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SPAN AND STABLY TRIVIAL BUNDLES

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E. Thomas [19] introduced the notion of span of a differentiable manifold (or of a vector bundle). The notion of span can be extended in an obvious way to *PL*-microbundles, topological microbundles and spherical fibrations. In the case of a vector bundle or a microbundle the dimension of the fibre will be referred to as its rank. A spherical fibration with fibre homotopically equivalent to S^{k-1} will be said to be of rank k . In this paper we study stably trivial objects of rank k over a *CW*-complex of dimension $\leq k$ from each of the above collections. Then we determine the span of such stably trivial objects over *CW*-complexes of a "special type" yielding generalizations of the Bredon-Kosinski, Thomas theorem on the span of a closed differentiable π -manifold [3], [19]. Though originally *PL*-microbundles were defined only over simplicial complexes, in this paper by a *PL*-microbundle of rank k over a *CW*-complex X we mean an element of the set $[X, BPL(k)]$ of homotopy classes of maps of X into $BPL(k)$.

Throughout this paper X will denote a *CW*-complex and X^k will denote the k -skeleton of X . We write $\xi \in \text{Vect}(X) \{PL \text{ mic}(X), \text{Topmic}(X) \text{ or } \text{Sph}(X)\}$ to denote that ξ is a vector bundle a *PL*-microbundle, a topological microbundle or a spherical fibration over X . We write ξ^k to denote that ξ is of rank k . We write $R(X)$ for any one of $\text{Vect}(X)$, $PL \text{ mic}(X)$, $\text{Topmic}(X)$ or $\text{Sph}(X)$. The trivial object of rank k in $R(X)$ will be denoted by $\epsilon_{k,X}^k$. We write $\xi \in R_+(X)$ to denote that ξ is orientable. We write $O_X^k, \theta_X^k, \epsilon_X^k$ and k_X respectively for the *trivial* vector bundle, *PL*-microbundle, topological microbundle and spherical fibration of rank k over X .

Section 2 is concerned with stably trivial elements $\xi^k \in R(X)$ when $\dim X \leq k$. In Section 3 we introduce the notion of a Gauss map for a $\xi \in R(X)$. If $\xi^k \in R(X)$ is stably trivial, $\dim X \leq k$ and $R \neq \text{Topmic}$ we prove the existence of a Gauss map for ξ . If $R = \text{Topmic}$ the same result is true whenever $k \neq 4$. In Section 4 we prove the main result of this paper (Theorem 4.3). An immediate consequence of this theorem the analogue of Bredon-Kosinski, Thomas theorem could be derived in all the categories *Diff*, *PL*, *Top* or *Poincare Complexes* with "obvious" exceptions.

1. The kernel of $\pi_k(B_k) \rightarrow \pi_k(B_{k+1})$. We write B_k for any one of $BSO(k)$, $BPL^+(k)$, $B\text{Top}^+(k)$ or $B\text{SH}(k)$. For our later results

we need information about the kernel of $\pi_k(B_k) \rightarrow \pi_k(B_{k+1})$. When $B_k \neq B\text{Top}^+(k)$ the kernel of $\pi_k(B_k) \rightarrow \pi_k(B_{k+1})$ is well-known. Using the results of Kirby-Siebenmann [13] and Lashof-Rotherberg [16] we get information about the kernel when $B_k = B\text{Top}^+(k)$, for $k \neq 4$. Let T_{S^k} , t_{S^k} , τ_{S^k} and λ_{S^k} denote the tangent vector bundle, tangent *PL*-microbundle, tangent microbundle and the tangent spherical fibration of S^k . Let

$$\begin{aligned} K_k &= \ker \pi_k(BSP(k)) \rightarrow \pi_k(BSO(k)), \\ C_k &= \ker \pi_k(BPL^+(k)) \rightarrow \pi_k(BPL^+(k+1)), \\ K_k &= \ker \pi_k(B\text{Top}^+(k)) \rightarrow \pi_k(B\text{Top}^+(k+1)) \end{aligned}$$

and

$$K''_k = \ker \pi_k(BSH(k)) \rightarrow \pi_k(BSH(k+1)).$$

It is well-known that the obvious map $\pi_k(BSO(k)) \rightarrow \pi_k(BSH(k))$ carries K_k isomorphically onto K''_k and that

$$(1) \quad K_k \simeq K''_k \simeq \begin{cases} Z & \text{if } k \text{ is even} \\ O & \text{if } k = 1, 3 \text{ or } 7 \\ Z_2 & \text{if } k \text{ is odd and } \neq 1, 3, 7. \end{cases}$$

with T_{S^k} (respy λ_{S^k}) as generator.

According to a result of W. M. Hirsch the map $\pi_k(BSO(k)) \rightarrow \pi_k(BPL^+(k))$ carries K_k onto C_k . A reference for this is [7]. Since the composite map $K_k \rightarrow C_k \rightarrow K''_k$ is an isomorphism, it follows that

$$(2) \quad K_k \simeq C_k \text{ and that } t_{S^k} \text{ generates } C_k.$$

PROPOSITION 1.1. *For $k \neq 4$, K'_k is cyclic and is generated by τ_{S^k} .*

$$(3) \quad \text{Moreover } K'_k \simeq \begin{cases} Z & \text{if } k \text{ is even and } \neq 4 \\ O & \text{if } k = 1, 3 \text{ or } 7 \\ Z_2 & \text{if } k \text{ is odd and } \neq 1, 3, 7. \end{cases}$$

Proof. Since the composite map $K_k \rightarrow K'_k \rightarrow K''_k$ is an isomorphism it follows that $K_k \rightarrow K'_k$ is an injection for all k .

Let $k \geq 5$. In the following commutative diagram where the horizontal rows are exact and the vertical maps are the obvious ones,

$$\begin{array}{ccccccc}
O \rightarrow & K_k & \rightarrow & \pi_k(BSO(k)) & \rightarrow & \pi_k(BSO(k+1)) \\
& \text{onto} \downarrow & & \downarrow & & \downarrow \\
O \rightarrow & C_k & \rightarrow & \pi_k(BPL^+(k)) & \rightarrow & \pi_k(BPL^+(k+1)) \\
& \downarrow & & \downarrow \text{onto} & & \downarrow \\
O \rightarrow & K_k & \rightarrow & \pi_k(B\text{Top}^+(k)) & \rightarrow & \pi_k(B\text{Top}^+(k+1))
\end{array}$$

DIAGRAM 1

the map $\pi_k(BPL^+(k)) \rightarrow \pi_k(B\text{Top}^+(k))$ is onto and $\pi_k(BPL^+(k+1)) \rightarrow \pi_k(B\text{Top}^+(k+1))$ for $k \geq 5$ by [13] or [16]. As already observed $K_k \rightarrow C_k$ is onto according to a result of M. W. Hirsch [7]. Standard diagram chasing using Diagram 1 yields $K_k \rightarrow K'_k$ is onto for $k \geq 5$.

For $k \leq 3$ it is known that $SO(k) \rightarrow \text{Top}^+(k)$ is a homotopy equivalence [15]. Hence for $k \leq 2$ we have $K_k \simeq K'_k$. When $k = 3$ we have $O = \pi_2(SO(3)) \simeq \pi_3(BSO(3)) \simeq \pi_3(B\text{Top}^+(3))$. Hence $K_3 = O = K'_3$. This completes the proof of 1.1.

2. Stably trivial elements $\xi \in R(X)$. Suppose $\dim X \leq k$ and $\xi^{k+1} \in R(X)$ is stably trivial. Then for $R \neq \text{Topmic}$ it is known that $\xi^{k+1} \simeq \epsilon_{R,X}^{k+1}$. This is actually a consequence of

$$(4) \quad \pi_i(B_{k+1}, B_k) = 0 \quad \text{for } i \leq k$$

whenever $B_k = BSO(k)$, $BPL^+(k)$ or $BSH(k)$. For $B_k = BSH(k)$, 4 is due to I. M. James [10]. When $B_k = BPL^+(k)$ it is due to Haefliger and Wall [7]. We write B_∞ to denote one of BSO , BPL^+ , $B\text{Top}^+$ or BSH .

LEMMA 2.1. *Let $\dim X \leq k$ and $\xi^{k+1} \in \text{Topmic}(X)$ be stably trivial. Then $\xi^{k+1} \simeq \epsilon_X^{k+1}$ whenever $k \neq 3$.*

Proof. From Kirby-Siebenmann [13] or Lashof-Rothenberg [16] we have $\pi_i(B\text{Top}^+(l+1), B\text{Top}^+(l)) = 0$ for $i \leq l$ and $l \geq 5$. As an immediate consequence of this and obstruction theory one gets $[X, B\text{Top}^+(k+1)] \rightarrow [X, B\text{Top}^+]$ to be an isomorphism for $k \geq 4$.

Now let $k \leq 2$. Since $\pi_i(B\text{Top}^+, BPL^+) \simeq \pi_{i-1}(\text{Top}^+, PL^+) = 0$ for $i \neq 4$, we see that $[X, BPL^+] \rightarrow [X, B\text{Top}^+]$ is an isomorphism. Also $SO(k+1) \rightarrow PL^+(k+1)$ and $PL^+(k+1) \rightarrow \text{Top}^+(k+1)$ are homotopy equivalences for $k \leq 2$. Hence each of the maps $[X, BSO(k+1)] \rightarrow [X, BPL^+(k+1)]$, $[X, BPL^+(k+1)] \rightarrow [X, B\text{Top}^+(k+1)]$ is an isomorphism. From 4 we see that $[X, BPL^+(k+1)] \rightarrow [X, BPL^+]$ is an isomorphism. Now Diagram 2 below immediately gives $[X, B\text{Top}^+(k+1)] \rightarrow [X, B\text{Top}^+]$ an isomorphism.

$$\begin{array}{ccc}
[X, BPL^+(k+1)] & \xrightarrow{\cong} & [X, BPL^+] \\
\cong \downarrow & & \cong \downarrow \\
[X, BTop^+(k+1)] & \longrightarrow & [X, BTop^+]
\end{array}$$

DIAGRAM 2

This completes the proof of Lemma 2.1.

PROPOSITION 2.2. *Let X be a CW-complex of dimension $\leq k$ where $k = 3$ or 7 . Let $\xi^k \in R_+(x)$ be such that $\xi^k | X^{k-1} \simeq \epsilon_{R, X^{k-1}}^k$. Then $\xi \simeq \epsilon_{R, X}^k$ whenever $R \neq \text{Sph}$.*

Proof. We have

$$(5) \quad O = \pi_3(BSO(3)) \simeq \pi_3(BPL^+(3)) \simeq \pi_3(BTop^+(3))$$

From results in Section 1 we see that $\ker \pi_7(B_7) \rightarrow \pi_7(B_8)$ is zero. From $\pi_i(B_{k+1}, B_k) = 0$ for $i \leq k$ and $k \geq 5$ it now follows that $\pi_7(B_7) \rightarrow \pi_7(B_8)$ and $\pi_7(B_8) \rightarrow \pi_7(B_\infty)$ are isomorphisms. From Bott [2] $\pi_6(SO) = 0$. From Hirsch and Mazur [8], [9] $\pi_7(BPL^+, BSO) \simeq \Gamma_6$ the group of concordance classes of smooth structures on S^6 . It is known [12] that $\Gamma_6 = 0$. Combining these with the result $\pi_7(BTop^+, BPL^+) = 0$ of Kirby-Siebenmann we get

$$(6) \quad O = \pi_7(BSO(7)) \simeq \pi_7(BPL^+(7)) \simeq \pi_7(BTop^+(7))$$

Let $\mu: X^{k-1} \rightarrow X$ denote the inclusion. If $X = X^{k-1} \cup_{i \in J} e_i^k$ we have a cofibration $\mu: X^{k-1} \rightarrow X$ with cofibre $\vee_{i \in J} S_i^k$. Let $c: X \rightarrow \vee_{i \in J} S_i^k$ be got by collapsing X^{k-1} to a point. In the Puppe exact sequence

$$\left[\vee_{i \in J} S_i^k, B_k \right] \xrightarrow{c^*} [X, B_k] \xrightarrow{\mu^*} [X^{k-1}, B_k]$$

we have $\mu^*(\xi^k) = 0$, since $\xi^k | X^{k-1}$ is trivial. Hence \exists an $x \in [\vee_{i \in J} S_i^k, B_k]$ such that $c^*(x) = \xi^k$. By 5 and 6, $\pi_k(B_k) = 0$ for $k = 3$ and 7 , whenever $B_k \neq BSH(k)$. Hence $x = 0$, which in turn yields $\xi^k = 0$ in $[X, B_k]$.

REMARKS.

2.3. If $F(k)$ denotes the subspace of $SH(k+1)$ consisting of base point preserving maps it is known [10] that

$$\pi_3(BSH(3)) \simeq \pi_2(SH(3)) \simeq \pi_2(F(3)) \simeq \pi_3(S^3) \simeq Z_2$$

and that

$$\pi_7(BSH(7)) \simeq \pi_6(SH(7)) \simeq \pi_6(F(7)) \simeq \pi_{13}(S^7) \simeq Z_2.$$

Let $k = 3$ or 7 . We have a CW structure X on S^k such that $X^{k-1} = *$ (base point). If $\xi^k \in \text{Sph}(X)$ is represented by the nonzero element of $[X, BSH(k)] \simeq \pi_{k-1}(SH(k)) \simeq Z_2$ then clearly $\xi^k|X^{k-1}$ is trivial, but ξ^k itself is not trivial.

2.4. Any $\xi^1 \in R_+(X)$ is trivial whatever be the dimension of X .

PROPOSITION 2.5. *Let $\eta^k \in R(X)$ be stably trivial and $\dim X \leq k$. Then*

$$\eta^k \oplus \epsilon_{R,X}^1 \simeq \epsilon_{R,X}^{k+1}.$$

Proof. As commented already, this is well-known when $R \neq \text{Topmic}$. For $R = \text{Topmic}$ and $k \neq 3$ this is an immediate consequence of Lemma 2.1. Let now $k = 3$. Then $\eta^3|X^2$ is stably trivial. From Lemma 2.1 applied to $\eta^3|X^2$ we get $\eta^3|X^3 \simeq \epsilon_{R,X^2}^3$. Now proposition 2.2 yields $\eta^3 \simeq \epsilon_{R,X}^3$. Hence $\eta \oplus \epsilon_{R,X}^1 \simeq \epsilon_{R,X}^3$.

3. Gauss maps.

DEFINITION 3.1. Let $\xi^k \in R(X)$. A map $f: X \rightarrow S^k$ will be called a Gauss map for ξ if $\xi = f^*(\tau_{R,S^k})$ in $R(X)$, where $\tau_{R,S^k} = T_{S^k}, t_{S^k}, \tau_{S^k}$ or λ_{S^k} according as $R = \text{Vect}, PL \text{ mic}, \text{Topmic}$ or Sph .

When $\xi \in R(X)$ admits of a Gauss map then necessarily ξ is stably trivial. The main result of this section is the following:

THEOREM 3.2. *Let $\dim X \leq k$ and $\xi^k \in R(X)$ stably trivial. There exists a Gauss map for ξ whatever be k if $R \neq \text{Topmic}$ and for $k \neq 4$ if $R = \text{Topmic}$.*

In the proof of this theorem we will be making use of the following lemma.

LEMMA 3.3. *Let Y be a CW complex of dimension $\leq k-1$. Then $[\Sigma Y, B_k] \rightarrow [\Sigma Y, B_{k+1}]$ is onto whatever be k if $B_k \neq B\text{Top}^+(k)$, and for $k \neq 3, 4$ if $B_k = B\text{Top}^+(k)$.*

Proof. Let $Y = Y^{k-2} \cup_{\nu \in J} e_{\nu}^{k-1}$, $i: Y^{k-2} \rightarrow Y$, $j: B_k \rightarrow B_{k+1}$ the inclusion maps and $h: Y \rightarrow \bigvee_{\nu \in J} S^{k-1}$ got by collapsing Y^{k-2} to a

point. Lemma 3.3 follows immediately by diagram chasing using the following commutative diagram coming from Puppe exact sequences where $(\Sigma h)^*$, $(\Sigma i)^*$ and all the j_* are group homomorphisms.

$$\begin{array}{ccccccc}
 \left[\Sigma \bigvee_{\nu \in J} S^{k-1}, B_k \right] & \xrightarrow{(\Sigma h)^*} & [\Sigma Y, B_k] & \xrightarrow{(\Sigma i)^*} & [\Sigma(Y^{k-2}), B_k] & \xrightarrow{\partial} & \left[\bigvee_{\nu \in J} S^{k-1}, B_k \right] \\
 \text{onto } a \downarrow j_* & & b \downarrow j_* & & c \downarrow j_* & & d \downarrow j_* \\
 \left[\Sigma \bigvee_{\nu \in J} S^{k-1}, B_{k+1} \right] & \xrightarrow{(\Sigma h)^*} & [\Sigma Y, B_{k+1}] & \longrightarrow & [\Sigma(Y^{k-2}), B_{k+1}] & \xrightarrow{F} & \left[\bigvee_{\nu \in J} S^{k-1}, B_{k+1} \right]
 \end{array}$$

DIAGRAM 3

Here the maps j_* marked by c and d are isomorphisms under the conditions in Lemma 3.3 and the j_* marked by a is onto.

Proof of Theorem 3.2. Let $X = X^{k-1} \cup_{\gamma \in J} e_{\gamma}^k$, $\mu: X^{k-1} \rightarrow X$ the inclusion and $c: X \rightarrow \bigvee_{\gamma \in J} S^k$ the map collapsing X^{k-1} to a point. Consider the following diagram where the horizontal rows are part of Puppe exact sequences of the confibration μ .

$$\begin{array}{ccccccc}
 [\Sigma(X^{k-1}), B_k] & \xrightarrow{\partial} & \left[\bigvee_{\gamma \in J} S^k, B_k \right] & \xrightarrow{c^*} & [X, B_k] & \xrightarrow{\mu^*} & [X^{k-1}, B_k] \\
 \downarrow j_* & & \downarrow j_* & & \downarrow j_* & & \downarrow j_* \\
 [\Sigma(X^{k-1}), B_{k+1}] & \xrightarrow{\partial} & \left[\bigvee_{\gamma \in J} S^k, B_{k+1} \right] & \xrightarrow{c^*} & [X, B_{k+1}] & \xrightarrow{\mu^*} & [X^{k-1}, B_{k+1}]
 \end{array}$$

DIAGRAM 4

By Lemma 2.1 we have $\mu^*(\xi^k) = 0$ in $[X^{k-1}, B_k]$ whenever $R \neq \text{Topmic}$ and $k-1 \neq 3$. By proposition 2.5, $j_*(\xi^k) = 0$ in $[X, B_{k+1}]$. From $\mu^*(\xi) = 0$ we get an element $u \in [\bigvee_{\gamma \in J} S^k, B_k]$ such that $c^*(\mu) = \xi$. Then $j_*(\mu) = x \in [\bigvee_{\gamma \in J} S^k, B_{k+1}]$ satisfies $c^*(x) = j_*(\xi) = 0$. Hence $\exists b \in [\Sigma(X^{k-1}), B_{k+1}]$ such that $x^b = 0$ where x^b is got from x by the action of $[\Sigma(X^{k-1}), B_{k+1}]$ on $[\bigvee_{\gamma \in J} S^k, B_{k+1}]$.

By Lemma 3.3, $\exists a \in [\Sigma(X^{k-1}), B_k]$ such that $j_*(a) = b$ except when $R = \text{Topmic}$ and $k = 3$ or 4 . Then the element $\mu' = \mu^a \in [\bigvee_{\gamma \in J} S^k, B_k]$ satisfies $j_*(\mu') = 0$ and $c^*(\mu') = \xi$. Identifying $[\bigvee_{\gamma \in J} S^k, B_k]$ with the direct product $\prod_{\gamma \in J} [S^k, B_k]$, μ' corresponds to an element $(\mu')_{\gamma \in J}$ where $\mu'_{\gamma} \in \ker j_*: \Pi_k(B_k) \rightarrow \Pi_k(B_{k+1})$. Using 1, 2, 3 of §1 we see that $\mu'_{\gamma} = d_{\gamma} \tau_{R, S^k}$ {for some $d_{\gamma} \in \mathbb{Z}$ if k is even, $d_{\gamma} \in \mathbb{Z}_2$ if k is odd}. Let $g_{\gamma}: S^k \rightarrow S^k$ be a map of degree d_{γ} and $\varphi: S^k \rightarrow B_k$ a classifying map for τ_{R, S^k} . Then clearly the composite map

$$\bigvee_{\gamma \in J} S^k \xrightarrow{vg_\gamma} \bigvee_{\gamma \in J} S^k \xrightarrow{\nabla} S^k \xrightarrow{\varphi} B_k \quad \text{represents} \quad \mu' = (\mu'_\gamma)_{\gamma \in J}.$$

From $c^*(\mu') = \xi$ it follows that $f^*(\tau_{R,S^k}) \simeq \xi$ where

$$f = \nabla \circ \left(\bigvee_{\gamma \in J} g_\gamma \right) \circ c : X \rightarrow S^k.$$

To complete the proof of Theorem 3.2 we have still to consider the case $R = \text{Topmic}$, $k = 3$. In this case $\xi|X^2$ is stably trivial of rank 3 over a 2-dimensional complex. By Lemma 2.1, $\xi|X^2 = \epsilon_{X^2}^3$. By Proposition 2.2, $\xi \simeq \epsilon_X^3$. Since $\tau_{S^3} \simeq \tau_{S^3}$ we have $f^*(\tau_{S^3}) \simeq \xi$. This completes the proof of Theorem 3.2.

4. Span of any $\xi \in R(X)$. We now recall the definition of span originally due to E. Thomas [19].

DEFINITION 4.1. Let $\xi \in R(X)$. The span of ξ is defined to be the largest integer l with the property $\xi \simeq \epsilon_{R,X}^l \oplus \eta$ for some $\eta \in R(X)$.

In this section we will be interested in complexes of the form $X = L \cup e^k$ where $\dim L \leq k - 1$. It is easy to see using the exact homology sequence of the pair (X, L) and the fact that $H_{k-1}(L)$ is free abelian that either $H_k(X) = 0$ or $H_k(X) \simeq Z$. If we further assume that $\text{Ext}(H_{k-1}(X), Z) = 0$ it follows from the universal co-efficient theorem that either $H^k(X) = 0$ or $H^k(X) \simeq Z$. By Hopf's classification theorem $[X, S^k] \simeq H^k(X)$. When $H_k(X) = 0$ every map $X \rightarrow S^k$ is homotopically trivial, when $H_k(X) \simeq Z$ the map $[f] \rightarrow \deg f$ provides an isomorphism of $[X, S^k]$ with l . Let $l \leq k$ and $\pi: V_{k+1,l+1} \rightarrow S^k$ denote the map which carries any orthonormal $(l+1)$ frame $(\tilde{v}_1, \dots, \tilde{v}_{l+1})$ in \mathbf{R}^{k+1} to the vector \tilde{v}_{l+1} . We will be considering mainly complexes $X = L \cup e^k$ with $\dim L \leq k - 1$ and satisfying the following condition:

(**) Suppose $\theta: X \rightarrow S^k$ is a map admitting of a lift $\varphi: X \rightarrow V_{k+1,l+1}$ (i.e. $\pi \circ \varphi = \theta$) and suppose $\deg \theta = 1$. Then $l \leq \sigma_k$, where $\sigma_k = 2^{c(k)} + 8d(k) - 1$ with $k+1 = 2^{c(k)} 16^{d(k)} b_k$, $0 \leq c(k) \leq 3$, $d(k) \geq 0$ and b_k odd.

DEFINITION 4.2. Let k be an integer ≥ 4 . A CW-complex X will be referred to as a "special complex" of dimension k

- (i) $X = L \cup e^k$ with $\dim L \leq k - 1$
- (ii) $\text{Ext}(H_{k-1}(X), Z) = 0$ and
- (iii) condition (**) is valid whenever k is odd.

Observe that when $H_k(X) = 0$ condition (**) is emptyly valid, since there are no maps $\theta: X \rightarrow S^k$ of degree 1 then.

THEOREM 4.3.

(A) Let $\xi^2 \in R_+(X)$ with X an arbitrary CW-complex. Then $\text{span } \xi = 0$ or 2 .

(B) Let $k = 1, 3$ or 7 and $\xi^k \in R(X)$ stably trivial with $\dim X \leq k$. Then $\text{span } \xi = k$.

(C) Let $k \geq 4$ and $k \neq 7$, X a special complex of dimension k and $\xi^k \in R(X)$ stably trivial. Then

- (i) $\text{span } \xi = \sigma_k$ or k whenever $R = \text{Vect}$
- (ii) if $R = PL \text{ mic}$ or Sph , $\text{span } \xi = \sigma_k$ or k whenever $k \neq 15$
- (iii) if $R = \text{Topmic}$, $\text{span } \xi = \sigma_k$ or k whenever $k \neq 4$ and 15 .

LEMMA 4.4. Let X be a CW-complex of dimension $\leq k$, ξ^k a vector bundle, $\alpha \in R(X)$ the object in $R(X)$ underlying ξ . Let l be any integer $\leq (k-1)/2$. Then $\alpha \simeq \beta \oplus \epsilon_{R,X}^l$ in $R(X)$ if and only if $\xi \simeq \eta \oplus O_X^l$ in $\text{Vect}(X)$.

Proof. Immediate consequence of a classical result of I. M. James [Proposition 1.2 in [10]] and obstruction theory.

LEMMA 4.5. The span of $\tau_{R,S^k} = \sigma_k$.

For $R = \text{Vect}$ this is a classical result of J. F. Adams [1]. For $R = \text{Topmic}$ this is Theorem 1.1 in [20]. For $R = PL \text{ mic}$ or Sph the proof is exactly similar to that of Theorem 1.1 in [20].

LEMMA 4.6. Let l be any integer $\leq (k-1)/2$, $f: X \rightarrow S^k$ a Gauss map for $\alpha^k \in R(X)$ and $\dim X \leq k$. Suppose $\alpha \simeq \beta \oplus \epsilon_{R,X}^l$. Then \exists a map $\varphi: X \rightarrow V_{k+l,l+1}$ such that $f = \pi \circ \varphi$.

Proof. This is an immediate consequence of Lemma 4.4 applied to the vector bundle $\xi^k = f^*(T_{S^k})$.

LEMMA 4.7. Let X be a CW-complex of dimension k satisfying conditions (i) and (ii) of Definition 4.2. Suppose k is odd, $H_k(X) \neq 0$ and a Gauss map $f: X \rightarrow S^k$ for $\xi^k \in R(X)$ has odd degree. Then any map $g: X \rightarrow S^k$ of degree 1 is a Gauss map for ξ .

Proof. This is an immediate consequence of the fact that $2\tau_{R,S^k} = 0$ in $\pi_k(B_k)$ whenever k is odd.

LEMMA 4.8. Let X be a CW-complex of dimension $k \geq 4$ and satisfying (i) and (ii) of Definition 4.2. Suppose k is even, a Gauss map $f: X \rightarrow S^k$ for $\xi^k \in R(X)$ has $\deg f \neq 0$. Then $\text{span } \xi = 0 = \sigma_k$.

Proof. Denote the span of ξ by $\sigma(\xi)$. If $\sigma(\xi) \neq 0$ we can find a $\eta^{k-1} \in R(X)$ such that $\xi \simeq \eta \oplus \epsilon_{R,X}^1$. Since $1 \leq (k-1)/2$, by Lemma 4.6 \exists a map $\varphi: X \rightarrow V_{k+1,2}$ satisfying $\pi \circ \varphi = f$. Since $H_k(V_{k+1,2}) \simeq Z_2$ it follows that $\deg f = 0$, contradicting the assumption $\deg f \neq 0$.

LEMMA 4.9. *Let X be a CW-complex of dimension k , satisfying conditions (i) and (ii) of Definition 4.2. Suppose $f: X \rightarrow S^k$ is a Gauss map for $\xi^k \in R(X)$. Then $\xi^k \simeq \epsilon_{R,X}^k$ whenever one of the following holds good.*

- (a) $H_k(X) = 0$
- (b) $H_k(X) \neq 0$ (hence $H_k(X) \simeq Z$) and $\deg f = 0$
- (c) $H_k(X) \neq 0$, k odd and $\deg f$ is even.

Proof. (a) and (b) are immediate consequences of Hopf's classification theorem. (c) is immediate from $2\tau_{R,S^k} = 0$ in $\pi_k(B_k)$ whenever k is odd.

Proof of Theorem 4.3. We write $\sigma(\xi)$ for the span of ξ .

(A) If $\sigma(\xi^2) \neq 0$, $\xi^2 \simeq \eta \oplus \epsilon_{R,X}^1$ for some $\eta^1 \in R_+(X)$. By Remark 2.4, $\eta^1 \simeq \epsilon_{R,X}^1$. Hence $\xi^2 \simeq \epsilon_{R,X}^2$. Thus $\sigma(\xi^2) = 2$.

(B) Immediate consequence of Theorem 3.2 and the fact $\tau_{R,S^k} \simeq \epsilon_{R,S^k}^k$ for $k = 1, 3, 7$.

(C) By Theorem 3.2, \exists a Gauss map $f: X \rightarrow S^k$ for ξ . If $H_k(X) = 0$, by Lemma 4.9 (a) we get $\sigma(\xi) = k$. If $H_k(X) \neq 0$ and $\deg f = 0$, by Lemma 4.9 (b) we get $\sigma(\xi) = k$. If k is odd and $\deg f$ is even by Lemma 4.9 (c) we get $\sigma(\xi) = k$. If $k \geq 4$ is even and $\deg f \neq 0$, by Lemma 4.8 we get $\sigma(\xi) = 0 = \sigma_k$.

Hence to complete the proof of (C) we have only to consider the case $k \geq 5$ odd and $k \neq 7$ and $\deg f$ odd. The existence of a Gauss map implies that $\sigma(\xi) \geq \sigma_k$. By Lemma 4.7, any map $g: X \rightarrow S^k$ of $\deg 1$ is a Gauss map for ξ . If possible let $\sigma(\xi) > \sigma_k$. For $R = \text{Vect}$ this means that \exists a map $\varphi: X \rightarrow V_{k+1,l+1}$ satisfying $\pi \circ \varphi = g$ for some $l > \sigma_k$, contradicting the validity of condition (**). Now suppose $R \neq \text{Vect}$. For $k \geq 5$ odd, $k \neq 7$ and 15 direct checking shows $\sigma_k + 1 \leq (k-1)/2$. If $\sigma(\xi) > \sigma_k$ then $\xi \simeq \eta \oplus \epsilon_{R,X}^l$ with $l = \sigma_k + 1$. From Lemma 4.6 we see that $\exists \varphi: X \rightarrow V_{k+1,l+1}$ such that $\pi \circ \varphi = g$, again contradicting (**).

5. Poincare complexes with $\nu_X = 0$. For any Poincare complex X let $\nu_X \in J(X)$ denote the spivak normal fibration of X . From the results of C.T.C. Wall [21], it follows that any Poincare complex X of formal dimension $k \neq 2$ is of the homotopy type of a CW-complex of dimension k and that if $k \neq 3$, X is homotopically

equivalent to $L \cup e^k$ with $\dim L \leq k - 1$. The methods employed in [5], [6] allow one to define unstable tangent spherical fibration for Poincare complexes of formal dimension $\neq 2$.

LEMMA 5.1. *Any connected Poincare complex X of formal dimension $k \geq 4$ with $\nu_X = 0$ is of the homotopy type of a "special complex" of dimension k (as given in Definition 4.2).*

Proof. From $H_{k-1}(X) \simeq H^1(X) \simeq \text{Hom}(H_1(X), Z)$ and finite generation of $H_1(X)$ we see that $H_{k-1}(X)$ is free abelian. Hence $\text{Ext}(H_{k-1}(X), Z) = 0$. As already commented X is of the homotopy type of $L \cup e^k$ where $\dim L \leq k - 1$. The Thom space of the normal fibration ν_k is reducible. Since $\nu_X = 0$ it follows that the Thom space of the trivial vector bundle σ_X^{k+1} is reducible. Suppose $k \geq 5$ is odd. By the Browder-Novikov theorem [4], [11] it now follows that \exists a closed C^∞ manifold M^k of dimension k and a homotopy equivalence $f: M^k \rightarrow X$ such that $f^*(O_X^{k+1}) = O_M^{k+1}$ is the stable normal bundle of M . This means M is a closed differentiable π -manifold. Lemma 5.1 is now an immediate consequence of Lemma 3.2 in [3].

For any PL (resp topological) manifold M the PL (resp topological) span of M is defined to be the span of the PL (resp topological) tangent microbundle of M . For a Poincare complex X the spherical span of X is defined to be the span of the unstable tangent spherical fibration of X . As an immediate consequence of Theorem 4.3 we get all the following results at one stroke.

THEOREM 5.2. (1) *Let M^k be a closed Diff, PL-or Top π -manifold of dimension k , with $k \neq 15$ in the case of a PL-manifold and $k \neq 4$ and 15 in the case of a topological manifold. Then the span (resp PL-span or Top span) of M is either σ_k or k .*

(2) *If X is a Poincare complex of formal dimension $k \neq 2$ and 15 with $\nu_X = 0$ in $J(X)$, then the spherical span of $X = \sigma_k$ or k .*

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