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**UNIFORM APPROXIMATION BY ELEMENTS OF A CONE OF  
REAL-VALUED FUNCTIONS**

WALTER ROTH

# UNIFORM APPROXIMATION BY ELEMENTS OF A CONE OF REAL-VALUED FUNCTIONS

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**The present paper gives conditions for approximation of all functions of a subspace  $M$  of  $C(K)$ , where  $K$  is a compact Hausdorff space by elements of an "admissible" subcone  $N$  of  $M$ . This implies generalizations of well-known theorems of Bauer and Stone-Weierstrass.**

**1. Preliminaries.** Let  $K$  be a compact Hausdorff space,  $M$  a linear space of real-valued continuous functions on  $K$ . A subcone  $N$  of  $M$  is said to be admissible if it separates the points of  $K$  and contains the constant functions. Denote by  $N^c$  the set of functions in  $C(K)$  that are the pointwise sup of finitely many functions in  $N$ . Let  $\leq_N$  be the relation on  $\Omega_K$  (the set of probability measures on  $K$ ) defined by:  $\mu \leq_N \nu$  iff for every  $f$  in  $N$   $\mu(f) \leq \nu(f)$ . Then  $\leq_N$  is an order relation on  $\Omega_K$ . For every  $\mu \in \Omega_K$  there exists an  $N^c$ -maximal (maximal with respect to  $\leq_N$ ) measure  $\nu \in \Omega_K$  such that  $\nu \geq_N \mu$  (see [1], I.5). The  $N^c$ -maximal measures live on every Baire set containing the Choquet boundary  $Ch_K(N)$  of  $N$ .

$Ch_K(N) = \{x \in K \mid \text{for every } \mu \in \Omega_K \text{ } \mu \geq_N \epsilon_x \text{ implies } \mu = \epsilon_x\}$  ( $\epsilon_x$  is the point measure at  $x$ ). For  $f \in C(K)$  the upper and lower envelopes are defined by

$$\bar{f}_N = \inf \{h \in -N \mid h > f\} \quad f_N = \sup \{h \in N \mid h < f\}$$

$\bar{f}_N$  resp.  $f_N$  are upper resp. lower-semi-continuous and coincide with  $f$  on  $Ch_K(N)$ . For every  $N^c$ -maximal measure  $\mu \in \Omega_K$  we have  $\mu(f) = \mu(\bar{f}_N) = \mu(f_N)$ .

## 2. An approximation-theorem: conditions for measures.

**THEOREM 1.** *Let  $M$  be a linear space of continuous real-valued functions on the compact Hausdorff space  $K$  and  $N$  an admissible cone.  $N$  is uniformly dense in  $M$  (dense with respect to the sup-norm) if and only if*

(1) For all  $N^c$ -maximal measures  $\mu, \nu$  in  $\Omega_K$   $\mu \underset{N}{>} \nu$  implies  $\mu \underset{M}{>} \nu$ , and one of the following conditions holds

(2a) Every  $M^c$ -maximal measure of  $\Omega_K$  is  $N^c$ -maximal.

(2b)  $f_N = f$  for every  $f$  in  $M$ .

If  $K$  is metrizable or if  $Ch_K(N)$  is compact (2a) and (2b) can be replaced by (2c)  $Ch_K(N) = Ch_K(M)$

The proof will use

LEMMA 1.  $N$  is uniformly dense in  $M$  if and only if for all  $\mu, \nu$  in  $\Omega_K$   $\mu \underset{N}{>} \nu$  implies  $\mu \underset{M}{>} \nu$ .

This Lemma is an immediate consequence of the Hahn-Banach Theorem and still holds if both  $N$  and  $M$  are admissible cones. Using the fact that the uniform closure of  $N$  in  $C(K)$  is given by

$\bar{N} = \{f \in C(K) \mid \text{such that for every measure } \lambda \text{ on } K \lambda(N) \geq 0 \text{ implies } \lambda(f) \geq 0.\}$

we decompose each such measure  $\lambda = \|\lambda\| (\mu - \nu)$ , where  $\mu, \nu \in \Omega_K$  and deduce Lemma 1.

To prove the theorem

( $\alpha$ ) suppose (1) and (2a) hold,  $\mu, \nu \in \Omega_K$  such that  $\mu \underset{N}{>} \nu$ .

Let  $\mu'$  be  $M^c$ -maximal and  $\mu' \underset{M^c}{>} \mu$   $\nu'$  be  $M^c$ -maximal and  $\nu' \underset{M^c}{>} \nu$ .

Then  $\mu'$  and  $\nu'$  are  $N^c$ -maximal (2a) and  $\mu' \underset{N^c}{>} \mu$ ,  $\nu' \underset{N^c}{>} \nu$  because

$N^c \subset M^c$ ).  $M$  is a linear space, therefore  $\underset{M}{<}$  implies  $\underset{M}{>}$  (indeed  $\underset{M}{<}$  is

an equivalence relation  $\widetilde{M}$ ) and we conclude

$$\begin{aligned} \mu' \widetilde{M} \mu \\ \nu' \widetilde{M} \nu \end{aligned} \Rightarrow \mu'(f) = \mu(f) \geq \nu(f) = \nu'(f) \text{ for every } f \text{ in } N.$$

i.e.  $\mu' \underset{N}{>} \nu'$  which implies by (1)  $\mu' \underset{M}{>} \nu'$ , hence  $\mu \underset{M}{>} \nu$ . From Lemma 1 we deduce  $M \subset \bar{N}$ .

(β) (2b) implies (2a) because for every  $f \in M$  the set  $\{h \in N^c \mid h < f\}$  is directed upward converging pointwise (2b) to the continuous function  $f$ . By Dini's Lemma the convergence is uniform, hence  $f \in \overline{N^c}$ ,  $M \subset \overline{N^c}$ ,  $M^c \subset \overline{N^c}$ .  $N^c$  and  $M^c$  define the same order relation on  $\Omega_K$ .

(γ) Suppose  $K$  is metrizable or  $Ch_K(N)$  is compact. In this case  $Ch_K(N) = Ch_K(M)$  implies (2a) because every measure in  $\Omega_K$  is  $N^c$ -maximal (resp.  $M^c$ -maximal) then if and only if it lives on  $Ch_K(N)$  (resp.  $Ch_K(M)$ ).

As a corollary of Theorem 1 we find a well-known theorem ([1], Th. II.4.5) on affine functions on a compact convex set:

COROLLARY 1. *Let  $A$  be the set of the affine continuous functions on the compact convex set  $X$ ,  $Ex(X)$  the extreme points of  $X$ ,  $f: Ex(X) \rightarrow \mathbf{R}$  a bounded continuous function (in the relative topology on  $Ex(X)$ ).*

*$f$  can be extended to a function in  $A$  if and only if*

$$(1) \quad \bar{f} = f \text{ on } \overline{Ex(X)} \quad \left( \begin{array}{l} \bar{f} = \inf \{g \in A \mid g > f \text{ on } Ex(X)\} \\ f = \sup \{g \in A \mid g < f \text{ on } Ex(X)\} \end{array} \right)$$

and

(2) *For all  $A^c$ -maximal measures  $\mu, \nu$  in  $\Omega_x$   $\mu \underset{A}{>} \nu$  implies  $\mu(\bar{f}) = \nu(\bar{f})$ .*

To prove the corollary suppose  $K = \overline{Ex(X)}$ ,  $N = A|_K$  and  $M = N + \mathbf{R} \cdot f$ , where  $f = f|_K = \bar{f}|_K \in C(K)$ . Clearly  $N$  is closed in  $C(K)$  and from Theorem 1 (2b) we see that  $N$  is uniformly dense in  $M$ , i.e.  $N = M$ .

REMARK. It is easy to show that if  $N$  is a linear space too, then condition (1) in Theorem 1 can be reduced to the comparison of maximal simplex measure (c.f. [1], Chapter I.6), i.e.

(1') For all  $N^c$ -maximal simplex measures  $\mu, \nu$  in  $\Omega_K$   $\mu \underset{N}{>} \nu$  implies  $\mu \underset{M}{>} \nu$ .

**3. Conditions for the order relation.** In this section we are going to replace (1) in Theorem 1 by conditions in terms of the order relation  $\cong$  in the function sets  $N$  and  $M$ . We use a simple generalization of a theorem of Cartier, Fell and Meyer:

LEMMA 2. Let  $M$  be a linear space of continuous real-valued functions on the compact Hausdorff space  $K$ ,  $N$  an admissible cone in  $M$ . Suppose that for every function  $f \in M$  the set

$$\{g \in N \mid g < f\}$$

is directed upward. Then for all  $\mu, \nu, \nu_1, \nu_2 \in \Omega_K, \lambda_1, \lambda_2 \geq 0$  such that  $\mu \underset{N}{>} \nu, \lambda_1 + \lambda_2 = 1$  and

$$\text{for every } f \text{ in } M \nu(f) = \lambda_1 \nu_1(f) + \lambda_2 \nu_2(f)$$

there exist  $\mu_1, \mu_2 \in \Omega_K$  such that  $\mu_1 \underset{N}{>} \nu_1, \mu_2 \underset{N}{>} \nu_2$  and

$$\text{for every } f \text{ in } M \mu(f) = \lambda_1 \mu_1(f) + \lambda_2 \mu_2(f).$$

To prove the lemma assume  $\lambda_1, \lambda_2 > 0$ . Define  $\phi: M \times M \rightarrow \mathbf{R}$  by

$$\phi(f, g) = \lambda_1 \nu_1(\bar{f}_N) + \lambda_2 \nu_2(\bar{g}_N)$$

Then

( $\alpha$ )  $\phi$  is positive subhomogeneous.

( $\beta$ ) For every  $f \in M$ , also  $-f \in M$  and therefore

$\{h \in N \mid h < -f\}$  is directed upward, hence

$\{h \in -N \mid h > f\}$  is directed downward, hence

$$\begin{aligned} \mu(\bar{f}_N) &= \inf \{\mu(h) \mid h \in -M, h > f\} \\ &\leq \inf \{\nu(h) \mid h \in -N, h > f\} = \nu(\bar{f}_N). \end{aligned}$$

( $\gamma$ ) Suppose  $F = \{(f, f) \in M \times M\}$  and

$$\psi_0: F \rightarrow \mathbf{R}, \quad \psi_0(f, f) = \mu(f).$$

Then  $\psi_0(f, f) = \mu(f) \leq \mu(\bar{f}_N) \leq \nu(\bar{f}_N) = \phi(f, f)$ .

( $\delta$ ) By the Hahn-Banach Theorem there exists a linear extension

$$\psi: M \times M \rightarrow \mathbf{R} \text{ of } \psi_0 \text{ such that } \psi \leq \phi.$$

( $\epsilon$ ) Let  $\mu_1, \mu_2$  be probability measures on  $K$  such that for every  $f$  in  $M$

$$\mu_1(f) = 1/\lambda_1 \psi(f, 0) \quad \text{and} \quad \mu_2(f) = 1/\lambda_2 \psi(0, f)$$

Then we conclude

For every  $f$  in  $M$   $\mu(f) = \psi_0(f, f) = \psi(f, f) = \lambda_1\mu_1(f) + \lambda_2\mu_2(f)$ .

For every  $f$  in  $-N$   $\mu_1(f) = 1/\lambda_1\psi(f, 0) \leq 1/\lambda_1\phi(f, 0) = \nu_1(\bar{f}_N) = \nu_1(f)$ .

Hence  $\nu_1 \underset{N}{\leq} \mu_1$  and  $\nu_2 \underset{N}{\leq} \mu_2$

which proves the lemma. Now we can state:

**THEOREM 2.** *Let  $M$  be a linear space of continuous real-valued functions on the compact Hausdorff space  $K$ ,  $N$  an admissible cone in  $M$ .  $N$  is uniformly dense in  $M$  if and only if*

(1) *For every  $f \in M$  the set*

$$\{g \in N \mid g < f\}$$

*is directed upward and*

(2)  $Ch_K(N) = Ch_K(M)$ .

*Proof.* Of course  $M \subset \bar{N}$  implies (1) and (2). Suppose now (1) and (2) hold and

$$R = \{(\nu, \mu) \in M' \times M' \mid \mu, \nu \geq 0, \|\mu\| = \|\nu\| = 1, \nu \underset{N}{\leq} \mu\}$$

$$S = \{(\mu, \mu) \in M' \times M' \mid \mu \geq 0, \|\mu\| = 1\}$$

Both  $R$  and  $S$  are compact convex in the weak topology on  $M' \times M'$ ,  $S \subset R$ . We show  $S = R$  and apply Lemma 1.

Lemma 2 shows that

$$Ex(R) \subset \{(\epsilon_x, \mu) \in R \mid x \in Ch_K(M)\}$$

because every  $\mu \in M'$ ,  $\mu \neq \epsilon_x$  for every  $x \in Ch_K(M)$ , is a convex combination of two different elements of  $M'$ . From  $Ch_K(N) = Ch_K(M)$  we conclude now

$$(\epsilon_x, \mu) \in Ex(R) \Rightarrow \epsilon_x \underset{N}{\leq} \mu \Rightarrow \epsilon_x = \mu \Rightarrow (\epsilon_x, \mu) \in S,$$

hence  $Ex(R) \subset S$ , which implies  $R = S$  and proves the theorem.

**4. Applications: The case  $M = C(X)$ .** If we choose  $M = C(X)$  Theorem 2 leads to generalizations of theorems of Bauer and of Stone-Weierstrass.

**THEOREM 3.** *Let  $N$  be an admissible cone of continuous real-valued functions on the compact Hausdorff space  $K$ . The following conditions are equivalent:*

- (1)  $N$  is uniformly dense in  $CK$ .
- (2)  $\bar{N}$  is maximum-stable, i.e.  $N^c \subset \bar{N}$ , and  $Ch_K(N) = Ch_K(N - N)$ .
- (3) For all  $f, g \in N$   $\inf\{h \in N \mid h > f \vee g\} \in \bar{N}$  ( $f \vee g$  is the point-wise sup of  $f$  and  $g$ )

$$\text{and } Ch_K(N) = Ch_K(N - N) \quad \text{and} \quad \overline{Ch_K(-N)} = K.$$

*Proof.* The implications (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (3) are trivial. Suppose (2) holds: Theorem 2 shows that  $N - N \subset \bar{N}$ , hence  $\bar{N}$  is a linear space.  $N^c \subset \bar{N}$  therefore implies  $(N^c - N^c) \subset \bar{N}$ . From the classical Stone-Weierstrass theorem we know that  $N^c - N^c$  is dense in  $C(K)$  which proves  $\bar{N} = C(K)$ .

Suppose now (3) holds. We show that  $N^c \subset \bar{N}$  and therefore (3) implies (2): Let  $f, g \in N$ . Then

$$h = \overline{f \vee g}_{(-N)} = \inf\{j \in N \mid j > f \vee g\} \in \bar{N}$$

Now  $h|_{Ch_K(-N)} = f \vee g|_{Ch_K(-N)}$  and  $\overline{Ch_K(-N)} = K$  imply  $h = f \vee g \in \bar{N}$ , hence  $N^c \subset \bar{N}$ .

Bauer's theorem (see [1], Th. II.4.1) is a special case of the following.

**THEOREM 4.** *Let  $N$  be a uniformly complete admissible cone of continuous real-valued functions on the compact Hausdorff space  $X$ ,  $K = Ch_X(N - N)$ . The following conditions are equivalent:*

- (1)  $N|_K = C(K)$ .
- (2) For all  $N^c$ -maximal measures  $\mu, \nu$  in  $\Omega_x$   $\mu \underset{N}{>} \nu$  implies  $\mu = \nu$  and  $Ch_X(N) = K$ .
- (3) For all  $f, g \in N$   $\inf\{h \in N \mid h > f \vee g\} \in N$  and  $Ch_X(N) = Ch_X(N - N)$  and  $Ch_X(-N) = K$ .

*Proof.* Clearly  $N|_K$  is closed in  $C(K)$ . (1)  $\Leftrightarrow$  (2) is an immediate consequence of Theorem 1. (1)  $\Rightarrow$  (3) is trivial. To prove the implication (3)  $\Rightarrow$  (1) we take  $f, g \in N$ . (3) guarantees the existence of an  $h \in N$  such that  $h = \inf\{j \in N \mid j > f \vee g\}$ . From  $\overline{Ch_K(-N)} = K$  we conclude now  $h|_K = f \vee g|_K$ . Hence  $N|_K$  is maximum stable and Theorem (3) proves  $N|_K = C(K)$ . (Obviously  $Ch_K(N|_K) = Ch_X(N)$ .)

### 5. Examples.

1. Suppose  $K$  is the finite set  $\{x_1, \dots, x_n\}$ ,  $N$  an admissible cone of functions (here real-valued finite sequences) on  $K$ . Then  $Ch_K(N)$  consists of all  $x_j \in K$  such that there exists  $f \in N$ , such that  $f(x_i) < f(x_j)$  for all  $i \neq j$ . (see [4], §7, Ex. 3). In this case Theorem 3 states:

$N = C(K) = \mathbf{R}^n$  iff for all  $f, g \in N$   $\inf\{h \in N \mid h > f \vee g\} \in N$  and for every  $x_j \in K$  there are  $f, g \in N$  such that if  $i \neq j$   $f(x_i) < f(x_j)$  and  $g(x_i) > g(x_j)$ .

2. Let  $K$  be the compact subset  $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0\}$  of  $\mathbf{R}$ . Then  $C(K)$  is the set of all convergent real-valued sequences.

$$N = \{(y)_n \in C(K) \mid (y)_n = (\alpha_1, \alpha_2, \dots, \alpha_k) + \lambda(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0) + r,$$

where  $r \in \mathbf{R}, k \in \mathbf{N}, \lambda \leq \min\{0, \alpha_1, \alpha_2, \dots, \alpha_k\}$

defines an admissible cone in  $C(K)$  with the properties:

(1)  $N \cap (-N) = \mathbf{R}$ :

Let  $f \in N \cap (-N)$ . Then

$$f = (\alpha_1, \dots, \alpha_k) + \lambda(1, \frac{1}{2}, \dots) + r = -(\beta_1, \dots, \beta_k) - \rho(1, \frac{1}{2}, \dots) = r'$$

$$\Rightarrow \lambda = -\rho = 0 \quad \text{because } \lambda, \rho \leq 0$$

$$\Rightarrow \alpha_1, \dots, \alpha_k \geq 0, \quad \beta_1, \dots, \beta_k \geq 0.$$

$$\Rightarrow \alpha_1 = -\beta_1 = 0, \dots \quad \alpha_k = -\beta_k = 0.$$

$$\Rightarrow f \in \mathbf{R}.$$

(2) Every  $x \in K$  is a maximum point of an element of  $N$ . (For  $x = 0$  choose  $\alpha_i = 0, \lambda = -1$ ). Therefore  $Ch_K(N) = K$  and  $Ch_K(N) = Ch_K(N - N) = K$ .

Every  $1/n \in K$  is a minimum point of an element of  $N$ , hence  $Ch_K(-N) = K$ .

(3)  $N$  is maximum-stable.

From Theorem 3 we conclude that  $N$  is uniformly dense in  $C(X)$ .

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