leq 1\}$ . Since  $B_a$  is  $w$ -compact,  $D$  is  $k$ -closed, and  $X_w \setminus A \subset D$ , it follows that  $B_a \cap D = B_a$ . Thus  $A \subset D$  and so  $D = X_w$ . So  $C(X_w)$  is a complete and  $B_r$ -complete algebra, but not a  $B$ -complete algebra since  $X_w$  is not a  $k$ -space.

The topological spaces of Examples 4.1 and 4.2 can be used to show that a closed subalgebra of a  $B$ -complete algebra need not be  $B_r$ -complete, and that the quotient of a  $B_r$ -complete algebra need not be  $B_r$ -complete.

EXAMPLE 4.4. Let  $X_v$  be as in Example 4.1 and let  $Y = \amalg K$  be the disjoint topological union of all compact  $K \subset X_v$ . Let  $f: Y \rightarrow X_v$  be the map which embeds each  $K$  identically into  $X_v$ . Since every compact subset of  $X_v$  is the image of some compact subset of  $Y$ , it follows that the induced homomorphism  $C(f): C(X_v) \rightarrow C(Y)$  is a topological embedding. Since  $C(X_v)$  is complete it is homeomorphic to a closed subalgebra of  $C(Y)$ . Since  $Y$  is locally compact  $C(Y)$  is  $B$ -complete, however,  $C(X_v)$  is not even a  $B_r$ -complete algebra.

EXAMPLE 4.5. Let  $X$ ,  $Y$  and  $f: X \rightarrow Y$  be as in Example 4.2. Consider the closed subset  $S = f(N \cup \{p\})$ , and let  $r: C(Y) \rightarrow C(S)$  be the restriction map. Since  $Y$  is regular and  $\sigma$ -compact it is normal, hence  $r$  is onto. By [7, Theorem 6.7]  $r$  is continuous and open. Since  $f(N)$  is a proper dense and  $k$ -closed subset of  $S$ ,  $C(S)$  is not a  $B_r$ -complete algebra, although  $C(Y)$  is  $B_r$ -complete.

We conclude this section by pointing out that  $B$ -complete algebras are not productive. This follows from the fact that  $B$ -complete spaces

are not productive [2, (1) page 48], and the fact that every  $B$ -complete space is a  $B$ -complete algebra if we take multiplication to be zero ( $xy = 0$  for all  $x, y$ ).

**5. A closed graph theorem for  $B_r$ -complete algebras.** A closed graph theorem can be obtained for  $B_r$ -complete algebras analogous to the one proved by T. Husain [3, pp. 95–96] for  $B$ -complete groups. Except for a few minor points, the proof is essentially the same and so only a sketch is given. Recall that a linear map  $f: E \rightarrow F$  is called *almost continuous* if  $\text{cl}(f^{-1}(V))$  is a neighborhood of zero whenever  $V$  is a neighborhood of zero, and  $f$  is said to have a *closed graph* if  $\{(x, f(x)): x \in E\}$  is a closed subset of  $E \times F$ .

**THEOREM 5.1.** *Let  $B$  be a  $B_r$ -complete algebra and let  $A$  be a topological algebra. Let  $h: A \rightarrow B$  be an almost continuous homomorphism having a closed graph and such that  $h(A)$  is dense in  $B$ . Then  $h$  is continuous.*

*Proof.* Let  $u$  be the initial topology of  $B$  and let  $v$  be the topology generated by the sets

$$U^* = \text{cl}(h(\text{cl}(h^{-1}(U))))$$

where  $U$  varies over all neighborhoods of zero in  $B_u$ . Since  $h(A)$  is dense in  $B_u$  each  $U^*$  is a neighborhood of zero in  $B_u$ , hence the identity map  $\text{id}: B_u \rightarrow B_v$  is continuous. Since the graph of  $h$  is closed, it follows as in [3] that  $v$  is a Hausdorff topology. Clearly  $v$  is a locally convex topology, and since  $V^2 \subset U$  implies that  $(V^*)^2 \subset U^*$  it follows that  $B_v$  is a topological algebra. Since  $h$  is almost continuous,  $h \circ \text{id}$  is continuous and consequently  $U^* \subset \text{cl}_v(U)$ , implying that  $\text{id}: B_u \rightarrow B_v$  is almost open. Since  $B_u$  is a  $B_r$ -complete algebra, the identity map is open and consequently  $h$  is continuous.

In closing we note that, in general, the assumption that  $h(A)$  is dense in  $B$  cannot be dropped. This is an immediate consequence of the fact that  $B_r$ -complete algebras are not closed hereditary.

Let  $B$  be a  $B_r$ -complete algebra having a closed subalgebra  $A$  which is not  $B_r$ -complete (e.g., Ex. 4.4). Then there exists another topology  $u$  on  $A$  such that  $A_u$  is a topological algebra and the identity map  $A \rightarrow A_u$  is continuous and almost open, but not open. It follows that the embedding  $A_u \rightarrow B$  is almost continuous and has a closed graph, but is not continuous.

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