ON THE GROUPS OF UNITS IN SEMIGROUPS OF PROBABILITY MEASURES

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We generalize Pym's decomposition \( w = \mu_E * w_H * \mu_F \) of idempotent probability measures to the decomposition \( \mu_E * \mathcal{H}(w_H) * \mu_F \) of the maximal groups of units in semigroup of probability measures on a compact semitopological semigroup. We also prove that \( \mathcal{H}(w) \equiv \mathcal{H}(w_H) \equiv N(H)/H \) algebraically and topologically. With these characterizations, we verify Rosenblatt's necessary and sufficient condition for the convergence of a convolution sequence \( (\nu^n)_{n \geq 1} \) of a probability measure \( \nu \) on a compact topological semigroup.

1. Introduction. Let \( S \) denote a compact semitopological semigroup (i.e., the multiplication is separately continuous) and \( (C(S),\| \|) \) the Banach space of all bounded real-valued continuous functions on \( S \). Then \( M^b(S) \) which is defined as the norm dual of \( C(S) \) is a Banach algebra under \( \| \mu \| = \sup \{|\mu(f)| : \|f\| \leq 1\} \) and the convolution * which is defined via \( \mu * \nu(f) = \int f(xy)\mu(dx)\nu(dy) \) on \( C(S) \). Let \( P(S) \) be the totality of probability measures on \( S \), which consists of all positive measures with norm 1 in \( M^b(S) \). Then \( P(S) \) is a compact semitopological semigroup under * and the weak* topology which is the topology of pointwise convergence on \( C(S) \) [4]. If \( S \) is topological (i.e., the multiplication is jointly continuous), then \( P(S) \) is topological (Prop. 4, [9]).

It is known that every compact semitopological semigroup has a minimal ideal which is not necessarily closed except in the case \( S \) is topological [7]. We thus introduce the following definition:

A compact semitopological semigroup is called topologically simple if its minimal ideal is dense in it.

For a subsemigroup \( T \) of \( S \), we use \( E(S) \) and \( M(T) \) to denote the totality of idempotents and the minimal ideal in \( S \) respectively. For a subsemigroup \( A \) of \( P(S) \), we write \( D(A) = \cup \{\text{supp } \mu : \mu \in A\} \) and \( \text{supp } A = \overline{D(A)} \), where \( \text{supp } \mu \) denotes the support of \( \mu \).

In the remainder, \( S \) will always denote a compact semitopological semigroup except mentioned especially.
2. The structure of an idempotent probability measure.

**Proposition 2.1.** Let $K$ be a compact topologically simple subsemigroup in $S$. Then

1. $E(M(K)) \neq \emptyset$

For $e \in E(M(K))$, we have

2. (a) $H = eKe$ is a compact topological subgroup with identity $e$

(b) $E = E(Ke)$ (resp. $F = E(eK)$) is a left (resp. right) zero compact topological subsemigroup

(c) $eE = Fe = e$, $FH = HE = H$ and $FE \subseteq H$

(d) $M(K) = EHF = [E, H, F]$ via

\[(x, g, y)(x', g', y') = (x, gyx'g', y')\]

(e) $Ke = (EHF)e = EH$ and $eK = e(EHF) = HF$

3. (a) $P(E)$ (resp. $P(F)$) is a left (resp. right) zero compact topological subsemigroup. In particular, $E(P(E)) = P(E)$ and $E(P(F)) = P(F)$

(b) $\delta_e^*P(E) = P(F)^*\delta_e = \delta_e$, where $\delta_e$ is the point-mass at $e$

(c) $P(F)^*P(E) \subseteq P(H)$. In particular,

$w_H * P(F)^*P(E) = P(F)^*P(E)^*w_H = w_H$

where $w_H^2 = w_H$ is the Haar measure on $H$

(d) $P(E) * w_H * P(F) \subseteq E(P(S))$.

**Proof.**

1. (See the proof of 3.4, p. 67, [1]).

2. (See p. 500, [7]; Thm. 2, p. 124, [3]).

3. (a) For $\mu, \nu \in P(E)$,

\[\mu * \nu(f) = \int f(xy)\mu(dx)\nu(dy) = \int f(x)\mu(dx)\nu(dy) = \mu(f).\]

Hence $P(E)$ is left zero. Furthermore, by 2(b) we see that $P(E)$ is a compact topological subsemigroup in $P(S)$.

(b) This follows from 2(c).

(d) Let $\mu = \mu_E * w_H * \mu_F \in P(E) * w_H * P(F)$. Then

\[\mu^2 = \mu_E * (w_H * \mu_F * \mu_E) * w_H * \mu_F = \mu_E * w_H * \mu_F.\]

**Lemma A.** $\text{supp}(\mu * \nu) = (\text{supp} \mu \text{ supp} \nu)$ in $P(S)$.

**Proof.** [4].

**Proposition 2.2.** Let $w^2 = w \in P(S)$. Then
1. supp \( w \) is a compact topologically simple subsemigroup
2. \( w = \mu_E \ast w_H \ast \mu_F \), where
   (a) \( H = e(\text{supp } w)e, E = E((\text{supp } w)e) \) and \( F = E(e(\text{supp } w)) \) for an \( e \in E(M(\text{supp } w)) \)
   (b) \( \mu_E \in P(E) \) with \( \text{supp } \mu_E = E \)
   (c) \( \mu_F \in P(E) \) with \( \text{supp } \mu_F = F \)
   (d) \( \omega_H = \omega_H \) is the Haar measure on \( H \)
3. \( w_H = \omega_H \ast \mu_F \ast \mu_E = \mu_F \ast \mu_E \ast \omega_H \)
4. \( w_H = w_H \ast \omega_H \ast \mu_F \ast \mu_E \ast \omega_H \)

**Proof.**
1. We refer it to (p. 500, [7]).
2. This is a result of 1 and Proposition 2.1.
3. This is a result of 3(c) in Proposition 2.1.
4. We prove the first equality only. As \( eE\omega_F \subseteq H \),

\[
w_H \ast \omega_H \ast \omega_H = w_H \ast (w_H \ast \mu_E \ast \omega_H \ast \mu_F \ast \omega_H) \ast \omega_H = w_H.
\]

**Proposition 2.3.** \( E(P(S)) = \bigcup \{P(E) \ast \omega_H \ast P(F) : K \text{ is a compact topologically simple subsemigroup}\} \).

3. A characterization of the maximal group of units. For \( e \in E(S) \) we denote by \( \mathcal{H}(e) \) the maximal group of units with identity \( e \) in the compact subsemigroup \( eSe \). We will see that \( \mathcal{H}(e) \) is in general a locally compact topological subgroup in the relative topology of \( S \) and \( \mathcal{H}(e) \) is closed and so compact in the case \( S \) is topological.

In this section, we maintain that \( w^2 = w = \mu_E \ast \omega_H \ast \mu_F \) is as in Proposition 2.2. In particular, \( H \) is a compact subgroup of \( \mathcal{H}(e) \).

**Lemma B.** \( \mathcal{H}(e) \) is a locally compact topological subgroup in the relative topology of \( S \). Furthermore, if \( S \) is topological, then \( \mathcal{H}(e) \) is a closed and hence compact subgroup.

**Proof.** As \( \mathcal{H}(e) \) is a topological subgroup in \( eSe \) (Cor. 6.3, pp. 282–283, [6]), \( \mathcal{H}(e) \) is a closed subsemigroup in \( eSe \) (3.1, p. 65, [1]). Without losing generality, we may assume that \( S = eSe = \mathcal{H}(e) \). Suppose that \( \mathcal{H}(e) \) is not locally compact. Then \( \mathcal{H}(e) \) is not open in \( S \). Thus if \( 0 \) is an open neighborhood of \( e \) in \( S \), then \( 0 \cap (S - \mathcal{H}(e)) \neq \emptyset \), for translation by an element of \( \mathcal{H}(e) \) is a homeomorphism of \( S \). Now, we choose a relatively compact open neighborhood \( U \) of \( e \) in \( S \). Then \( (U \cap \mathcal{H}(e))^{-1} \) is open in \( \mathcal{H}(e) \) and contains \( e \), so there is an open neighborhood \( V \) of \( e \) in \( S \) so that \( V \cap \mathcal{H}(e) = (U \cap \mathcal{H}(e))^{-1} \). Then \( U \cap V \) is an open neighborhood of \( e \) in \( S \) so that \( (U \cap V) \cap \mathcal{H}(e) \) is symmetric (i.e., \( h \in (U \cap V) \cap \mathcal{H}(e) \) iff \( h^{-1} \in (U \cap V) \cap \mathcal{H}(e) \)). Since \((U \cap V) \cap (S - \mathcal{H}(e)) \neq \emptyset \), there is an \( x \)}
in it. Hence there is a net \((h_\alpha)\) in \(\mathcal{H}(e)\) with \(h_\alpha \to x\). Since \(h_\alpha\) is eventually in \(U \cap V \subseteq \hat{U}\), there is an \(y \in U \cap V\) so that \(h_\beta^{-1} \to y\) for some subnet \((h_\beta)\). In particular,

\[ xy = \lim h_\beta h_\beta^{-1} = e \]

and

\[ yx = \lim h_\beta^{-1} h_\beta = e. \]

this contradicts the fact that \(x \in S - \mathcal{H}(e)\). Hence \(\mathcal{H}(e)\) is locally compact in the relative topology. For the last statement, we refer it to (2.3, p. 17, [5]).

**Proposition 3.1.** The following statements hold:

1. \(\mathcal{H}(w_H) = \{ w_H \ast \delta_x : x \in N(H) \}\), where \(N(H)\) is the normalizer of \(H\) in \(\mathcal{H}(e)\) and \(\delta_x\) are the point-masses

2. The maps \(\mathcal{H}(w) \xrightarrow{\alpha} \mathcal{H}(w_H)\) defined via

\[ \alpha(\mu) = (w_H \ast \mu \ast \mu \ast \mu \ast w_H) = w_H \ast \mu \ast w_H \]

and

\[ \beta(\nu) = \mu \ast \nu \ast \mu \ast \mu \]

are mutually inverse continuous group-morphisms.

**Proof.** 1. We prove it in three steps:

(i) \(\text{supp } \mu \subseteq eSe\) for all \(\mu \in \mathcal{H}(w_H)\).

(ii) Let \(\mu \in \mathcal{H}(w_H)\), then there exists a \(\nu \in \mathcal{H}(w_H)\) so that \(\mu \ast \nu = \nu \ast \mu = w_H\). Hence for given \(a \in \text{supp } \mu\) and \(b \in \text{supp } \nu\) \(\delta_{ab} \ast w_H = \delta_{ba} \ast w_H = w_H\) and thus \(abH = abH = H = baH = baH\) or \(ab = ba = b\) for some \(g, h \in H\); let \(x = h^{-1}a\) and \(x' = agh^{-1}\), then \(xb = bx' = e\) and so \(x' = ex' = (xb)x' = x(bx') = x\). Furthermore,

\[ \mu \ast \delta_b = (w_H \ast \mu) \ast \delta_b = w_H \ast (\mu \ast \delta_b) = w_H \]

and so \(\mu = w_H \ast \delta_x = w_H \ast \delta_x \ast w_H\). By (Thm. 1, p. 124, [3]) and Lemma A, we obtain that \(Hx = Hx = HxH = HxH\). This implies \(x \in N(H)\).

(iii) The converse of (ii) follows from the fact that \(w_H \ast \delta_x = \delta_x \ast w_H = w_H \ast \delta_x \ast w_H\).

2. We prove it in two steps:
ON THE GROUPS OF UNITS 307

(i) \( \alpha(\mu_1, \mu_2) = w_H \ast \mu_F \ast \mu_1 \ast \mu_2 \ast \mu_E \ast w_H \)
\( = w_H \ast \mu_F \ast \mu_1 \ast w \ast \mu_2 \ast \mu_E \ast w_H \)
\( = w_H \ast \mu_F \ast \mu_1 \ast \mu_E \ast w_H^2 \ast \mu_F \ast \mu_2 \ast w_H \)
\( = \alpha(\mu_1) \alpha(\mu_2), \)
\( \beta(v_1, v_2) = \mu_E \ast v_1 \ast v_2 \ast \mu_F \)
\( = \mu_E \ast v_1 \ast w_H \ast v_2 \ast \mu_F \)
\( = \mu_E \ast v_1 \ast \mu_F \ast \mu_E \ast w_H \ast v_2 \ast \mu_F \)
\( = \beta(v_1) \beta(v_2). \)

(ii) \( \alpha \circ \beta(v) = \alpha(\mu_E \ast v \ast \mu_F) \)
\( = w_H \ast \mu_F \ast \mu_E \ast v \ast \mu_F \ast \mu_E \ast w_H \)
\( = w_H \ast v \ast w_H \)
\( = v, \)
\( \beta \circ \alpha(\mu) = \beta(w_H \ast \mu_F \ast \mu \ast \mu_E \ast w_H) \)
\( = \mu_E \ast w_H \ast \mu_F \ast \mu \ast \mu_E \ast w_H \ast \mu_F \)
\( = w \ast \mu \ast w \)
\( = \mu. \)

**Proposition 3.2.** The following statements hold:
1. \( D(\mathcal{H}(w_H)) = N(H) \) and \( \text{supp}(\mathcal{H}(w_H)) = \overline{N(H)} \)
2. \( D(\mathcal{H}(w)) = E(N(H))F = [E, N(H), F] \)
3. \( \text{supp}(\mathcal{H}(w)) = E(N(H))F = [E, N(H), F] \).

**Proof.** 1. This follows from Proposition 3.1. 1.
2. This follows from Proposition 3.1. 2 and the above statement.
3. This follows from 2.

So far, we have only an algebraic characterization of \( \mathcal{H}(w) \). In the remainder, we will characterize \( \mathcal{H}(w) \) and its subgroups topologically.

**Proposition 3.3.** The map \( \eta: N(H)/H \rightarrow \mathcal{H}(w_H) \) defined via
\( \eta(xH) = w_H \ast \delta_x (= \delta_x \ast w_H) \)
is a topological isomorphism.

**Proof.** We observe first that \( \eta \) is a well-defined algebraic isomorphism. Hence it remains to show that \( \eta \) is an open map. To
each \( f \in C(S) \), \( F_f(x) = \int f(xy)w_H(dy) \) is a bounded continuous function constant on each orbit \( xH \) in the compact orbit space \( eSe/H \). Without losing generality, we may assume that \( eSe = S \). Suppose that \( a_aH \to aH \) in \( N(H)/H \). Then

\[
\delta_{a_a} * w_H(f) = F_f(a_aH) \to F_f(aH) = \delta_a * w_H(f).
\]

Hence \( \eta \) is a continuous group-morphism. Suppose that \( a_aH \not\to aH \). Since \( N(H)/H \) is compact, there is a subnet \( (a_bH) \) which converges to a \( bH \not= aH \). By Urysohn's Lemma, there is a continuous function \( F: S \to [0, 1] \) with \( F(aH) = 0 \) and \( F(bH) = 1 \). Clearly,

\[
\delta_{a_a} * w_H(fop) = Fop(a_a) = F(a_aH) \not\to F(aH)
\]

where \( fop : S \to S/H \) is the orbit map. Hence \( \eta \) is a topological isomorphism.

The following example shows that not all \( \mathcal{K}(w) \) are compact:

**Example.** Let \( S = R \cup \{\infty\} \) be the one-point compactification of the additive group of real numbers. Then \( S \) is a compact semitopological semigroup and \( \mathcal{K}(\delta_0) = \{\delta_x : x \in R\} \) which is not compact.

4. **On a limit theorem.** Rosenblatt has proved a necessary and sufficient condition for the convergence of a convolution sequence \( (\nu^n)_{n \geq 1} \) of a probability measure \( \nu \) on a compact topological semigroup (Thm. 1, p. 152, [8]). We will see one side of his condition is an immediate result of our characterizations of the groups of units.

**Proposition 4.1.** Let \( \nu \in P(S) \). Then \( 1/n(\nu + \nu^2 + \cdots + \nu^n) \) converges to an idempotent probability measure \( L(\nu) \in P(S) \) so that

1. \( \nu^m * L(\nu) = L(\nu)^m \nu^n = L(\nu) \) for all \( m, n \geq 1 \)
2. \( \text{supp } L(\nu) = M(T) \), where \( T \) is a closed subsemigroup generated by \( \nu \), i.e., \( T = \cup \{\text{supp } \nu^n : n \geq 1\} \).

**Proof.** (See Thm. 3, [2]).

In the remainder, we maintain that \( \Sigma(\nu) = \{\nu^n : n \geq 1\} \), \( K(\nu) = M(\Sigma(\nu)) \) and \( L(\nu) = \lim 1/n(\nu + \nu^2 + \cdots + \nu^n) = \mu_x * w_G * \mu_Y \). Without losing generality, we may assume that \( S \) is generated by \( \nu \), i.e., \( S = T \). Then \( \text{supp } L(\nu) = M(S) \), \( G = eSe = e(\text{supp } L(\nu))e \), \( X = E(Se) = E(\text{supp } L(\nu)e) = \text{supp } \mu_x \) and
\[ Y = E(eS) = E(e(\text{supp } L(v))) = \mu_y \]

for an \( e \in E(M(\text{supp } L(v))) \) (cf. 3.5, p. 67, [1]). In particular, \( \mathcal{H}(L(v)) = \mu_x \{ w_G \} \mu_y = \{ L(v) \} \).

**Lemma C.** \( K(v) \) is a compact commutative topological subgroup in \( P(S) \).

**Proof.** (See the proof of 3.4, p. 67, [1]).

Let \( w^2 = w = \mu'_x \mu_H^* \mu'_F \in K(v) \). In particular, \( K(v) \) is a compact subgroup of \( \mathcal{H}(w) \). Then.

**Lemma D.** The following statements hold:
1. \( E(M(\text{supp } w)) = E(D(\mathcal{H}(w))) = E(D(K(v))) \)
2. \( D(K(v)) \subseteq M(S) \subseteq \text{supp } K(v) \). In particular, \( \text{supp } K(v) = M(S) \)
3. \( E(M(\text{supp } w)) \subseteq M(S) \).

**Proof.** 1. This follows from the fact that
\[ E([E, H, F]) = [E, \{ e \}, F] = E([E, N(H), F]) = E(D(\mathcal{H}(w))). \]

2. As \( K(v) \) is an ideal in \( \Sigma(v) \), \( D(K(v))D(\Sigma(v)) \subseteq D(K(v)) \subseteq \text{supp}(K(v)) \) and so \( \text{supp}(K(v)) \) is a closed ideal in \( S \) (See 3.1, p. 65, [1]), in particular, \( \text{supp}(K(v)) \supseteq M(S) \). On the other hand, \( D(K(v)) = M(\text{supp}(K(v))) \) (See 3.1, p. 65, [1]) and thus \( M(S) \supseteq D(K(v)) \).

**Lemma E.** The following statements hold:
1. \( \nu^* w = w^* \nu \in K(v) \)
2. \( L(\nu)^* w = w^* L(\nu) = L(\nu) \)
3. There exists an \( e^2 = e \in M(\text{supp } w) \cap M(S) \)
4. \( H = e(\text{supp } w)e \subseteq eSe = G \)
5. \( E = E((\text{supp } w)e) = E(Se) = X \)
6. \( F = E(e(\text{supp } w)) = E(eS) = Y \)
7. \( YX \subseteq H \)
8. \( w = \mu_\nu^* w_H^* \mu_\nu \) with \( \text{supp } \mu_\nu^* = X \) and \( \text{supp } \mu_\nu = Y \).

**Proof.** 1. This follows from the fact that \( K(v) = M(\Sigma(v)) \).
2. This follows from Proposition 4.1.
2. This follows from Lemma D.
4. This is trivial.
5. Let \( w = \mu_\nu^* w_H^* \mu_\nu \). That \( L(\nu) = w^* L(\nu) = L(\nu)^* w \) implies
By Propositions 2.1 and 2.2, $EGY = XGY$ and $E = X$.

6. Similarly.
7. This follows from 5 and 6.
8. This is done in the proof of 2.

**Lemma F.** The following statements are equivalent

1. $v^\ast w = w^\ast v \neq w$
2. $K(v) \neq \{w\}$
3. $w \neq L(v)$
4. $H$ is a proper closed normal subgroup in $G$ (i.e., $N(H) = G$) so that $G = \bigcup \{g^nH : n \geq 1\}$ for some $g \in G - H$.

**Proof.** 1 $\Rightarrow$ 2. This is trivial.

2 $\Rightarrow$ 3. Suppose that $w = L(v)$. Then $K(v) = H(L(v)) = \{L(v)\}$. This is a contradiction. Hence $w \neq L(v)$.

3 $\Rightarrow$ 1. Suppose that $w = w^\ast v = v^\ast w$. Then

$$w = w^\ast (1/n(v + v^2 + \cdots + v^n)) = 1/n(v + v^2 + \cdots + v^n)^\ast w$$

for all $n \geq 1$. In particular, $w = w^\ast L(v) = L(v)^\ast w = L(v)$.

1 $\Rightarrow$ 4. There is a $g \in N(H) - H$ so that $w^\ast v = \mu_{\delta_g}(w^\ast H) \ast \mu_{\delta_g}$.

Let $H(w) \rightleftharpoons \beta H(w_H)$ be the mutually inverse continuous morphisms of Proposition 3.2. Then

$$w^\ast v^n = (w^\ast v)^n = (\beta \circ \alpha (w^\ast v))^n$$

$$= \beta((\alpha (w^\ast v))^n)$$

$$= \beta((w_H \ast \delta_{\varepsilon})^n)$$

$$= \beta(w_H \ast \delta_{\varepsilon}^n)$$

$$= \mu_{\delta_g}(w_H \ast \delta_{\varepsilon}^n) \ast \mu_{\delta_g}.$$ 

Furthermore, $\bigcup_{n \geq 1} (supp v^n supp w) = (\bigcup_{n \geq 1} supp v^n)(supp w)$ and

$$\bigcup supp v^n (supp w) \supseteq \bigcup supp v^n (supp w)$$

$$= S(supp w) = (XGY)(XHY) = XGY$$
ON THE GROUPS OF UNITS

(cf. 3.1, p. 55, [1] for the inclusion). This implies \( w^*v \) generates \( XGY \) and thus \( \alpha(w^*v) = w_H^*\delta_g \) generates \( G \), i.e., \( G = \bigcup\{g^*H; n \geq 1\} \). That \( N(H) = G \) follows easily.

4 \( \Rightarrow \) 2. Suppose \( K(v) = \{w\} \). Then \( w = L(v) \), in particular, \( H = G \).

**Proposition 4.2.** The following statements are equivalent:
1. \( H = G \).
2. \( L(v) = w \).
3. \( K(v) = \{w\} \).
4. \( w^*v = v^*w = w \).

**Proposition 4.3.** If \( (v^*)_{n \geq 1} \) converges, then any statement of Proposition 4.2 holds. The converse holds on compact topological semigroups only.

**Proof.** The first statement is trivial. For the converse part, we refer to (p. 380, [2]).

**Theorem** (Rosenblatt). Let \( S \) be a compact topological semigroup generated by \( v \). Then \( (v^*)_{n \geq 1} \) does not converge iff there is a proper closed normal subgroup \( H \) of \( G \) such that

\[
[X, H, Y] \supp v = [X, Hg, Y]
\]

for some \( g \in G - H \) with \( G = \bigcup\{g^*H; n \geq 1\} \).

**Proof.** It remains to show the "if" part which we refer to (Thm. 1, p. 152, [8]).

**Acknowledgements.** The author wishes to thank Drs. Karl H. Hofmann, John R. Liukkonen and Michael W. Mislove for many helpful suggestions.

**References**


Received August 9, 1974 and in revised form March 20, 1975.

NATIONAL TSING HUA UNIVERSITY,
TAIWAN 300
<table>
<thead>
<tr>
<th>Authors</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Waleed A. Al-Salam and A. Verma</td>
<td>A fractional Leibniz q-formula</td>
<td>1</td>
</tr>
<tr>
<td>Robert A. Bekes</td>
<td>Algebraically irreducible representations of $L_1(G)$</td>
<td>11</td>
</tr>
<tr>
<td>Thomas Theodore Bowman</td>
<td>Construction functors for topological semigroups</td>
<td>27</td>
</tr>
<tr>
<td>Stephen LaVern Campbell</td>
<td>Operator-valued inner functions analytic on the closed disc. II</td>
<td>37</td>
</tr>
<tr>
<td>Leonard Eliezer Dor and Edward Wilfred Odell, Jr.</td>
<td>Monotone bases in $L_p$</td>
<td>51</td>
</tr>
<tr>
<td>Yukiyoshi Ebihara, Mitsuhiro Nakao and Tokumori Nanbu</td>
<td>On the existence of global classical solution of initial-boundary value problem for $cmu - u^3 = f$</td>
<td>63</td>
</tr>
<tr>
<td>Y. Gordon</td>
<td>Unconditional Schauder decompositions of normed ideals of operators between some $l_p$-spaces</td>
<td>71</td>
</tr>
<tr>
<td>Gary Grefsrud</td>
<td>Oscillatory properties of solutions of certain nth order functional differential equations</td>
<td>83</td>
</tr>
<tr>
<td>Irvin Roy Hentzel</td>
<td>Generalized right alternative rings</td>
<td>95</td>
</tr>
<tr>
<td>Zensiro Goseki and Thomas Benny Rushing</td>
<td>Embeddings of shape classes of compacta in the trivial range</td>
<td>103</td>
</tr>
<tr>
<td>Emil Grosswald</td>
<td>Brownian motion and sets of multiplicity</td>
<td>111</td>
</tr>
<tr>
<td>Donald LaTorre</td>
<td>A construction of the idempotent-separating congruences on a bisimple orthodox semigroup</td>
<td>115</td>
</tr>
<tr>
<td>Pjek-Hwee Lee</td>
<td>On subrings of rings with involution</td>
<td>131</td>
</tr>
<tr>
<td>Marvin David Marcus and H. Minc</td>
<td>On two theorems of Frobenius</td>
<td>149</td>
</tr>
<tr>
<td>Michael Douglas Miller</td>
<td>On the lattice of normal subgroups of a direct product</td>
<td>153</td>
</tr>
<tr>
<td>Grattan Patrick Murphy</td>
<td>A metric basis characterization of Euclidean space</td>
<td>159</td>
</tr>
<tr>
<td>Roy Martin Rakestraw</td>
<td>A representation theorem for real convex functions</td>
<td>165</td>
</tr>
<tr>
<td>Louis Jackson Ratliff, Jr.</td>
<td>On Rees localities and $H_1$-local rings</td>
<td>169</td>
</tr>
<tr>
<td>Simeon Reich</td>
<td>Fixed point iterations of nonexpansive mappings</td>
<td>195</td>
</tr>
<tr>
<td>Domenico Rosa</td>
<td>$B$-complete and $B_r$-complete topological algebras</td>
<td>199</td>
</tr>
<tr>
<td>Walter Roth</td>
<td>Uniform approximation by elements of a cone of real-valued functions</td>
<td>209</td>
</tr>
<tr>
<td>Helmut R. Salzmann</td>
<td>Homogene kompakte projektive Ebenen</td>
<td>217</td>
</tr>
<tr>
<td>Jerrold Norman Siegel</td>
<td>On a space between $BH$ and $B_\infty$</td>
<td>235</td>
</tr>
<tr>
<td>Robert C. Sine</td>
<td>On local uniform mean convergence for Markov operators</td>
<td>247</td>
</tr>
<tr>
<td>James D. Stafney</td>
<td>Set approximation by lemniscates and the spectrum of an operator on an interpolation space</td>
<td>253</td>
</tr>
<tr>
<td>Árpád Száz</td>
<td>Convolution multipliers and distributions</td>
<td>267</td>
</tr>
<tr>
<td>Kalathoor Varadarajan</td>
<td>Span and stably trivial bundles</td>
<td>277</td>
</tr>
<tr>
<td>Robert Breckenridge Warfield, Jr.</td>
<td>Countably generated modules over commutative Artinian rings</td>
<td>289</td>
</tr>
<tr>
<td>John Yuan</td>
<td>On the groups of units in semigroups of probability measures</td>
<td>303</td>
</tr>
</tbody>
</table>