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**ON THE GROUPS OF UNITS IN SEMIGROUPS OF  
PROBABILITY MEASURES**

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## ON THE GROUPS OF UNITS IN SEMIGROUPS OF PROBABILITY MEASURES

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We generalize Pym's decomposition  $w = \mu_E * w_H * \mu_F$  of idempotent probability measures to the decomposition  $\mu_E * \mathcal{H}(w_H) * \mu_F$  of the maximal groups of units in semigroup of probability measures on a compact semitopological semigroup. We also prove that  $\mathcal{H}(w) \cong \mathcal{H}(w_H) \cong N(H)/H$  algebraically and topologically. With these characterizations, we verify Rosenblatt's necessary and sufficient condition for the convergence of a convolution sequence  $(\nu^n)_{n \geq 1}$  of a probability measure  $\nu$  on a compact topological semigroup.

**1. Introduction.** Let  $S$  denote a compact semitopological semigroup (i.e., the multiplication is separately continuous) and  $(C(S), \| \cdot \|)$  the Banach space of all bounded real-valued continuous functions on  $S$ . Then  $M^b(S)$  which is defined as the norm dual of  $C(S)$  is a Banach algebra under  $\| \mu \| = \sup\{|\mu(f)| : \|f\| \leq 1\}$  and the convolution  $*$  which is defined via  $\mu * \nu(f) = \int f(xy)\mu(dx)\nu(dy)$  on  $C(S)$ . Let  $P(S)$  be the totality of probability measures on  $S$ , which consists of all positive measures with norm 1 in  $M^b(S)$ . Then  $P(S)$  is a compact semitopological semigroup under  $*$  and the weak\* topology which is the topology of pointwise convergence on  $C(S)$  [4]. If  $S$  is topological (i.e., the multiplication is jointly continuous), then  $P(S)$  is topological (Prop. 4, [9]).

It is known that every compact semitopological semigroup has a minimal ideal which is not necessarily closed except in the case  $S$  is topological [7]. We thus introduce the following definition:

A compact semitopological semigroup is called topologically simple if its minimal ideal is dense in it.

For a subsemigroup  $T$  of  $S$ , we use  $E(S)$  and  $M(T)$  to denote the totality of idempotents and the minimal ideal in  $S$  respectively. For a subsemigroup  $A$  of  $P(S)$ , we write  $D(A) = \cup\{\text{supp } \mu : \mu \in A\}$  and  $\text{supp } A = \overline{D(A)}$ , where  $\text{supp } \mu$  denotes the support of  $\mu$ .

In the remainder,  $S$  will always denote a compact semitopological semigroup except mentioned especially.

## 2. The structure of an idempotent probability measure.

PROPOSITION 2.1. *Let  $K$  be a compact topologically simple subsemigroup in  $S$ . Then*

1.  $E(M(K)) \neq \emptyset$

For  $e \in E(M(K))$ , we have

2. (a)  $H = eKe$  is a compact topological subgroup with identity  $e$
- (b)  $E = E(Ke)$  (resp.  $F = E(eK)$ ) is a left (resp. right) zero compact topological subsemigroup
- (c)  $eE = Fe = e$ ,  $FH = HE = H$  and  $FE \subseteq H$
- (d)  $M(K) = EHF = [E, H, F]$  via

$$(x, g, y)(x', g', y') = (x, gyx'g', y')$$

(e)  $Ke = (EHF)e = EH$  and  $eK = e(EHF) = HF$

3. (a)  $P(E)$  (resp.  $P(F)$ ) is a left (resp. right) zero compact topological subsemigroup. In particular,  $E(P(E)) = P(E)$  and  $E(P(F)) = P(F)$
- (b)  $\delta_e^* P(E) = P(F)^* \delta_e = \delta_e$ , where  $\delta_e$  is the point-mass at  $e$
- (c)  $P(F)^* P(E) \subseteq P(H)$ . In particular,

$$w_H * P(F)^* P(E) = P(F)^* P(E)^* w_H = w_H,$$

where  $w_H^2 = w_H$  is the Haar measure on  $H$

- (d)  $P(E)^* w_H * P(F) \subseteq E(P(S))$ .

*Proof.* 1. (See the proof of 3.4, p. 67, [1]).

2. (See p. 500, [7]; Thm. 2, p. 124, [3]).

3. (a) For  $\mu, \nu \in P(E)$ ,

$$\mu * \nu(f) = \int f(xy) \mu(dx) \nu(dy) = \int f(x) \mu(dx) \nu(dy) = \mu(f).$$

Hence  $P(E)$  is left zero. Furthermore, by 2(b) we see that  $P(E)$  is a compact topological subsemigroup in  $P(S)$ .

(b) This follows from 2(c).

(d) Let  $\mu = \mu_E * w_H * \mu_F \in P(E)^* w_H * P(F)$ . Then

$$\mu^2 = \mu_E * (w_H * \mu_F * \mu_E) * w_H * \mu_F = \mu_E * w_H * \mu_F.$$

LEMMA A.  $\text{supp}(\mu * \nu) = \overline{(\text{supp } \mu \text{ supp } \nu)}$  in  $P(S)$ .

*Proof.* [4].

PROPOSITION 2.2. *Let  $w^2 = w \in P(S)$ . Then*

1.  $\text{supp } w$  is a compact topologically simple subsemigroup
2.  $w = \mu_E * w_H * \mu_F$ , where
  - (a)  $H = e(\text{supp } w)e$ ,  $E = E((\text{supp } w)e)$  and  $F = E(e(\text{supp } w))$  for an  $e \in E(M(\text{supp } w))$
  - (b)  $\mu_E \in P(E)$  with  $\text{supp } \mu_E = E$
  - (c)  $\mu_F \in P(E)$  with  $\text{supp } \mu_F = F$
  - (d)  $w_H^2 = w_H$  is the Haar measure on  $H$
3.  $w_H = w_H * \mu_F * \mu_E = \mu_F * \mu_E * w_H$
4.  $w_H = w_H * w * w_H = w_H * \mu_F * w * \mu_E * w_H$ .

- Proof.*
1. We refer it to (p. 500, [7]).
  2. This is a result of 1 and Proposition 2.1.
  3. This is a result of 3(c) in Proposition 2.1.
  4. We prove the first equality only. As  $eEHFe \subseteq H$ ,

$$w_H * w * w_H = w_H * (w_H * \mu_E * w_H * \mu_F * w_H) * w_H = w_H.$$

PROPOSITION 2.3.  $E(P(S)) = \cup \{P(E) * w_H * P(F) : K \text{ is a compact topologically simple subsemigroup}\}$ .

**3. A characterization of the maximal group of units.** For  $e \in E(S)$  we denote by  $\mathcal{H}(e)$  the maximal group of units with identity  $e$  in the compact subsemigroup  $eSe$ . We will see that  $\mathcal{H}(e)$  is in general a locally compact topological subgroup in the relative topology of  $S$  and  $\mathcal{H}(e)$  is closed and so compact in the case  $S$  is topological.

In this section, we maintain that  $w^2 = w = \mu_E * w_H * \mu_F$  is as in Proposition 2.2. In particular,  $H$  is a compact subgroup of  $\mathcal{H}(e)$ .

LEMMA B.  $\mathcal{H}(e)$  is a locally compact topological subgroup in the relative topology of  $S$ . Furthermore, if  $S$  is topological, then  $\mathcal{H}(e)$  is a closed and hence compact subgroup.

*Proof.* As  $\mathcal{H}(e)$  is a topological subgroup in  $eSe$  (Cor. 6.3, pp. 282–283, [6]),  $\mathcal{H}(e)$  is a closed subsemigroup in  $eSe$  (3.1, p. 65, [1]). Without losing generality, we may assume that  $S = eSe = \mathcal{H}(e)$ . Suppose that  $\mathcal{H}(e)$  is not locally compact. Then  $\mathcal{H}(e)$  is not open in  $S$ . Thus if  $0$  is an open neighborhood of  $e$  in  $S$ , then  $0 \cap (S - \mathcal{H}(e)) \neq \emptyset$ , for translation by an element of  $\mathcal{H}(e)$  is a homeomorphism of  $S$ . Now, we choose a relatively compact open neighborhood  $U$  of  $e$  in  $S$ . Then  $(U \cap \mathcal{H}(e))^{-1}$  is open in  $\mathcal{H}(e)$  and contains  $e$ , so there is an open neighborhood  $V$  of  $e$  in  $S$  so that  $V \cap \mathcal{H}(e) = (U \cap \mathcal{H}(e))^{-1}$ . Then  $U \cap V$  is an open neighborhood of  $e$  in  $S$  so that  $(U \cap V) \cap \mathcal{H}(e)$  is symmetric (i.e.,  $h \in (U \cap V) \cap \mathcal{H}(e)$  iff  $h^{-1} \in (U \cap V) \cap \mathcal{H}(e)$ ). Since  $(U \cap V) \cap (S - \mathcal{H}(e)) \neq \emptyset$ , there is an  $x$

in it. Hence there is a net  $(h_\alpha)$  in  $\mathcal{H}(e)$  with  $h_\alpha \rightarrow x$ . Since  $h_\alpha$  is eventually in  $U \cap V \subseteq \bar{U}$ , there is an  $y \in U \cap V$  so that  $h_\beta^{-1} \rightarrow y$  for some subnet  $(h_\beta)$ . In particular,

$$xy = \lim h_\beta h_\beta^{-1} = e$$

and

$$yx = \lim h_\beta^{-1} h_\beta = e.$$

this contradicts the fact that  $x \in S - \mathcal{H}(e)$ . Hence  $\mathcal{H}(e)$  is locally compact in the relative topology. For the last statement, we refer it to (2.3, p. 17, [5]).

PROPOSITION 3.1. *The following statements hold:*

1.  $\mathcal{H}(w_H) = \{w_H * \delta_x : x \in N(H)\}$ , where  $N(H)$  is the normalizer of  $H$  in  $\mathcal{H}(e)$  and  $\delta_x$  are the point-masses

2. The maps  $\mathcal{H}(w) \xrightarrow[\beta]{\alpha} \mathcal{H}(w_H)$  defined via

$$\alpha(\mu) = (w_H * \mu_F) * \mu * (\mu_E * w_H) = w_H * \mu * w_H$$

and

$$\beta(\nu) = \mu_E * \nu * \mu_F$$

are mutually inverse continuous group-morphisms.

*Proof.* 1. We prove it in three steps:

(i)  $\text{supp } \mu \subseteq eSe$  for all  $\mu \in \mathcal{H}(w_H)$ .

(ii) Let  $\mu \in \mathcal{H}(w_H)$ , then there exists a  $\nu \in \mathcal{H}(w_H)$  so that  $\mu * \nu = \nu * \mu = w_H$ . Hence for given  $\underline{a} \in \text{supp } \mu$  and  $b \in \text{supp } \nu$   $\delta_{ab} * w_H = \delta_{ba} * w_H = w_H$  and thus  $abH = abH = H = baH = baH$  or  $ab = bag = h$  for some  $g, h \in H$ : let  $x = h^{-1}a$  and  $x' = agh^{-1}$ , then  $xb = bx' = e$  and so  $x' = ex' = (xb)x' = x(bx') = x$ . Furthermore,

$$\mu * \delta_b = (w_H * \mu) * \delta_b = w_H * (\mu * \delta_b) = w_H$$

and so  $\mu = w_H * \delta_x = w_H * \underline{\delta_x} * w_H$ . By (Thm. 1, p. 124, [3]) and Lemma A, we obtain that  $Hx = \underline{Hx} = HxH = HxH$ . This implies  $x \in N(H)$ .

(iii) The converse of (ii) follows from the fact that  $w_H * \delta_x = \delta_x * w_H = w_H * \delta_x * w_H$ .

2. We prove it in two steps:

$$\begin{aligned}
\text{(i)} \quad \alpha(\mu_1 \mu_2) &= w_H * \mu_F * \mu_1 * u_2 * \mu_E * w_H \\
&= w_H * \mu_F * \mu_1 * w * \mu_2 * \mu_E * w_H \\
&= w_H * \mu_F * \mu_1 * \mu_E * w_H^2 * \mu_F * \mu_2 * w_H \\
&= \alpha(\mu_1)\alpha(\mu_2), \\
\beta(\nu_1 \nu_2) &= \mu_E * \nu_1 * \nu_2 * \mu_F \\
&= \mu_E * \nu_1 * w_H * \nu_2 * \mu_F \\
&= \mu_E * \nu_1 * \mu_F * \mu_E * w_H * \nu_2 * \mu_F \\
&= \beta(\nu_1)\beta(\nu_2). \\
\text{(ii)} \quad \alpha \circ \beta(\nu) &= \alpha(\mu_E * \nu * \mu_F) \\
&= w_H * \mu_F * \mu_E * \nu * \mu_F * \mu_E * w_H \\
&= w_H * \nu * w_H \\
&= \nu, \\
\beta \circ \alpha(\mu) &= \beta(w_H * \mu_F * \mu * \mu_E * w_H) \\
&= \mu_E * w_H * \mu_F * \mu * \mu_E * w_H * \mu_F \\
&= w * \mu * w \\
&= \mu.
\end{aligned}$$

PROPOSITION 3.2. *The following statements hold:*

1.  $D(\mathcal{H}(w_H)) = N(H)$  and  $\text{supp}(\mathcal{H}(w_H)) = \overline{N(H)}$
2.  $D(\mathcal{H}(w)) = E(N(H))F = \overline{[E, N(H), F]}$
3.  $\text{supp}(\mathcal{H}(w)) = E(N(H))F = \overline{[E, N(H), F]}$ .

*Proof.* 1. This follows from Proposition 3.1. 1.

2. This follows from Proposition 3.1. 2 and the above statement.

3. This follows from 2.

So far, we have only an algebraic characterization of  $\mathcal{H}(w)$ . In the remainder, we will characterize  $\mathcal{H}(w)$  and its subgroups topologically.

PROPOSITION 3.3. *The map  $\eta: N(H)/H \rightarrow \mathcal{H}(w_H)$  defined via*

$$\eta(xH) = w_H * \delta_x (= \delta_x * w_H)$$

*is a topological isomorphism.*

*Proof.* We observe first that  $\eta$  is a well-defined algebraic isomorphism. Hence it remains to show that  $\eta$  is an open map. To

each  $f \in C(S)$ ,  $F_f(x) = \int f(xy)w_H(dy)$  is a bounded continuous function constant on each orbit  $xH$  in the compact orbit space  $eSe/H$ . Without losing generality, we may assume that  $eSe = S$ . Suppose that  $a_\alpha H \rightarrow aH$  in  $N(H)/H$ . Then

$$\delta_{a_\alpha} * w_H(f) = F_f(a_\alpha H) \rightarrow F_f(aH) = \delta_a * w_H(f).$$

Hence  $\eta$  is a continuous group-morphism. Suppose that  $a_\alpha H \not\rightarrow aH$ . Since  $\overline{N(H)}/H$  is compact, there is a subnet  $(a_\beta H)$  which converges to a  $bH \neq aH$ . By Urysohn's Lemma, there is a continuous function  $F: S \rightarrow [0, 1]$  with  $F(aH) = 0$  and  $F(bH) = 1$ . Clearly,

$$\begin{aligned} \delta_{a_\alpha} * w_H(fop) &= Fop(a_\alpha) = F(a_\alpha H) \not\rightarrow F(aH) \\ &= Fop(a) = \delta_a * w_H(Fop), \end{aligned}$$

where  $p: S \rightarrow S/H$  is the orbit map. Hence  $\eta$  is a topological isomorphism.

The following example shows that not all  $\mathcal{H}(w)$  are compact:

EXAMPLE. Let  $S = R \cup \{\infty\}$  be the one-point compactification of the additive group of real numbers. Then  $S$  is a compact semitopological semigroup and  $\mathcal{H}(\delta_0) = \{\delta_x: x \in R\}$  which is not compact.

**4. On a limit theorem.** Rosenblatt has proved a necessary and sufficient condition for the convergence of a convolution sequence  $(\nu^n)_{n \geq 1}$  of a probability measure  $\nu$  on a compact topological semigroup (Thm. 1, p. 152, [8]). We will see one side of his condition is an immediate result of our characterizations of the groups of units.

PROPOSITION 4.1. *Let  $\nu \in P(S)$ . Then  $1/n(\nu + \nu^2 + \dots + \nu^n)$  converges to an idempotent probability measure  $L(\nu) \in P(S)$  so that*

1.  $\nu^m * L(\nu) = \underline{L(\nu)} * \nu^m = L(\nu)$  for all  $m, n \geq 1$
2.  $\text{supp } L(\nu) = \underline{M(T)}$ , where  $T$  is a closed subsemigroup generated by  $\nu$ , i.e.,  $T = \cup \{\text{supp } \nu^n: n \geq 1\}$ .

*Proof.* (See Thm. 3, [2]).

In the remainder, we maintain that  $\Sigma(\nu) = \{\nu^n: n \geq 1\}^-$ ,  $K(\nu) = M(\Sigma(\nu))$  and  $L(\nu) = \lim 1/n(\nu + \nu^2 + \dots + \nu^n) = \mu_X * w_G * \mu_Y$ . Without losing generality, we may assume that  $S$  is generated by  $\nu$ , i.e.,  $S = T$ . Then  $\text{supp } L(\nu) = \underline{M(S)}$ ,  $G = eSe = e(\text{supp } L(\nu))e$ ,  $X = E(Se) = E((\text{supp } L(\nu))e) = \text{supp } \mu_X$  and

$$Y = E(eS) = E(e(\text{supp } L(\nu))) = \mu_Y$$

for an  $e \in E(M(\text{supp } L(\nu)))$  (cf. 3.5, p. 67, [1]). In particular,  $\mathcal{H}(L(\nu)) = \mu_X * \{w_G\} * \mu_Y = \{L(\nu)\}$ .

LEMMA C.  $K(\nu)$  is a compact commutative topological subgroup in  $P(S)$ .

*Proof.* (See the proof of 3.4, p. 67, [1]).

Let  $w^2 = w = \mu'_E * w_H * \mu'_F \in K(\nu)$ . In particular,  $K(\nu)$  is a compact subgroup of  $\mathcal{H}(w)$ . Then.

LEMMA D. *The following statements hold:*

1.  $E(M(\text{supp } w)) = E(D(\mathcal{H}(w))) = E(D(K(\nu)))$
2.  $D(K(\nu)) \subseteq M(S) \subseteq \text{supp } K(\nu)$ . In particular,  $\text{supp } K(\nu) = M(S)$
3.  $E(M(\text{supp } w)) \subseteq M(S)$ .

*Proof.* 1. This follows from the fact that

$$E([E, H, F]) = [E, \{e\}, F] = E([E, N(H), F]) = E(D(\mathcal{H}(w))).$$

2. As  $K(\nu)$  is an ideal in  $\Sigma(\nu)$ ,  $D(K(\nu))D(\Sigma(\nu)) \subseteq D(K(\nu)) \subseteq \text{supp}(K(\nu))$  and so  $\text{supp}(K(\nu))$  is a closed ideal in  $S$  (See 3.1, p. 65, [1]), in particular,  $\text{supp}(K(\nu)) \supseteq M(S)$ . On the other hand,  $D(K(\nu)) = M(\text{supp}(K(\nu)))$  (See 3.1, p. 65, [1]) and thus  $M(S) \supseteq D(K(\nu))$ .

LEMMA E. *The following statements hold:*

1.  $\nu * w = w * \nu \in K(\nu)$
2.  $L(\nu) * w = w * L(\nu) = L(\nu)$
3. *There exists an  $e^2 = e \in M(\text{supp } w) \cap M(S)$*
4.  $H = e(\text{supp } w)e \subseteq eSe = G$
5.  $E = E((\text{supp } w)e) = E(Se) = X$
6.  $F = E(e(\text{supp } w)) = E(eS) = Y$
7.  $YX \subseteq H$
8.  $w = \mu'_X * w_H * \mu'_Y$  with  $\text{supp } \mu'_X = X$  and  $\text{supp } \mu'_Y = Y$ .

*Proof.* 1. This follows from the fact that  $K(\nu) = M(\Sigma(\nu))$ .

2. This follows from Proposition 4.1.

2. This follows from Lemma D.

4. This is trivial.

5. Let  $w = \mu'_E * w_H * \mu'_F$ . That  $L(\nu) = w * L(\nu) = L(\nu) * w$  implies



$$\begin{aligned}\mu_X * w_G * \mu &= (\mu'_E * w_H * \mu'_F) * (\mu_X * w_G * \mu_Y) \\ &= \mu'_E * w_G * \mu_Y \in P(X) * w_G * P(Y).\end{aligned}$$

By Propositions 2.1 and 2.2,  $\overline{EGY} = \overline{XGY}$  and  $E = X$ .

6. Similarly.
7. This follows from 5 and 6.
8. This is done in the proof of 2.

LEMMA F. *The following statements are equivalent*

1.  $\nu * w = w * \nu \neq w$
2.  $K(\nu) \neq \{w\}$
3.  $w \neq L(\nu)$
4.  $H$  is a proper closed normal subgroup in  $G$  (i.e.,  $N(H) = G$ ) so that  $G = \cup \{g^n H : n \geq 1\}$  for some  $g \in G - H$ .

*Proof.*  $1 \Rightarrow 2$ . This is trivial.

$2 \Rightarrow 3$ . Suppose that  $w = L(\nu)$ . Then  $K(\nu) = \mathcal{H}(L(\nu)) = \{L(\nu)\}$ . This is a contradiction. Hence  $w \neq L(\nu)$ .

$3 \Rightarrow 1$ . Suppose that  $w = w * \nu = \nu * w$ . Then

$$w = w * (1/n(\nu + \nu^2 + \cdots + \nu^n)) = 1/n(\nu + \nu^2 + \cdots + \nu^n) * w$$

for all  $n \geq 1$ . In particular,  $w = w * L(\nu) = L(\nu) * w = L(\nu)$ .

$1 \Rightarrow 4$ . There is a  $g \in N(H) - H$  so that  $w * \nu = \mu'_X * (w_H * \delta_g) * \mu'_Y$ .

Let  $\mathcal{H}(w) \xrightleftharpoons[\beta]{\alpha} \mathcal{H}(w_H)$  be the mutually inverse continuous morphisms of Proposition 3.2. Then

$$\begin{aligned}w * \nu^n &= (w * \nu)^n = (\beta \circ \alpha(w * \nu))^n \\ &= \beta((\alpha(w * \nu))^n) \\ &= \beta((w_H * \delta_g)^n) \\ &= \beta(w_H * \delta_{g^n}) \\ &= \mu'_X * (w_H * \delta_{g^n}) * \mu'_Y.\end{aligned}$$

Furthermore,  $\cup_{n \geq 1} (\text{supp } \nu^n \text{ supp } w) = (\cup_{n \geq 1} \text{supp } \nu^n)(\text{supp } w)$  and

$$\begin{aligned}\overline{(\cup \text{supp } \nu^n)(\text{supp } w)} &\supseteq \overline{(\cup \text{supp } \nu^n)}(\text{supp } w) \\ &= S(\text{supp } w) = \overline{(XGY)(XHY)} = \overline{XGY}\end{aligned}$$

(cf. 3.1, p. 55, [1] for the inclusion). This implies  $w * \nu$  generates  $\overline{XGY}$  and thus  $\alpha(w * \nu) = w_H * \delta_g$  generates  $G$ , i.e.,  $G = \overline{\cup \{g^n H : n \geq 1\}}$ . That  $N(H) = G$  follows easily.

$4 \Rightarrow 2$ . Suppose  $K(\nu) = \{w\}$ . Then  $w = L(\nu)$ , in particular,  $H = G$ .

PROPOSITION 4.2. *The following statements are equivalent:*

1.  $H = G$ .
2.  $L(\nu) = w$ .
3.  $K(\nu) = \{w\}$ .
4.  $w * \nu = \nu * w = w$ .

PROPOSITION 4.3. *If  $(\nu^n)_{n \geq 1}$  converges, then any statement of Proposition 4.2 holds. The converse holds on compact topological semigroups only.*

*Proof.* The first statement is trivial. For the converse part, we refer to (p. 380, [2]).

THEOREM (Rosenblatt). *Let  $S$  be a compact topological semigroup generated by  $\nu$ . Then  $(\nu^n)_{n \geq 1}$  does not converge iff there is a proper closed normal subgroup  $H$  of  $G$  such that*

$$[X, H, Y] \text{ supp } \nu = [X, Hg, Y]$$

for some  $g \in G - H$  with  $G = \overline{\cup \{g^n H : n \geq 1\}}$ .

*Proof.* It remains to show the "if" part which we refer to (Thm. 1, p. 152, [8]).

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