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We generalize Pym's decomposition $w = \mu_E * w_H * \mu_F$ of idempotent probability measures to the decomposition $\mu_E * \mathcal{H}(w_H) * \mu_F$ of the maximal groups of units in semigroup of probability measures on a compact semitopological semigroup. We also prove that $\mathcal{H}(w) \cong \mathcal{H}(w_H) \cong N(H)/H$ algebraically and topologically. With these characterizations, we verify Rosenblatt's necessary and sufficient condition for the convergence of a convolution sequence $(\nu^n)_{n \geqq 1}$ of a probability measure ν on a compact topological semigroup.

1. Introduction. Let S denote a compact semitopological semigroup (i.e., the multiplication is separately continuous) and $(C(S), \| \ \|)$ the Banach space of all bounded real-valued continuous functions on S. Then $M^b(S)$ which is defined as the norm dual of $C(S)$ is a Banach algebra under $\|\mu\| = \sup\{|\mu(f)| : \|f\| \leqq 1\}$ and the convolution $*$ which is defined via $\mu * \nu(f) = \int f(xy)\mu(dx)\nu(dy)$ on $C(S)$. Let $P(S)$ be the totality of probability measures on S, which consists of all positive measures with norm 1 in $M^b(S)$. Then $P(S)$ is a compact semitopological semigroup under $*$ and the weak* topology which is the topology of pointwise convergence on $C(S)$ **[4]**. If S is topological (i.e., the multiplication is jointly continuous), then $P(S)$ is topological (Prop. 4, **[9]**).

It is known that every compact semitopological semigroup has a minimal ideal which is not necessarily closed except in the case S is topological **[7]**. We thus introduce the following definition:

> A compact semitopological semigroup is called topologically simple if its minimal ideal is dense in it.

For a subsemigroup T of S, we use $E(S)$ and $M(T)$ to denote the totality of idempotents and the minimal ideal in S respectively. For a subsemigroup A of $P(S)$, we write $D(A) = \cup\{\text{supp}\,\mu : \mu \in A\}$ and $\text{supp}\,A = \overline{D(A)}$, where $\text{supp}\,\mu$ denotes the support of μ.

In the remainder, S will always denote a compact semitopological semigroup except mentioned especially.

2. The structure of an idempotent probability measure.

PROPOSITION 2.1. *Let K be a compact topologically simple subsemigroup in S. Then*
1. $E(M(K)) \neq \varnothing$

For $e \in E(M(K))$, we have
2. (a) $H = eKe$ *is a compact topological subgroup with identity* e
   (b) $E = E(Ke)$ (resp. $F = E(eK)$) *is a left (resp. right) zero compact topological subsemigroup*
   (c) $eE = Fe = e$, $FH = HE = H$ *and* $FE \subseteq H$
   (d) $M(K) = EHF = [E, H, F]$ *via*

$$(x, g, y)(x', g', y') = (x, gyx'g', y')$$

   (e) $Ke = (EHF)e = EH$ *and* $eK = e(EHF) = HF$
3. (a) $P(E)$ *(resp. $P(F)$) is a left (resp. right) zero compact topological subsemigroup. In particular,* $E(P(E)) = P(E)$ *and* $E(P(F)) = P(F)$
   (b) $\delta_e^* P(E) = P(F)^* \delta_e = \delta_e$, *where δ_e is the point-mass at* e
   (c) $P(F)^* P(E) \subseteq P(H)$. *In particular,*

$$w_H * P(F)^* P(E) = P(F)^* P(E)^* w_H = w_H,$$

where $w_H^2 = w_H$ is the Haar measure on H
   (d) $P(E) * w_H * P(F) \subseteq E(P(S))$.

Proof. 1. (See the proof of 3.4, p. 67, **[1]**).
2. (See p. 500, **[7]**; Thm. 2, p. 124, **[3]**).
3. (a) For $\mu, \nu \in P(E)$,

$$\mu * \nu(f) = \int f(xy)\mu(dx)\nu(dy) = \int f(x)\mu(dx)\nu(dy) = \mu(f).$$

Hence $P(E)$ is left zero. Furthermore, by 2(b) we see that $P(E)$ is a compact topological subsemigroup in $P(S)$.
   (b) This follows from 2(c).
   (d) Let $\mu = \mu_E * w_H * \mu_F \in P(E) * w_H * P(F)$. Then

$$\mu^2 = \mu_E * (w_H * \mu_F * \mu_E) * w_H * \mu_F = \mu_E * w_H * \mu_F.$$

LEMMA A. $\operatorname{supp}(\mu * \nu) = \overline{(\operatorname{supp} \mu \ \operatorname{supp} \nu)}$ *in* $P(S)$.

Proof. **[4]**.

PROPOSITION 2.2. *Let $w^2 = w \in P(S)$. Then*

1. supp w *is a compact topologically simple subsemigroup*
2. $w = \mu_E * w_H * \mu_F$, *where*

(a) $H = e(\text{supp } w)e$, $E = E((\text{supp } w)e)$ *and* $F = E(e(\text{supp } w))$ *for an* $e \in E(M(\text{supp } w))$

(b) $\mu_E \in P(E)$ *with* $\text{supp } \mu_E = E$

(c) $\mu_F \in P(E)$ *with* $\text{supp } \mu_F = F$

(d) $w_H^2 = w_H$ *is the Haar measure on* H

3. $w_H = w_H * \mu_F * \mu_E = \mu_F * \mu_E * w_H$
4. $w_H = w_H * w * w_H = w_H * \mu_F * w * \mu_E * w_H$.

Proof. 1. We refer it to (p. 500, [7]).

2. This is a result of 1 and Proposition 2.1.
3. This is a result of 3(c) in Proposition 2.1.
4. We prove the first equality only. As $eEHFe \subseteq H$,

$$w_H * w * w_H = w_H * (w_H * \mu_E * w_H * \mu_F * w_H) * w_H = w_H.$$

PROPOSITION 2.3. $E(P(S)) = \cup\{P(E)^* w_H * P(F)$: *$K$ is a compact topologically simple subsemigroup*$\}$.

3. A characterization of the maximal group of units. For $e \in E(S)$ we denote by $\mathcal{H}(e)$ the maximal group of units with identity e in the compact subsemigroup eSe. We will see that $\mathcal{H}(e)$ is in general a locally compact topological subgroup in the relative topology of S and $\mathcal{H}(e)$ is closed and so compact in the case S is topological.

In this section, we maintain that $w^2 = w = \mu_E * w_H * \mu_F$ is as in Proposition 2.2. In particular, H is a compact subgroup of $\mathcal{H}(e)$.

LEMMA B. *$\mathcal{H}(e)$ is a locally compact topological subgroup in the relative topology of S. Furthermore, if S is topological, then $\mathcal{H}(e)$ is a closed and hence compact subgroup.*

Proof. As $\mathcal{H}(e)$ is a topological subgroup in eSe (Cor. 6.3, pp. 282–283, [6]), $\mathcal{H}(e)$ is a closed subsemigroup in eSe (3.1, p. 65, [1]). Without losing generality, we may assume that $S = eSe = \mathcal{H}(e)$. Suppose that $\mathcal{H}(e)$ is not locally compact. Then $\mathcal{H}(e)$ is not open in S. Thus if 0 is an open neighborhood of e in S, then $0 \cap (S - \mathcal{H}(e)) \neq \emptyset$, for translation by an element of $\mathcal{H}(e)$ is a homeomorphism of S. Now, we choose a relatively compact open neighborhood U of e in S. Then $(U \cap \mathcal{H}(e))^{-1}$ is open in $\mathcal{H}(e)$ and contains e, so there is an open neighborhood V of e in S so that $V \cap \mathcal{H}(e) = (U \cap \mathcal{H}(e))^{-1}$. Then $U \cap V$ is an open neighborhood of e in S so that $(U \cap V) \cap \mathcal{H}(e)$ is symmetric (i.e., $h \in (U \cap V) \cap \mathcal{H}(e)$ iff $h^{-1} \in (U \cap V) \cap \mathcal{H}(e)$). Since $(U \cap V) \cap (S - \mathcal{H}(e)) \neq \emptyset$, there is an x

in it. Hence there is a net (h_α) in $\overline{\mathcal{H}(e)}$ with $h_\alpha \to x$. Since h_α is eventually in $U \cap V \subseteq \bar{U}$, there is an $y \in \overline{U \cap V}$ so that $h_\beta^{-1} \to y$ for some subnet (h_β). In particular,

$$xy = \lim h_\beta h_\beta^{-1} = e$$

and

$$yx = \lim h_\beta^{-1} h_\beta = e.$$

this contradicts the fact that $x \in S - \mathcal{H}(e)$. Hence $\mathcal{H}(e)$ is locally compact in the relative topology. For the last statement, we refer it to (2.3, p. 17, **[5]**).

PROPOSITION 3.1. *The following statements hold*:

1. $\mathcal{H}(w_H) = \{w_H * \delta_x : x \in N(H)\}$, *where* $N(H)$ *is the normalizer of* H *in* $\mathcal{H}(e)$ *and* δ_x *are the point-masses*

2. *The maps* $\mathcal{H}(w) \overset{\alpha}{\underset{\beta}{\rightleftarrows}} \mathcal{H}(w_H)$ *defined via*

$$\alpha(\mu) = (w_H * \mu_F) * \mu * (\mu_E * w_H) = w_H * \mu * w_H$$

and

$$\beta(\nu) = \mu_E * \nu * \mu_F$$

are mutually inverse continuous group-morphisms.

Proof. 1. We prove it in three steps:

(i) supp $\mu \subseteq eSe$ for all $\mu \in \mathcal{H}(w_H)$.

(ii) Let $\mu \in \mathcal{H}(w_H)$, then there exists a $\nu \in \mathcal{H}(w_H)$ so that $\mu * \nu = \nu * \mu = w_H$. Hence for given $a \in$ supp μ and $b \in$ supp ν $\delta_{ab} * w_H = \delta_{ba} * w_H = w_H$ and thus $\overline{abH} = abH = H = baH = \overline{baH}$ or $ab = bag = h$ for some $g, h \in H$: let $x = h^{-1}a$ and $x' = agh^{-1}$, then $xb = bx' = e$ and so $x' = ex' = (xb)x' = x(bx') = x$. Furthermore,

$$\mu * \delta_b = (w_H * \mu) * \delta_b = w_H * (\mu * \delta_b) = w_H$$

and so $\mu = w_H * \delta_x = w_H * \delta_x * w_H$. By (Thm. 1, p. 124, **[3]**) and Lemma A, we obtain that $Hx = \overline{Hx} = \overline{HxH} = HxH$. This implies $x \in N(H)$.

(iii) The converse of (ii) follows from the fact that $w_H * \delta_x = \delta_x * w_H = w_H * \delta_x * w_H$.

2. We prove it in two steps:

$$\text{(i)} \qquad \alpha(\mu_1 \mu_2) = w_H * \mu_F * \mu_1 * u_2 * \mu_E * w_H$$
$$= w_H * \mu_F * \mu_1 * w * \mu_2 * \mu_E * w_H$$
$$= w_H * \mu_F * \mu_1 * \mu_E * w_H^2 * \mu_F * \mu_2 * w_H$$
$$= \alpha(\mu_1)\alpha(\mu_2),$$
$$\beta(\nu_1 \nu_2) = \mu_E * \nu_1 * \nu_2 * \mu_F$$
$$= \mu_E * \nu_1 * w_H * \nu_2 * \mu_F$$
$$= \mu_E * \nu_1 * \mu_F * \mu_E * w_H * \nu_2 * \mu_F$$
$$= \beta(\nu_1)\beta(\nu_2).$$
$$\text{(ii)} \qquad \alpha \circ \beta(\nu) = \alpha(\mu_E * \nu * \mu_F)$$
$$= w_H * \mu_F * \mu_E * \nu * \mu_F * \mu_E * w_H$$
$$= w_H * \nu * w_H$$
$$= \nu,$$
$$\beta \circ \alpha(\mu) = \beta(w_H * \mu_F * \mu * \mu_E * w_H)$$
$$= \mu_E * w_H * \mu_F * \mu * \mu_E * w_H * \mu_F$$
$$= w * \mu * w$$
$$= \mu.$$

PROPOSITION 3.2. *The following statements hold:*
1. $D(\mathscr{H}(w_H)) = N(H)$ *and* $\operatorname{supp}(\mathscr{H}(w_H)) = \overline{N(H)}$
2. $D(\mathscr{H}(w)) = E(N(H))F = [E, N(H), F]$
3. $\operatorname{supp}(\mathscr{H}(w)) = \overline{E(N(H))F} = \overline{[E, N(H), F]}$.

Proof. 1. This follows from Proposition 3.1. 1.
2. This follows from Proposition 3.1. 2 and the above statement.
3. This follows from 2.

So far, we have only an algebraic characterization of $\mathscr{H}(w)$. In the remainder, we will characterize $\mathscr{H}(w)$ and its subgroups topologically.

PROPOSITION 3.3. *The map* $\eta: N(H)/H \to \mathscr{H}(w_H)$ *defined via*

$$\eta(xH) = w_H * \delta_x (= \delta_x * w_H)$$

is a topological isomorphism.

Proof. We observe first that η is a well-defined algebraic isomorphism. Hence it remains to show that η is an open map. To each $f \in C(S)$, $F_f(x) = \int f(xy) w_H(dy)$ is a bounded continuous function constant on each orbit xH in the compact orbit space eSe/H. Without losing generality, we may assume that $eSe = S$. Suppose that $a_\alpha H \to aH$ in $N(H)/H$. Then

$$\delta_{a_\alpha} * w_H(f) = F_f(a_\alpha H) \to F_f(aH) = \delta_a * w_H(f).$$

Hence η is a continuous group-morphism. Suppose that $a_\alpha H \not\to aH$. Since $N(H)/H$ is compact, there is a subnet $(a_\beta H)$ which converges to a $bH \neq aH$. By Urysohn's Lemma, there is a continuous function $F: S \to [0, 1]$ with $F(aH) = 0$ and $F(bH) = 1$. Clearly,

$$\begin{aligned}\delta_{a_\alpha} * w_H(fop) = Fop(a_\alpha) = F(a_\alpha H) &\not\to F(aH) \\ &= Fop(a) = \delta_a * w_H(Fop),\end{aligned}$$

where $p: S \to S/H$ is the orbit map. Hence η is a topological isomorphism.

The following example shows that not all $\mathcal{H}(w)$ are compact:

Example. Let $S = R \cup \{\infty\}$ be the one-point compactification of the additive group of real numbers. Then S is a compact semitopological semigroup and $\mathcal{H}(\delta_0) = \{\delta_x : x \in R\}$ which is not compact.

4. On a limit theorem. Rosenblatt has proved a necessary and sufficient condition for the convergence of a convolution sequence $(\nu^n)_{n \geqq 1}$ of a probability measure ν on a compact topological semigroup (Thm. 1, p. 152, **[8]**). We will see one side of his condition is an immediate result of our characterizations of the groups of units.

Proposition 4.1. *Let $\nu \in P(S)$. Then $1/n(\nu + \nu^2 + \cdots + \nu^n)$ converges to an idempotent probability measure $L(\nu) \in P(S)$ so that*

1. $\nu^m * L(\nu) = L(\nu) * \nu^n = L(\nu)$ *for all* $m, n \geqq 1$
2. supp $L(\nu) = M(T)$, *where T is a closed subsemigroup generated by* ν, i.e., $T = \cup\{\text{supp } \nu^n : n \geqq 1\}$.

Proof. (See Thm. 3, **[2]**).

In the remainder, we maintain that $\Sigma(\nu) = \{\nu^n : n \geqq 1\}^-$, $K(\nu) = M(\Sigma(\nu))$ and $L(\nu) = \lim 1/n(\nu + \nu^2 + \cdots + \nu^n) = \mu_X * w_G * \mu_Y$. Without losing generality, we may assume that S is generated by ν, i.e., $S = T$. Then supp $L(\nu) = M(S)$, $G = eSe = e(\text{supp } L(\nu))e$, $X = E(Se) = E((\text{supp } L(\nu))e) = \text{supp } \mu_X$ and

$$Y = E(eS) = E(e(\text{supp}\, L(\nu))) = \mu_Y$$

for an $e \in E(M(\text{supp}\, L(\nu)))$ (cf. 3.5, p. 67, **[1]**). In particular, $\mathcal{H}(L(\nu)) = \mu_X * \{w_G\} * \mu_Y = \{L(\nu)\}$.

LEMMA C. *$K(\nu)$ is a compact commutative topological subgroup in $P(S)$.*

Proof. (See the proof of 3.4, p. 67, **[1]**).

Let $w^2 = w = \mu'_E * w_H * \mu'_F \in K(\nu)$. In particular, $K(\nu)$ is a compact subgroup of $\mathcal{H}(w)$. Then.

LEMMA D. *The following statements hold*:
1. $E(M(\text{supp}\, w)) = E(D(\mathcal{H}(w))) = E(D(K(\nu)))$
2. $D(K(\nu)) \subseteq M(S) \subseteq \text{supp}\, K(\nu)$. *In particular,* $\text{supp}\, K(\nu) = M(S)$
3. $E(M(\text{supp}\, w)) \subseteq M(S)$.

Proof. 1. This follows from the fact that

$$E([E, H, F]) = [E, \{e\}, F] = E([E, N(H), F]) = E(D(\mathcal{H}(w))).$$

2. As $K(\nu)$ is an ideal in $\Sigma(\nu)$, $D(K(\nu))D(\Sigma(\nu)) \subseteq D(K(\nu)) \subseteq \text{supp}(K(\nu))$ and so $\text{supp}(K(\nu))$ is a closed ideal in S (See 3.1, p. 65, **[1]**), in particular, $\text{supp}(K(\nu)) \supseteq M(S)$. On the other hand, $D(K(\nu)) = M(\text{supp}(K(\nu)))$ (See 3.1, p. 65, **[1]**) and thus $M(S) \supseteq D(K(\nu))$.

LEMMA E. *The following statements hold*:
1. $\nu * w = w * \nu \in K(\nu)$
2. $L(\nu) * w = w * L(\nu) = L(\nu)$
3. *There exists an* $e^2 = e \in M(\text{supp}\, w) \cap M(S)$
4. $H = e(\text{supp}\, w)e \subseteq eSe = G$
5. $E = E((\text{supp}\, w)e) = E(Se) = X$
6. $F = E(e(\text{supp}\, w)) = E(eS) = Y$
7. $YX \subseteq H$
8. $w = \mu'_X * w_H * \mu'_Y$ *with* $\text{supp}\, \mu'_X = X$ *and* $\text{supp}\, \mu'_Y = Y$.

Proof. 1. This follows from the fact that $K(\nu) = M(\Sigma(\nu))$.
2. This follows from Proposition 4.1.
2. This follows from Lemma D.
4. This is trivial.
5. Let $w = \mu'_E * w_H * \mu'_F$. That $L(\nu) = w * L(\nu) = L(\nu) * w$ implies

$$\mu_X * w_G * \mu = (\mu'_E * w_H * \mu'_F) * (\mu_X * w_G * \mu_Y)$$
$$= \mu'_E * w_G * \mu_Y \in P(X) * w_G * P(Y).$$

By Propositions 2.1 and 2.2, $\overline{EGY} = \overline{XGY}$ and $E = X$.

6. Similarly.
7. This follows from 5 and 6.
8. This is done in the proof of 2.

LEMMA F. *The following statements are equivalent*

1. $\nu * w = w * \nu \neq w$
2. $K(\nu) \neq \{w\}$
3. $w \neq L(\nu)$
4. *H is a proper closed normal subgroup in G* (i.e., $N(H) = G$) *so that* $G = \overline{\cup\{g^n H: n \geqq 1\}}$ *for some* $g \in G - H$.

Proof. $1 \Rightarrow 2$. This is trivial.

$2 \Rightarrow 3$. Suppose that $w = L(\nu)$. Then $K(\nu) = \mathcal{H}(L(\nu)) = \{L(\nu)\}$. This is a contradiction. Hence $w \neq L(\nu)$.

$3 \Rightarrow 1$. Suppose that $w = w * \nu = \nu * w$. Then

$$w = w * (1/n(\nu + \nu^2 + \cdots + \nu^n)) = 1/n(\nu + \nu^2 + \cdots + \nu^n) * w$$

for all $n \geqq 1$. In particular, $w = w * L(\nu) = L(\nu) * w = L(\nu)$.

$1 \Rightarrow 4$. There is a $g \in N(H) - H$ so that $w * \nu = \mu'_X * (w_H * \delta_g) * \mu'_Y$. Let $\mathcal{H}(w) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{H}(w_H)$ be the mutually inverse continuous morphisms of Proposition 3.2. Then

$$\begin{aligned} w * \nu^n = (w * \nu)^n &= (\beta \circ \alpha(w * \nu))^n \\ &= \beta((\alpha(w * \nu))^n) \\ &= \beta((w_H * \delta_g)^n) \\ &= \beta(w_H * \delta_{g^n}) \\ &= \mu'_X * (w_H * \delta_{g^n}) * \mu'_Y. \end{aligned}$$

Furthermore, $\bigcup_{n \geqq 1}(\operatorname{supp} \nu^n \operatorname{supp} w) = (\bigcup_{n \geqq 1} \operatorname{supp} \nu^n)(\operatorname{supp} w)$ and

$$\overline{(\cup \operatorname{supp} \nu^n)(\operatorname{supp} w)} \supseteq \overline{(\cup \operatorname{supp} \nu^n)}(\operatorname{supp} w)$$
$$= S(\operatorname{supp} w) = \overline{(XGY)}\,\overline{(XHY)} = \overline{XGY}$$

(cf. 3.1, p. 55, **[1]** for the inclusion). This implies $w * \nu$ generates $\overline{XGY}$ and thus $\alpha(w * \nu) = w_H * \delta_g$ generates G, i.e., $G = \cup\{g^n H\colon n \geqq 1\}$. That $N(H) = G$ follows easily.

$4 \Rightarrow 2$. Suppose $K(\nu) = \{w\}$. Then $w = L(\nu)$, in particular, $H = G$.

PROPOSITION 4.2. *The following statements are equivalent*:
1. $H = G$.
2. $L(\nu) = w$.
3. $K(\nu) = \{w\}$.
4. $w * \nu = \nu * w = w$.

PROPOSITION 4.3. *If* $(\nu^n)_{n \geqq 1}$ *converges, then any statement of Proposition* 4.2 *holds. The converse holds on compact topological semigroups only.*

Proof. The first statement is trivial. For the converse part, we refer to (p. 380, **[2]**).

THEOREM (Rosenblatt). *Let S be a compact topological semigroup generated by* ν. *Then* $(\nu^n)_{n \geqq 1}$ *does not converge iff there is a proper closed normal subgroup H of G such that*

$$[X, H, Y] \operatorname{supp} \nu = [X, Hg, Y]$$

for some $g \in G - H$ *with* $G = \overline{\cup\{g^n H\colon n \geqq 1\}}$.

Proof. It remains to show the "if" part which we refer to (Thm. 1, p. 152, **[8]**).

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