

# Pacific Journal of Mathematics

**ON A SPACE BETWEEN  $BH$  AND  $B_\infty$**

JERROLD NORMAN SIEGEL

## ON A SPACE BETWEEN $BH$ AND $B_\infty$

JERROLD SIEGEL

Let  $F \subseteq E_\infty \xrightarrow{p} B_\infty$  be the classifying space of fibre homotopy equivalence classes of fibrations with fiber  $F$ . An obstruction to the existence of a retraction  $B_\infty \xrightarrow{r} E_\infty$  is developed. This obstruction is shown to vanish when  $F$  is a stable space. Consequences of the existence of a retraction are indicated.

One asks the following question: Let  $F \subseteq E_\infty \xrightarrow{p_\infty} B_\infty$  be a universal fibration for  $\mathcal{H}(-, F)$ , for which  $F$  do there exist retractions  $r: B_\infty \rightarrow E_\infty$ ?

An obstruction theoretic answer to this question is given below. The method is first to observe that  $F$  must be a loop space. Then construct a fibration  $F \subseteq E_+ \xrightarrow{p_+} B_+$ , admitting a retraction  $r: B_+ \rightarrow E_+$ , that is "universal" with respect to this property. Finally, compute the fibre of the classifying map  $B_+ \rightarrow B_\infty$ . As an application of this computation it is shown that a retraction exists whenever  $F$  is a stable space.

Our interest in this question grew out of an observation in [7]. In particular, let  $L(\pi, n)$  be the classifying space for  $\mathcal{H}(-, K(\pi, n))$ , (fibre homotopy equivalence classes of fibrations with fibre  $K(\pi, n)$ ). Let  $K(\pi, n) \subseteq E_\infty \xrightarrow{p} L(\pi, n)$  be the universal fibration [1], in [7] it was observed that there is a retraction of  $L(\pi, n)$  to  $E_\infty$ . This observation was used in [6] and implicitly in [7] to facilitate the computation of the cohomology of  $L(\pi, n)$ . 2.13, below, generalizes that part of the main result of [7] and seems to give the appropriate setting for that theorem.

We would like to thank the referee for his helpful comments and especially for alerting us to the difficulties in the work of Allaud and Dold.

**1. Preliminaries.** We follow Allaud [1], (see Dold [2] for an alternate treatment)<sup>1</sup>. In addition, we make the blanket assumption that all spaces are in  $\mathcal{C}$  the category of spaces of the homotopy type of countable connected C.W complexes (resp.  $\mathcal{C}_0$  base pointed) according to [1], [5] the constructions below do not carry us out of these categories.

---

<sup>1</sup> Peter May has noted a difficulty in both works. In particular, neither work supplies well defined set valued functors for the theory of fibrations. These, of course, are required for application of Browns theorem. May will discuss various remedies in a forthcoming monograph.

DEFINITION 1.1. Let  $F$  be in  $\mathcal{C}$  and let  $(X, x_0)$  be in  $\mathcal{C}_0$ . A *fibre space over  $X$  with fibre  $F$*  (written  $(\mathcal{E}, g)$ ) consists of a sequence of spaces and maps.  $F \xrightarrow{g} E \xrightarrow{p} X$  such that:

- (a) The triple  $\mathcal{E} = (E, p, X)$  is a fibre space.
- (b)  $g: F \rightarrow p^{-1}(x_0)$  is a homotopy equivalence.

DEFINITION 1.1a. If we further assume:

- (1)  $(F, b_0) \in \mathcal{C}_0$
- (2)  $\mathcal{E}^0 = (E, p, X, s)$  is a fibre space with cross-section.
- (3)  $g: (F, b_0) \rightarrow (p^{-1}(x_0), s(x_0))$  is a base point preserving homotopy equivalence.

We call  $(\mathcal{E}^0, g)$  a *fibre space with cross-section*.

DEFINITION 1.2. Let  $(\mathcal{E}_1, g_1)$  and  $(\mathcal{E}_2, g_2)$  be fibre spaces. A *fibre map*  $(\mathcal{E}_1, g_1) \xrightarrow{f} (\mathcal{E}_2, g_2)$  is a triple of maps  $(g, \bar{f}, f')$

$$\begin{array}{ccccc} F_1 & \xrightarrow{g_1} & E_1 & \xrightarrow{p_1} & X_1 \\ g \downarrow & & I \bar{f} \downarrow & & II f' \downarrow \\ F_2 & \xrightarrow{g_2} & E_2 & \xrightarrow{p_2} & X_2 \end{array}$$

such that  $I$  commutes up to homotopy and  $II$  commutes. We denote the category of fibre spaces and maps by  $\mathcal{F}$ .

DEFINITION 1.2a. If  $(\mathcal{E}_1^0, g_1)$  and  $(\mathcal{E}_2^0, g_2)$  are fibre spaces with cross-section and  $(\mathcal{E}_1^0, g_1) \xrightarrow{f} (\mathcal{E}_2^0, g_2)$  is a fibre map such that  $\bar{f}s_1 = s_2f'$  then  $f$  is called a *section preserving fibre map*. We denote the resulting category by  $\mathcal{F}^0$ .

DEFINITION 1.3. Let  $(\mathcal{E}, g)$  be a fibre space over  $X$  with fibre  $F$ . Let  $f: Y \rightarrow X$  be a base pointed map. We have the usual induced fibration  $(f^{-1}(\mathcal{E}), g)$  with fibre  $F$ .  $f$  induces  $(f^{-1}(\mathcal{E}), g) \xrightarrow{f} (\mathcal{E}, g)$  in  $\mathcal{F}$ .

DEFINITION 1.3a. If  $(\mathcal{E}^0, g)$  carries a cross-section then so does  $f^{-1}(\mathcal{E}^0, g)$  and  $f$  is in  $\mathcal{F}^0$ .

DEFINITION 1.4. Restricting to a fixed fibre  $F$  there are contravariant functors  $\mathcal{H}(-, F)$  and  $\mathcal{H}_0(-, F)$  from  $\mathcal{C}_0$  to the category of sets.  $\mathcal{H}(X, F)$  (resp.  $\mathcal{H}_0(X, F)$ ), is the set of fibre homotopy equiva-

lence classes of fibrations over  $X$  (resp. base section preserving) with fibre  $F$ .

We now state two theorems, the first of which appears in [1]. The second can be proved by mimicking that proof. We offer an alternative proof below.

**THEOREM 1.5.** *Given  $F$ , there exists a space  $B_\infty \in \mathcal{C}_0$  and a fibre space  $(\mathcal{E}_\infty, g_\infty)$  over  $B_\infty$  with fibre  $F$  such that the natural transformation  $[X, B_\infty]_0 \xrightarrow{T} \mathcal{H}(X, F)$  given by  $T[f] = (f^{-1}(\mathcal{E}_\infty), g_\infty)$  is a 1-1 onto set map.*

**THEOREM 1.5a.** *There exists a space  $B_+ \in \mathcal{C}_0$  and a fibre space  $(\mathcal{E}_+, g_+)$  over  $B_+$  with fibre  $F$  such that the natural transformation  $[X, B_+]_0 \xrightarrow{T_0} \mathcal{H}_0(X, F)$  given by  $T_0[f] = (f^{-1}(\mathcal{E}_+), g_+)$  is a 1-1 onto set map.*

We prove 1.5a by observing that, in fact,  $B_+$  is the total space of the universal fibration over  $B_\infty$ .

**DEFINITION 1.6.** If  $\mathcal{E} = (E, p, X)$  is a fibre space with fibre  $F$ . Let  $E^F$  consist of all maps from  $F$  into  $E$  which are a homotopy equivalence into some  $p^{-1}(x)$ ,  $x \in X$ . Fixing  $b \in F$ , let  $\tilde{p}f = pf(b)$ . Set  $\mathcal{E}^F = (E^F, \tilde{p}, X)$ .  $\mathcal{E}^F$  is a principal  $H(F)$ -fibration where  $H(F)$  is the space of homotopy equivalences of  $F$  to itself.

**DEFINITION 1.6a.** If  $\mathcal{E}_0 = (E, p, X, s)$  is a fibre space with base pointed fibre  $(F, b)$  and cross-section  $s$ . Let  $E_0^F \subseteq E^F$  consist of all maps  $f$  such that  $f$  is a base point preserving homotopy equivalence into  $(p^{-1}(x), s(x))$  for some  $x$ .  $\mathcal{E}_0^F$  is a principal  $H_0(F)$ -fibration. Where  $H_0(F)$  is the space of base point preserving homotopy equivalences.

We also need the following construction.

Let  $E \xrightarrow{p} X$  be a fibration with fibre  $F$ . Let  $E^2 \xrightarrow{p} E$  be the “pull back” over  $E$  of  $E \xrightarrow{p} X$

$$\begin{array}{ccc} E^2 & \xrightarrow{t} & E \\ s \Downarrow p^2 \nearrow & & \downarrow p \\ E & \xrightarrow{p} & X \end{array}$$

The cross-section  $s$  is induced by the identity map. Fixing  $b \in F$  we have a map  $\alpha : (E^2)_0^F \rightarrow E^F$  given by  $\alpha(f) = t \cdot f$ .

**LEMMA 1.7.**  *$\alpha$  is a homeomorphism of  $(E^2)_0^F$  to  $E^F$ .*

*Proof.* The inverse of  $\alpha$  is  $\beta: E^F \rightarrow (E^2)_0^F$  given by

$$\beta(f) = f(b) \times f: F \rightarrow E \times E.$$

**THEOREM 1.8.** *Let  $E_\infty \xrightarrow{p_\infty} B_\infty$  as in 1.5 then  $(E_\infty^2)_0^F \xrightarrow{p_\infty^2} E_\infty$  is a universal principal  $H_0(F)$ -fibration.*

*Proof.* By 1.7  $\alpha_*: \pi_i((E_\infty^2)_0^F) \cong \pi_i(E_\infty^F)$ . By Theorem 4.1 of [1]  $\pi_i(E_\infty^F) = 0$ .

**COROLLARY 1.9.**  $E_\infty^2 \xrightarrow{p_\infty^2} E_\infty$  is universal for  $\mathcal{H}_0(-, F)$ .

*Proof.* Given  $\mathcal{E}^0 = E \xrightleftharpoons[s]{p} X$  we have, using the universality of  $B_\infty$

$$\begin{array}{ccc} E & \longrightarrow & E_\infty \\ p \downarrow s & & \downarrow p_\infty \\ X & \longrightarrow & B_\infty \end{array}$$

hence  $\mathcal{E}^0$  is induced by a map into  $E_\infty$ . But using 1.8 and trivial modifications of the arguments in [4] §7, we have  $\mathcal{E}^0$  and  $\mathcal{E}^{0'}$  are base section fibre homotopy equivalent iff the induced maps into  $E_\infty$  are homotopic.

This explicit model will facilitate later computations. We will also need the following observation about fibrations with cross-sections.

**DEFINITION 1.10.** Let  $(\mathcal{E}^0, g)$  be a fibration over  $X$  with fibre  $(F, b)$  and cross-section  $s$ . There is the usual associated fibration  $(\Omega \mathcal{E}^0, \Omega g)$  over  $X$  with fibre  $\Omega(F)$  where  $(\Omega p)^{-1}(x) = \Omega(p^{-1}(x))$  based at  $s(x)$ .

**LEMMA 1.11.** *Let  $(\mathcal{E}^0, g) = F \xrightarrow{s} E \xrightleftharpoons[p]{p} X$  be a fibration with cross-section. Let  $F_s \subseteq X \xrightarrow{s} E$  be the fibration of the map  $s$ , then the pull back*

$$\begin{array}{ccc} F_s \subseteq s^{-1}(X) & \dashrightarrow & X \\ p' \downarrow & & \downarrow s \\ X & \xrightarrow{s} & E \end{array}$$

*is fibre homotopy equivalent with cross-section to  $(\Omega \mathcal{E}^0, \Omega g)$ .*

*Proof.* First note  $s^{-1}(X) \subseteq E^I$  and  $s^{-1}(X) = \{f: I \rightarrow E \mid f(0) \text{ and } f(1) \in s(X)\}$  hence there is the natural inclusion  $\Omega(\mathcal{E}^0) \subseteq s^{-1}(X)$  covering  $X$ , where  $\Omega(\mathcal{E}^0)$  here denoted the total space of  $(\Omega \mathcal{E}^0, \Omega(g))$ . There is map  $\gamma: s^{-1}(x) \rightarrow \Omega E^{\text{free}}$  given by  $\gamma(f) = (spf)^{-1} * f$ , where  $*$  denotes path composition. Since  $(\Omega p)\gamma = (pf)^{-1} * pf$ ,  $\gamma$  can be deformed to a fibre map  $\gamma': s^{-1}(x) \rightarrow \Omega(\mathcal{E}^0)$  such that  $\gamma'i = id$ . By a standard argument one shows  $i\gamma$  is cross-section preserving fibre homotopic to the identity.

We close this section with one final technical observation. The proof is straightforward and we leave it undone.

The following notation is useful. Given a map  $X \xrightarrow{f} Y$  we let  $X_f = X \times_{f \times 0} Y^I$ , and let  $X_f \xrightarrow{f'} Y$  be the standard replacement of  $X \xrightarrow{f} Y$  by a fibration.

Let  $F \xrightarrow{g} E \xrightleftharpoons[p]{p} B$  be a fibration with retraction. We have the following diagram.

$$\begin{array}{ccc} E & = & E \longrightarrow E_p \\ r \Downarrow p & & r' \Downarrow p \quad \Downarrow \\ B & \xrightarrow{i} & B_r = B_r \end{array}$$

THEOREM 1.12.

$$\begin{array}{ccc} E & \longrightarrow & i^{-1}(E_p) \\ \downarrow p & & \downarrow p' \\ B & = & B \end{array}$$

is a fibre homotopy equivalence.

**2. The Space  $B_+$ .** We will be concerned with the question of when there is a retraction of  $B_\infty$  to  $E_\infty$ . We begin with certain simple technical facts we will need in the following.

THEOREM 2.1. *Let  $F \rightarrow E \xrightleftharpoons[p]{p} B$  be a fibration with  $r$  a retraction to the total space. Then the composition  $\Omega F_r \rightarrow \Omega B \rightarrow F$  is a homotopy equivalence.*

*Proof.* Consider the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_i(E) & \xrightarrow{p_*} & \pi_i(B) & \xrightarrow{\partial} & \pi_{i-1}(F) \longrightarrow 0 \\
 & & \parallel & & \parallel & & \\
 0 & \longleftarrow & \pi_i(E) & \xleftarrow{r_*} & \pi_i(B) & \xleftarrow{i_*} & \pi_i(F_r) \longleftarrow 0
 \end{array}$$

We show  $\partial i_*$  is an isomorphism by a simple diagram chase.  $f$  is then the geometric realization of this map.

$\partial i_*$  is a monomorphism.  $\partial i_*x = 0 = \exists y \cdot \partial \cdot p_*y = i_*x$  but  $r_*p_*y = y$  so  $y = r_*p_*y = r_*i_*x = 0$  so  $i_*x = 0 = x = 0$ .

$\partial i_*$  is an epimorphism. Let  $z \in \pi_{i-1}(F)$ ,  $z = \partial y = \partial(y - p_*r_*y)$  but  $r_*(y - p_*r_*y) = 0$  so  $\exists x \cdot \partial \cdot i_*x = y - p_*r_*y$ .

LEMMA 2.2. *Let  $H$  be an  $H$ -space. Let  $(\mathcal{E}_\infty, g_\infty)$  be the universal fibration for  $H$  then the sequence  $0 \rightarrow \pi_i(E_\infty) \rightarrow \pi_i(B_\infty) \rightarrow \pi_{i-1}(H) \rightarrow 0$  is exact and splits.*

*Proof.* Let  $H \rightarrow * \rightarrow BH$  be the universal principal  $H$ -fibration.

Consider the diagram.

$$\begin{array}{ccccc}
 H & \longrightarrow & * & \longrightarrow & BH \\
 \parallel & & \downarrow & & \downarrow \\
 H & \longrightarrow & E_\infty & \longrightarrow & B_\infty
 \end{array}$$

This leads to the following diagram of homotopy group.

$$\begin{array}{ccc}
 \pi_i(BH) & \xrightarrow[\cong]{\partial} & \pi_{i-1}(H) \\
 \tau_* \downarrow & & \parallel \\
 \pi_i(B_\infty) & \xrightarrow{\partial_\infty} & \pi_{i-1}(H)
 \end{array}$$

$\partial_\infty$  is therefore split by  $\tau_*\partial^{-1}$ .

2.3. Let  $BH \rightarrow E_+ \xrightleftharpoons[p_*]{s} B_+$  be the universal fibration for  $\mathcal{H}_0(-, BH)$ . By 2.1  $H$  is the fibre of  $s$ . Setting  $B_+ = E_+$ ,  $E_+ = B_+$ ,  $s = p_+$  and  $p_+ = r_+$  we have the following diagram.

$$\begin{array}{ccccc}
 H & & = & H & \\
 \downarrow & & & & \downarrow \\
 T' & \xrightarrow{i'} & E_+ & \xrightarrow{\sigma} & E_\infty \\
 g \downarrow & & r_+ \Downarrow p_+ & & \downarrow p_\infty \\
 T & \xrightarrow{i} & B_+ & \xrightarrow{\rho} & B_\infty
 \end{array}$$

where  $T$  is the fibre of  $\rho$  and  $T'$  is the fibre of  $\sigma$ .

It is the space  $T$  that we are interested in since, as we now know, it measures the obstruction to retracting  $B_\infty$  to  $E_\infty$ .

**THEOREM 2.4.** *Let  $H \rightarrow E \xrightarrow{p} X$  be a fibration such that  $r$  is a retraction of  $X$  to  $E$  then the classifying map  $f: X \rightarrow B_\infty$  factors through  $B_+$ . That is, there exists  $f': X \rightarrow B_+$  and a diagram of fibre maps.*

$$\begin{array}{ccccc}
 E & \xrightarrow{f'} & E_+ & \longrightarrow & E_\infty \\
 r \Downarrow p & & r_+ \Downarrow p_+ & & \downarrow p_\infty \\
 X & \xrightarrow{f'} & B_+ & \xrightarrow{\rho} & B_\infty
 \end{array}$$

with  $\rho f' \sim f$  and  $\bar{f}' r \sim r_+ f'$ .

Moreover  $f'$  induces a unique map on homotopy. That is if  $\rho f'_1 \sim \rho f'_2$  then  $(f'_1)_* = (f'_2)_*: \pi_i(X) \rightarrow \pi_i(B_+)$ .

*Proof.* Consider the fibration

$$\begin{array}{c}
 X_r \\
 \bar{p} \Downarrow r' \\
 E
 \end{array}$$

where  $\bar{p} \sim p$  is a cross-section. Since  $B_+ = E_+$ , we have the fibre map

$$\begin{array}{ccc}
 X_r & \xrightarrow{f'} & B_+ \\
 \bar{p} \Downarrow r' & & p_+ \Downarrow r_+ \\
 E & \xrightarrow{f} & E_+
 \end{array}$$

and the associated diagram of fibrations

$$\begin{array}{ccc} E_{\tilde{p}} & \xrightarrow{f} & E_+ \\ \tilde{p}' \downarrow & & \downarrow p_+ \\ X_r & \xrightarrow{f'} & B_+ \end{array}$$

but  $\tilde{p} \sim p$  implies that  $E_p \xrightarrow{p'} X_r$  is fibre homotopy equivalent to  $E_{\tilde{p}} \xrightarrow{p'} X_r$ , and by 1.12 we have

$$\begin{array}{ccc} E & \longrightarrow & E_p \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{i} & X_r \end{array}$$

a fibre map. Let  $f' = i\bar{f}'$ .

In order to verify uniqueness consider the diagram

$$\begin{array}{ccccccc} 0 \rightarrow \pi_i(E) & \xrightarrow[p_*]{p_*} & \pi_i(X) & \xrightarrow{\partial} & \pi_{i-1}(H) & \rightarrow & 0 \\ & & \parallel \bar{f}' & & \parallel f' & & \parallel \\ 0 \rightarrow \pi_i(E_+) & \xrightarrow[p_*]{r_*} & \pi_i(B_+) & \xrightarrow{\partial} & \pi_{i-1}(H) & \rightarrow & 0 \end{array}$$

Since  $\bar{f}'r \sim r_*f'$  and  $E_+$  is universal for  $H_0(-, BH)$  we have  $\bar{f}'$  is unique (indeed  $\bar{f}'$  is unique up to homotopy). The uniqueness of  $f'$  follows from the splitting of the diagram.

It is reasonable to expect that  $f'$  is unique up to homotopy, however  $B_+$  is not classifying space in sense of Brown [2]. Fibre spaces admitting retractions do not form a suitable category.

To infer that  $f'$  is unique up to homotopy from the above requires knowledge the two fibre maps that are homotopic along the base are homotopic in the total space. In general of course this is not the case.

**THEOREM 2.5.** *The fibration  $H \rightarrow E_\infty \xrightarrow{p_\infty} B_\infty$  admits a retraction  $r$ , iff the fibration  $T \rightarrow B_+ \xrightarrow{p} B_\infty$  admits a cross-section,  $s$ .*

*Proof.*

$\Rightarrow$  : Let  $E_\infty \xrightarrow{p_\infty} B_\infty$  admit a retraction. Therefore, by 2.4 we can factor the identity map  $\text{id}: B_\infty \rightarrow B_\infty$  through  $\rho$ . That is find  $s': B_\infty \rightarrow B_+$  with  $\rho s' \sim \text{id}$ . Let  $s$  be a cross-section homotopic to  $s'$ .

$\Leftarrow$  : Consider the diagram.

$$\begin{array}{ccccc} E_\infty & \xrightarrow{i'} & E_+ & \xrightarrow{\sigma} & E_\infty \\ p_\infty \downarrow & & p_+ \Downarrow r_+ & & p_\infty \downarrow \phi \\ B_\infty & \xrightarrow{s} & B_+ & \xrightarrow{\rho} & B_\infty \end{array}$$

*Note.* Here we use the fact that  $B_\infty$  is universal to conclude that  $s^{-1}(E_+) \rightarrow B_\infty$  is fibre-homotopy equivalent to  $E_\infty \xrightarrow{p_\infty} B_\infty$ .

Define

$$\begin{aligned} \bar{r} &= \sigma r_+ s. \\ \bar{r} p_\infty &= \sigma r_+ s p_\infty \\ &= \sigma r_+ p_+ t \\ &= \sigma t \end{aligned}$$

but  $\sigma t$  is a homotopy equivalence so set  $r = (\sigma t)^{-1} \bar{r}$

We now study the problem of sectioning the fibration  $T \rightarrow B_+ \xrightarrow{\rho} B_\infty$ .

Consider again the diagram.

$$(2.6) \quad \begin{array}{ccccc} & & H & = & H \\ & & \downarrow & & \downarrow \\ T' & \xrightarrow{i'} & E_+ & \xrightarrow{\sigma} & E_\infty \\ g \downarrow & & r_+ \Downarrow p_+ & & \downarrow p_\infty \\ T & \xrightarrow{i} & B_+ & \xrightarrow{\rho} & B_\infty \end{array}$$

**THEOREM 2.7.**  $g: T' = T$

*Proof.* By symmetry of fibre products the fibre of  $\zeta$  and the fibre of  $\sigma$  are the same.

*Description of T'.* Given a fibration  $(\mathcal{E}_0, g)$  with cross-section. There is the “loop” fibre map.

$$\begin{array}{ccccc}
 (F^F)_0 & \xrightarrow{g} & E_o^F & \xrightarrow{p} & B \\
 \downarrow \Omega(F) & & \downarrow \Omega(F) & & \parallel \\
 \Omega(F) & \xrightarrow{\Omega g} & \Omega(\mathcal{E}^0) & \xrightarrow{\Omega(p)} & B
 \end{array}$$

Given by fibre-wise looping of the mapping space. We apply this construction to  $BH \rightarrow B_+ \xrightarrow[r_+]{\tau} E_+$  (see 2.3) in conjunction with 1.11 to obtain the following diagram.

$$\begin{array}{ccccc}
 (BH)_0^{BH} & \xrightarrow{\Omega} & (H^H)_0 & = & (H^H)_0 \\
 \downarrow & & \downarrow & & \downarrow \\
 B_+^{BH} & \longrightarrow & (\Omega(B_+))^H & \longrightarrow & (E_\infty^2)_b^H \\
 \downarrow r_+ & & \downarrow & & \downarrow \\
 E_+ & = & E_+ & \xrightarrow{\rho} & E_\infty
 \end{array}$$

Using the outer edges of 2.4 combined with 1.8 yield the following commutative diagram of homotopy group.

$$\begin{array}{ccc}
 \pi_n(E_+) & \xrightarrow{\sigma_n} & \pi_n(E_\infty) \\
 \partial \parallel & & \parallel \partial \\
 \pi_{n-1}((BH)_0^{BH}) & \xrightarrow{\Omega} & \pi_{n-1}(H_0^H)
 \end{array}$$

and this plus the usual elementary considerations about mapping spaces of fibre spaces shows.

**THEOREM 2.9.**  $\Omega(T')$  is homotopy equivalent to the fibre of  $BH_0^{BH} \xrightarrow{\Omega} H_0^H$  which we denote by  $L$ .

*Description of L.* We begin by noting that the map  $\Omega$  is equivalent to the map  $(BH)_{\mathbb{D}}^{BH} \xrightarrow{\tau^*} (BH)_\tau^{\Omega BH}$  induced by the canonical

map.  $S \Omega BH \xrightarrow{\tau} BH$ . The subscripts refer to the appropriate component of mapping spaces.

Considering  $\tau$  as an inclusion we have the well known lemma:

LEMMA 2.10.  $(BH)_{\text{ID}}^{BH} \xrightarrow{\tau^*} (BH)_\tau^{S \Omega BH}$  is a fibration with fibre  $L = \{f \in (BH)^{BH} \mid f/S \Omega BH = \tau\}$ .

If we further assume  $BH$  is a loop space then for any  $X$  all components of  $BH^X$  have the same homotopy type. Indeed

THEOREM 2.11. Let  $BH$  be a loop space then  $L$  is homotopy equivalent to  $(BH^{BH/S \Omega BH})_0$  where  $(BH/S \Omega BH)_0$  is the cofibre of  $\tau$ .

*Proof.* This is a standard argument using associativity and the existence of a homotopy inverse.

EXAMPLE 1.12. Suppose we define an  $n - 1$  connected space  $H$  to be stable if  $\pi_i(H) = 0, i > 2n - 3$  and  $BH$  is a loop space.  $S \Omega BH \rightarrow BH$  is an isomorphism in homology through dimension  $2n - 2$ . But  $\pi_i(BH) = 0, i > 2n - 2$  so

$$\pi_j(L) = \pi_j((BH^{BH/S \Omega BH})_0) = [S^j \wedge (BH/S \Omega BH), BH]_0 = 0, \quad j \geq 0.$$

We therefore have

THEOREM 2.13. Let  $H$  be stable then  $\pi_j(B_+) \cong^p \pi_j(B_\infty)$ , all  $j$ . Hence,  $\rho$  is a homotopy equivalence.

This generalizes 2.2 of [7].

*Final Remarks 2.14.* The importance of the existence of a retraction to the total space was discussed in [6] and [7]. In particular, the existence of a retraction implies  $E_{\infty}^{p,q} = 0, q > 0$  in the spectral sequence of the fibration  $H \subseteq E_+ \xrightarrow{p} B_+$ . Indeed,  $H^*(E_+) = E_{\infty}^{*,0}$  splits off.

Since  $H \subseteq * \xrightarrow{p} BH$  of course admits a retraction to the total space. The classifying map  $BH \rightarrow B_\infty$  factors through  $B_+$ .

2.15.

$$\begin{array}{ccccc} H \subseteq * & \longrightarrow & E_+ & \longrightarrow & E_\infty \\ & & \downarrow & & \downarrow \\ & & \downarrow p_+ & & \downarrow p_\infty \\ BH & \longrightarrow & B_+ & \xrightarrow{\rho} & B_\infty \end{array}$$

Using 2.15, the observations about spectral sequences above and the comparison theorem, one would like to conclude

$$H^*(B_+) \cong H^*(E_+) \oplus H^*(BH).$$

In general this is not true because the coefficient system  $H^*(F)$  over  $B_+$  is not trivial. However, the machinery of [6] and [7] carries over to this more general situation and allows specific computations to be carried out. Hopefully, such information can be used to infer information about  $B_\infty$  and /or the classifying map  $BH \rightarrow B_\infty$  in such specific situations.

#### REFERENCES

1. G. Allaud, *On the Classification of Fibre Spaces*, Math. Zeitschr., **92** (1966), 110–125.
2. E. H. Brown, *Cohomology Theories*, Ann. of Math., **75** (1962), 467–484.
3. A. Dold, *Halberacte Homopifunktoren*, Lecture Notes in Mathematics. 12, Springer-Verlag, Berlin-New York, 1966.
4. A. Dold, R. Lashof, *Principal quasifibrations and fibre homotopy equivalence of bundles*, Illinois J. Math., **3** (1959), 285–305.
5. J. Milnor, *On Spaces having the homotopy type of C.W. complexes*, Trans. Amer. Math. Soc., **90** (1959), 272–280.
6. J. Siegel, *Cohomology operations in local coefficient theory*, Illinois J. Math., **15** (1971), 52–63.
7. ———, *k-invariants in local coefficient theory*, Proc. Amer. Math. Soc., **29** (1971), 169–174.

Received January 7, 1974 and in revised form April 25, 1974.

UNIVERSITY OF MISSOURI-ST. LOUIS

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, California 90024

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. A. BEAUMONT

University of Washington  
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION  
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

Copyright © 1975 Pacific Journal of Mathematics  
All Rights Reserved

# Pacific Journal of Mathematics

Vol. 60, No. 2

October, 1975

Waleed A. Al-Salam and A. Verma, <i>A fractional Leibniz <math>q</math>-formula</i> . . . . .	1
Robert A. Bekes, <i>Algebraically irreducible representations of <math>L_1(G)</math></i> . . . . .	11
Thomas Theodore Bowman, <i>Construction functors for topological semigroups</i> . . . . .	27
Stephen LaVern Campbell, <i>Operator-valued inner functions analytic on the closed disc. II</i> . . . . .	37
Leonard Eliezer Dor and Edward Wilfred Odell, Jr., <i>Monotone bases in <math>L_p</math></i> . . . . .	51
Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu, <i>On the existence of global classical solution of initial-boundary value problem for <math>cmu - u^3 = f</math></i> . . . . .	63
Y. Gordon, <i>Unconditional Schauder decompositions of normed ideals of operators between some <math>l_p</math>-spaces</i> . . . . .	71
Gary Grefsrud, <i>Oscillatory properties of solutions of certain <math>n</math>th order functional differential equations</i> . . . . .	83
Irvin Roy Hentzel, <i>Generalized right alternative rings</i> . . . . .	95
Zensiro Goseki and Thomas Benny Rushing, <i>Embeddings of shape classes of compacta in the trivial range</i> . . . . .	103
Emil Grosswald, <i>Brownian motion and sets of multiplicity</i> . . . . .	111
Donald LaTorre, <i>A construction of the idempotent-separating congruences on a bisimple orthodox semigroup</i> . . . . .	115
Pjek-Hwee Lee, <i>On subrings of rings with involution</i> . . . . .	131
Marvin David Marcus and H. Minc, <i>On two theorems of Frobenius</i> . . . . .	149
Michael Douglas Miller, <i>On the lattice of normal subgroups of a direct product</i> . . . . .	153
Grattan Patrick Murphy, <i>A metric basis characterization of Euclidean space</i> . . . . .	159
Roy Martin Rakestraw, <i>A representation theorem for real convex functions</i> . . . . .	165
Louis Jackson Ratliff, Jr., <i>On Rees localities and <math>H_i</math>-local rings</i> . . . . .	169
Simeon Reich, <i>Fixed point iterations of nonexpansive mappings</i> . . . . .	195
Domenico Rosa, <i><math>B</math>-complete and <math>B_r</math>-complete topological algebras</i> . . . . .	199
Walter Roth, <i>Uniform approximation by elements of a cone of real-valued functions</i> . . . . .	209
Helmut R. Salzmann, <i>Homogene kompakte projektive Ebenen</i> . . . . .	217
Jerrold Norman Siegel, <i>On a space between <math>BH</math> and <math>B_\infty</math></i> . . . . .	235
Robert C. Sine, <i>On local uniform mean convergence for Markov operators</i> . . . . .	247
James D. Stafney, <i>Set approximation by lemniscates and the spectrum of an operator on an interpolation space</i> . . . . .	253
Árpád Szász, <i>Convolution multipliers and distributions</i> . . . . .	267
Kalathoor Varadarajan, <i>Span and stably trivial bundles</i> . . . . .	277
Robert Breckenridge Warfield, Jr., <i>Countably generated modules over commutative Artinian rings</i> . . . . .	289
John Yuan, <i>On the groups of units in semigroups of probability measures</i> . . . . .	303