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FULL CONVEX l-SUBGROUPS AND THE EXISTENCE OF a^*-CLOSURES OF LATTICE ORDERED GROUPS

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An affirmative answer to the question of whether an arbitrary lattice-ordered group has an a^*-closure is the main result of this paper. This result is obtained by first introducing the notion of a full convex l-subgroup which is closely analogous to the notion of a closed convex l-subgroup.

The first two sections of this paper are a development of the basic properties of full convex l-subgroups. In §3 we define an f-extension of an l-group, the direct analogue of the definition of an a^*-extension. It is the existence of f-closures which we prove in §4; the existence of a^*-closures is a corollary to the proof. We believe the study of full convex l-subgroups will continue to enrich the theory of lattice-ordered groups.

G and H will denote lattice-ordered groups throughout. $G \leqq H$ will mean that G is an l-subgroup of H. For a subset X of G, $\mathrm{Cn}(G, X)$ and $\mathrm{Cl}(G, X)$ will denote the smallest convex l-subgroup of G containing X and the smallest closed convex l-subgroup of G containing X, respectively. This notation will be shortened whenever the result is unambiguous; for example, $\mathrm{Cn}(G, \{x\})$ and $\mathrm{Cn}(G, X \cup \{x\})$ may be written $\mathrm{Cn}(x)$ and $\mathrm{Cn}(X \cup \{x\})$.

1. Full convex l-subgroups. A set X of positive elements of G is *full* if, for positive y, $\mathrm{Cl}(x) = \mathrm{Cl}(y)$ and $x \in X$ imply $y \in X$. A convex l-subgroup will be said to be full if the set of its positive elements is full.

THEOREM 1.1. *For a convex l-subgroup C of the l-group G the following are equivalent:*

(i) *C is full.*

(ii) *C is a union of closed convex l-subgroups.*

(iii) $C = \cup \{\mathrm{Cl}(c) \mid 1 \leqq c \in C\}$.

(iv) *If D is a finitely generated l-subgroup of C then* $\mathrm{Cl}(D) \subseteqq C$.

(v) *For each $c \in C$, the strictly positive elements of each positive coset of* $\mathrm{Cn}(c)$ *which lies outside C have a strictly positive lower bound.*

Proof. The equivalence of the first four conditions is clear upon recollecting that every finitely generated convex l-subgroup is generated by a single element. To show that (iii) implies (v), let c be a member of C, let X be the strictly positive elements of some positive coset of $\mathrm{Cn}(c)$ which lies outside C, and let y be any member of X. If X has no strictly positive lower bound then $\inf X = 1$ whence $\sup\{yx^{-1} \mid x \in X\} = y$. Since $yx^{-1} \in \mathrm{Cn}(c)$ for all $x \in X$, $y \in \mathrm{Cl}(c)$, contradicting (iii). Now suppose (iii) does not hold; let g satisfy $1 < g \in \mathrm{Cl}(c) - C$ for some $c \in C$. Now $g = \sup X$ for some set X of positive elements of $\mathrm{Cn}(c)$. Therefore $1 = \inf\{x^{-1}g \mid x \in X\}$. Since this last is a set of positive elements of $\mathrm{Cn}(c)g$, (v) does not hold.

For any subset X of G let $\mathrm{Fl}(G, X)$ be the smallest full convex l-subgroup of G containing X. A set X of elements of a partially ordered set is *upper* (*lower*) *directed* if for all x and y in X there is a z in X such that $z \geqq x$ and $z \geqq y$($z \leqq x$ and $z \leqq y$).

LEMMA 1.2. *If X is an upper directed set of positive elements of G then* $\mathrm{Fl}(X) = \bigcup\{\mathrm{Cl}(x) \mid x \in X\}$.

Proof. By Theorem 1.1 part (ii) it is enough to show that $A = \bigcup\{\mathrm{Cl}(x) \mid x \in X\}$ is a convex l-subgroup of G. Note that $1 \leqq g \leqq a \in A$ implies $y \in A$ since this implication is true for each $\mathrm{Cl}(x)$. For the same reason, $g \in A$ if and only if $|g| \in A$. Consider first positive elements a and b of A; say $a \in \mathrm{Cl}(x)$ and $b \in \mathrm{Cl}(y)$ for x and y in X. Then $ab \in \mathrm{Cl}(z)$ where z is an element of X exceeding x and y. Now for arbitrary a and b from A we have $|a|$, $|b| \in A$ so that $|a|\,|b|\,|a| \in A$ which implies $|ab| \in A$ and $ab \in A$. The arguments for $a \vee b \in A$ and $a \wedge b \in A$ are similar.

If Fl is replaced by Cl or Cn in parts (ii) and (iii) of the next lemma, the resulting statements are known to be true (ref. Proposition 3.4 of [4], lemma 3.2 of [3]).

LEMMA 1.3. *For an l-group G,*

(i) $\mathrm{Cl}(g) = \mathrm{Fl}(g)$ *for any* $g \in G$.

(ii) $\mathrm{Fl}(A \cap B) = \mathrm{Fl}(\mathrm{A}) \cap \mathrm{Fl}(B)$ *for convex l-subgroups A and B.*

(iii) $\mathrm{Fl}(A \cup \{a\}) \cap \mathrm{Fl}(A \cup \{b\}) = \mathrm{Fl}(A \cup \{a \wedge b\})$ *for positive elements a and b and convex l-subgroups A of G.*

Proof. (i) is a result of Theorem 1.1 part (ii). If $1 < x \in \mathrm{Fl}(A) \cap \mathrm{Fl}(B)$ then by Lemma 1.2 there are positive elements a in A and b in B such that

$$x \in \mathrm{Cl}(a) \cap \mathrm{Cl}(b) = \mathrm{Cl}(a \wedge b) \subseteqq \mathrm{Fl}(A \cap B)\,.$$

Since the opposite containment is clear, (ii) is proved. Part (iii) follows from part (ii) and the statement which results from replacing Fl by Cn in (iii).

2. Full prime convex l-subgroups. The theorem for closed convex l-subgroups analogous to the following theorem is known to be true.

THEOREM 2.1. *The full convex l-subgroup K of G is prime if and only if the full convex l-subgroups containing K form a totally ordered set.*

Proof. If K is prime then *all* convex l-subgroups containing K form a totally ordered set. If K is not prime there are elements a and b not belonging to K such that $a \wedge b = 1$. Then

$$\mathrm{Fl}\,(K \cup \{a\}) \cap \mathrm{Fl}\,(K \cup \{b\}) = \mathrm{Fl}\,(K \cup \{a \wedge b\}) = \mathrm{Fl}\,(K) = K\,.$$

Therefore if $\mathrm{Fl}\,(K \cup \{a\}) \subseteqq \mathrm{Fl}\,(K \cup \{b\})$ then $\mathrm{Fl}\,(K \cup \{a\}) = K$, contradicting $a \notin K$. Similarly $\mathrm{Fl}\,(K \cup \{b\})$ cannot be contained in $\mathrm{Fl}\,(K \cup \{a\})$.

THEOREM 2.2. *Suppose S is a lower directed set of positive elements of G and D is maximal among convex l-subgroups of G which do not intersect S. Then D is prime and if S is full then so is D.*

Proof. Suppose $a \wedge b = 1$ but neither a nor b is in D. Then there are elements s and t of S such that $s \in \mathrm{Cn}\,(D \cup \{a\})$ and $t \in \mathrm{Cn}\,(D \cup \{b\})$. Let v be a member of S beneath t and s. By the Cn version of Lemma 1.3 (iii), $v \in \mathrm{Cn}\,(D \cup \{a\}) \cap \mathrm{Cn}\,(D \cup \{b\}) = D$, a contradiction. If S is full but D is not, then $\mathrm{Fl}(D)$ must properly contain D and therefore must intersect S. By Lemma 1.2. $\mathrm{Fl}\,(D) = \cup\,\{\mathrm{Cl}\,(d) \mid 1 < d \in D\}$ so there must be positive elements $d \in D$ and $s \in S$ with $s \in \mathrm{Cl}\,(d)$. Therefore $\mathrm{Cl}\,(s) = \mathrm{Cl}\,(s \wedge d)$. The fullness of S implies $d \wedge s \in S$, contradicting $S \cap D = \varnothing$.

If "full" is replaced by "closed" in any one of the next four propositions, a false statement results. These properties represent important differences between the full prime and the closed prime convex l-subgroups.

COROLLARY 2.3. *Suppose S is a lower directed set of positive elements of G and D is maximal among full convex l-subgroups of G which do not intersect S. Then D is prime.*

Proof. Let $T = \{1 \leqq g \in G \mid \mathrm{Cl}\,(g) = \mathrm{Cl}\,(s)$ for some $s \in S\}$. T is full, lower directed, and contains S but does not intersect D. By Zorn's Lemma, let C be maximal among convex l-subgroups of G which contain D but do not intersect T. Theorem 2.2 assures us that C must be not only full but also prime. The maximality of D forces $C = D$.

COROLLARY 2.4. *Every full convex l-subgroup is an intersection of prime full convex l-subgroups.*

THEOREM 2.5. *If C is a full convex l-subgroup of G and P is minimal among the prime convex l-subgroups containing D, then P is full.*

Proof. Let S be the set of positive elements not in P. S is lower directed since P is prime. By Zorn's lemma let D be maximal among full convex l-subgroups containing C and not intersecting S. Since $C \subseteqq D \subseteqq P$ and D is prime by Corollary 2.3, $P = D$.

COROLLARY 2.6. *Minimal prime convex l-subgroups are full.*

An ideal N of G is closed if and only if for all convex l-subgroups Q of G containing N, Q is closed whenever Q/N is closed (Lemma 3.4 of [1]). If "closed" is replaced by "full" in the previous statement, the result is false. The next two corollaries, however, are partial analogues.

COROLLARY 2.7. *If N is a closed ideal of G and K/N is full in G/N then K is full in G.*

Proof. Theorem 1.1 part (ii) with the theorem cited above.

COROLLARY 2.8. *If N is a full ideal of G and Q/N is minimal prime in G/N then Q is full in G.*

Proof. Q/N is minimal prime in G/N if and only if Q is minimal among prime convex l-subgroups of G containing N. Such a Q must be full by corollary 2.6.

In [3] Byrd and Lloyd prove that every convex l-subgroup containing a closed prime convex l-subgroup is closed and prime. The failure of the analogous phenomenon for full prime convex l-subgroups constitutes another important distinction between closed and full prime convex l-subgroups.

COROLLARY 2.9. *For an l-group G, every convex l-subgroup of G containing a full prime convex l-subgroup is itself full if and only if every convex l-subgroup is full.*

Proof. If every convex l-subgroup containing a full prime convex l-subgroup is full, then, since minimal prime convex l-subgroups are full, every prime convex l-subgroup is full. Since every convex l-subgroup is an intersection of prime convex l-subgroups, every one is full.

3. f-extensions. The methods and results of this section are closely analogous to those of §1 of [1].

Suppose G is an l-subgroup of H. If every pair of distinct full convex l-subgroups of H have distinct intersections with G then we say H is an *f-extension* of G and write $G \prec H$. Every a-extension is an f-extension and every f-extension is an a^*-extension.

Suppose $G \leqq H$, X is a set of positive elements of G, and $g \in G$. In the next several lemmas it will be necessary to distinguish between $g = \sup X$ in G and $g = \sup X$ in H. The first notation means that every element *of* G exceeding all members of X must exceed g. The second means every element *of* H exceeding all members of X must exceed g. $g = \sup X$ in H implies $g = \sup X$ in G but not conversely.

LEMMA 3.1. *Suppose* $G \leqq H$. *Then*

(i) $\mathrm{Cl}(H, g) \cap G \subseteqq \mathrm{Cl}(G, g)$ *for all positive* g *in* G.

(ii) $\mathrm{Fl}(H, K) \cap G = K$ *for* K *a full convex l-subgroup of* G.

(iii) $\mathrm{Fl}(H, K \cap G) \subseteqq K$ *and* $\mathrm{Fl}(H, K \cap G) \cap G = K \cap G$ *for* K *a full convex l-subgroup of* H.

Proof. If $1 < x \in \mathrm{Cl}(H, g) \cap G$ then $x = \sup \{x \wedge g^n \mid n = 1, 2, \cdots\}$ in H. Therefore $x = \sup \{x \wedge g^n\}$ in G so $x \in \mathrm{Cl}(G, g)$. (ii) follows from (i) and Lemma 1.2. (iii) is clear.

$\mathscr{F}(G)$, $\mathscr{K}(G)$, and $\mathscr{C}(G)$ will denote the complete distributive lattices of full convex l-subgroups of G, of closed convex l-subgroups of G, and of convex l-subgroups of G, respectively. $\mathscr{G}(G)$ will denote the distributive lattice $\{\mathrm{Cl}(g) \mid g \in G\}$. A subset I of a lattice L is an *ideal* if I is upper directed and $l \leqq k \in I$ implies $l \in I$ for all l in L.

LEMMA 3.2. *The ideals of the lattice* $\mathscr{G}(G)$ *are in one-to-one correspondence with* $\mathscr{F}(G)$.

Proof. If I is an ideal of $\mathscr{G}(G)$ then $\cup I \in \mathscr{F}(G)$ by Lemma 1.2. Conversely, $I = \{\mathrm{Cl}(g) 1 < g \in K\}$ is an ideal of $\mathscr{G}(G)$ for any convex l-subgroup K of G, and $\cup I = K$ if K is full.

A convex l-subgroup G of H is *large in* H if every nontrivial convex l-subgroup of H has nontrivial intersection with G.

THEOREM 3.3. *Suppose* $G \leqq H$, $K \in \mathscr{F}(H)$, $M \in \mathscr{F}(G)$. *Define* $K\tau = K \cap G$ *and* $M\delta = \mathrm{Fl}\,(H, M)$. *Then the following are equivalent:*

(i) $G \prec H$ *(τ is one-to-one).*

(ii) *τ is a lattice isomorphism from $\mathscr{F}(H)$ onto $\mathscr{F}(G)$.*

(iii) *δ maps $\mathscr{F}(G)$ onto $\mathscr{F}(H)$.*

(iv) *For every positive h in H there is a positive g in G such that* $\mathrm{Cl}\,(H, h) = \mathrm{Cl}\,(H, g)$.

(v) *τ is a lattice isomorphism from $\mathscr{G}(H)$ onto $\mathscr{G}(G)$.*

Proof. (i) implies (ii). If $G \prec H$ then every nontrivial full convex l-subgroup of H has nontrivial intersection with G. By the corollary to Theorem 1.7 of [1], G is large in H. By Lemma 1.8 of [1], if X is a subset of G then $g = \sup X$ in G if and only if $g = \sup X$ in H. Therefore $\mathrm{Cl}\,(H, g) \cap G = \mathrm{Cl}\,(G, g)$ for g in G, so τ must map $\mathscr{F}(H)$ into $\mathscr{F}(G)$. Lemma 3.1 part (ii) now yields (ii).

(ii) implies (iii) follows from Lemma 3.1 part (ii). If (iii) holds then for each positive h in H there is some full convex l-subgroup K of G such that

$$\mathrm{Cl}\,(H, h) = K\delta = \mathrm{Fl}\,(H, K) = \cup \{\mathrm{Cl}\,(H, k) \mid 1 < k \in K\} .$$

This is only possible if there is some positive $g \in K$ with $\mathrm{Cl}\,(H, g) = \mathrm{Cl}\,(H, h)$; that is, if (iv) holds.

To show that (iv) implies (i) suppose J and K are full convex l-subgroups of H having identical intersection with G, and that $1 < k \in K$. By (iv) let g satisfy $1 < g \in G$ and $\mathrm{Cl}\,(H, g) = \mathrm{Cl}\,(H, k)$. Now $g \in G \cap K = G \cap J$ so $k \in \mathrm{Cl}\,(\mathrm{H}, g) \subseteqq J$. That is, $K \subseteqq J$. A symmetrical argument gives $J \subseteqq K$.

Thus far we have the equivalence of the first four conditions. That (ii) implies (v) is clear since an element K of $\mathscr{G}$ may be distinguished in the lattice $\mathscr{F}$ by the lattice property: for every subset X of $\mathscr{F}$, if $K \subseteqq \mathrm{Fl}\,(\cup X)$ then there is a finite subset Y of X such that $K \subseteqq \mathrm{Fl}\,(\cup Y)$. Conversely, if τ is a lattice isomorphism from $\mathscr{G}(H)$ onto $\mathscr{G}(G)$ it may be extended to a lattice isomorphism of $\mathscr{F}(H)$ onto $\mathscr{F}(G)$ by Lemma 3.2.

COROLLARY 3.4. *Suppose* $G \leqq H \leqq K$. *Then* $G \prec K$ *if and only if* $G \prec H$ *and* $H \prec K$.

4. **Existence of f-closures and a^*-closures.** Suppose U is a class of l-groups containing H. If H has no proper f-extensions in U then H is said to be *f-closed relative to U*. If $G \in U$, $G \prec H$, and H is f-closed relative to U then we say that H is an *f-closure of G relative to U*. The purpose of this section is to show the existence

of f-closures relative to various classes (Theorem 4.10). The general procedure is that of §2 of [1].

THEOREM 4.1. *The union of l-groups which is totally ordered by $\prec$ is an f-extension of each l-group in the set.*

Proof. Suppose G is a member of X, a set of l-groups totally ordered by $\prec$. Let J and K be full convex l-subgroups of $\bigcup X$ such that $J \cap G = K \cap G$. For the sake of contradiction assume $J \neq K$ whence $J \cap M \neq K \cap M$ for some $M \in X$ with $G \prec M$. Now $(J \cap M) \cap G = (K \cap M) \cap G$ so $J \cap M$ and $K \cap M$ cannot both be in $\mathscr{F}(M)$; let x and y be positive members of M with $x \in K$ but $y \in \mathrm{Cl}(M, x) - K$. That is, $y = \sup\{x^n \wedge y \mid n = 1, 2, \cdots\}$ in M but not in $\bigcup X$. Therefore there must be a positive z in $\bigcup X - M$ with $y > z \geqq x^n \wedge y$ for all n. Let N be a member of X containing z. Then $y \in \mathrm{Cl}(M, x) - (\mathrm{Cl}(N, x) \cap M)$, contradicting $M \prec N$ by Theorem 3.3 part (v).

The next several lemmas have as their goal the establishment of a cardinality bound on G dependent only on $\mathscr{F}(G)$ (Theorem 4.8). For this purpose we first consider $A(T)$, the l-group of order-preserving permutations of the totally ordered set T (ref. [6]). An l-subgroup G of $A(T)$ is said to be *transitive on* T if for every s and t in T there is some g in G such that $(s)g = t$. For fixed t in T, $G_t = \{g \in G \mid (t)g = t\}$, a prime convex l-subgroup of G.

LEMMA 4.2. *Suppose G is a transitive l-subgroup of $A(T)$ for some totally ordered set T. Suppose $s \in T$ and $S = \{t \in T \mid G_s = G_t\}$. Then for r and v in S there is a unique θ in $A(T)$ such that $(r)\theta = v$ and $\theta g = g\theta$ for all g in G.*

Proof. For each t in T define $(t)\theta = (v)g$ for some g in G such that $(r)g = t$. It is routine to verify that θ is well-defind and has the required properties, and that these properties specify θ uniquely.

The next result relies heavily on the methods of Khuon [7]. $|X|$ denotes the cardinality of the set X, $P(X)$ denotes the set of subsets of X, X^Y denotes the set of all maps from Y into X, and $\boldsymbol{R}$ denotes the set of real numbers.

THEOREM 4.3. *Suppose G is a transitive l-subgroup of $A(T)$ for some totally ordered set T, and that $s \in T$. Let β be $|\{G_t \mid t \in T\}|$, γ be $|\{Q \in \mathscr{C}(G) \mid G_s \subseteqq Q\}|$, and δ be* $\max(\beta, \boldsymbol{R}^\gamma)$. *Then $|T| \leqq \delta$ and $|G| \leqq |P(\delta)|$.*

Proof. Let $S = \{t \in T \mid G_t = G_s\}$. For each $r \in S$ let θ_r be the unique member of $A(T)$ which takes s to r and which commutes with every member of G. Let $Z = \{\theta_r \mid r \in S\}$. It is routine to verify that Z is a totally ordered l-subgroup of $A(T)$ and that the map $r \longrightarrow \theta_r$ is an order isomorphism from S onto Z. By a result of Conrad [5], $|Z| \leqq |\boldsymbol{R}^{\mathscr{C}(Z)}|$.

Claim. $|\mathscr{C}(Z)| \leqq \gamma$.

Proof of Claim. For $X \in \mathscr{C}(Z)$ let $V = \{r \in S \mid \theta_r \in X\}$. Notice that X is transitive on V and that $V\theta = V$ for all θ in X. Let $T(X)$ be the smallest convex subset of T containing V.

$T(X)$ is a convex G-block; that is, $(T(X))g \cap T(X)$ is either empty or $T(X)$ for each positive g in G. If not, elements t, u, and v from $T(X)$ and w from $T - T(X)$ can be found such that $(t)g = u$ and $(v)g = w$ for some positive g in G. The symmetry of the argument and the convexity of $T(X)$ allow us to assume $t \leqq u < v < w$. Let q and r from V satisfy $q \leqq t$ and $v \leqq r$. Let $\theta \in X$ take q to r. Then

$$q < w = (v)q = (v)\theta^{-1}g\theta \leqq (r)\theta^{-1}g\theta = (q)g\theta \leqq (t)g\theta = (u)\theta < (r)\theta \ .$$

The outer members of this inequality are in V, which implies $w \in T(X)$, a contradiction.

The correspondence $X \longrightarrow T(X)$ is one-to-one. For if X and Y are distinct members of $\mathscr{C}(Z)$ and $1 < \theta_r \in Y - X$, then, because Z is totally ordered, $X \subseteqq Y$ and $\theta_r > \theta_t$ for all $\theta_t \in X$. Therefore $t \in T(Y) - T(X)$, which is to say $T(X)$ and $T(Y)$ are distinct. Since distinct convex G-blocks correspond to distinct convex l-subgroups of G containing G_s, the claim is proved.

To complete the proof of the lemma observe that $S \cap Sg$ is either empty or S for every g in G. By the transitivity of G on T, the translates of S partition T into disjoint order isomorphic classes, each containing no more than $\boldsymbol{R}^{\gamma}$ elements. Since β is the number of distinct classes, the result follows.

LEMMA 4.4. *For* $C \in \mathscr{C}(G)$, $|\mathscr{F}(C)| \leqq |\mathscr{F}(G)|$.

Proof. The map $K \longrightarrow \mathrm{Fl}\,(G, K)$ is one-to-one from $\mathscr{F}(C)$ into $\mathscr{F}(G)$ by Lemma 3.1 part (ii).

LEMMA 4.5. *Suppose* $\cap \mathscr{M} = 1$ *where* $\mathscr{M}$ *is the set of maximal convex l-subgroups of G. Then* $|G| \leqq \max\,(|\boldsymbol{R}|, |P(\mathscr{F}(G))|)$.

Proof. With each positive g in G associate the map $\hat{g}$ defined by $(M)\hat{g} = Mg$ for each M in $\mathscr{M}$. $\hat{g}$ is a member of $\Pi\{G/M \mid M \in \mathscr{M}\}$, the set theoretic product of the totally ordered sets G/M. The association $g \to \hat{g}$ is one-to-one since $g \neq h$ implies $gh^{-1} \notin M$ for some $M \in \mathscr{M}$ which gives $Mg \neq Mh$. Therefore it is enough to bound $\Pi G/M$.

$|\mathscr{M}| \leqq |\mathscr{F}(G)|$ since distinct maximal convex l-subgroups contain distinct minimal prime convex l-subgroups, each of which is full by Corollary 2.6.

Fix $M \in \mathscr{M}$. Let N be $\cap \{g^{-1}Mg \mid 1 \leqq g \in G\}$, H be G/N, T be G/M and s be $M \in T$. We wish to apply Theorem 4.3 to H viewed as a transitive l-subgroup of $A(T)$. The stabilizers $\{H_t \mid t \in T\}$ are conjugates of M and therefore in $\mathscr{M}$. In the terminology of Theorem 4.3, $\beta \leqq |\mathscr{F}(G)|$ and $\gamma = 1$ from which it follows that $|G/M| \leqq \max(|\boldsymbol{R}|, |\mathscr{F}(G)|)$. Finally,

$$|\Pi G/M| \leqq |(G/M)^{\mathscr{F}(G)}| \leqq \max(|\boldsymbol{R}|, |P(\mathscr{F}(G))|) .$$

A positive member g of G is a *strong unit* if $\mathrm{Cn}(G, g) = G$. If an l-group has a strong unit, then every convex l-subgroup is contained in a maximal convex l-subgroup. For each positive g in G define $L(g)$ to be $\mathrm{Cl}(G, M)$ where M is the intersection of the maximal convex l-subgroups of $\mathrm{Cn}(G, g)$. Conrad and Bleier point out (discussion preceding Lemma 2.6 of [1]) that $g \notin L(g) \subsetneqq \mathrm{Cn}(G, g)$. This fact implies that $L(g) = \mathrm{Cl}(\mathrm{Cn}(G, g), M)$. The normality of M in $\mathrm{Cn}(G, g)$ implies that $L(g)$ is normal in $\mathrm{Cn}(G, g)$.

LEMMA 4.6. *Let g be a positive element of G, and let $L(g)$ be as above. Then the cardinality of the set of cosets of $L(g)$ in* $\mathrm{Cl}(G, g)$ *is bounded by max* $(|\boldsymbol{R}|, |P(\mathscr{F}(G))|)$.

Proof. Let H be $\mathrm{Cl}(G, g)/L(g)$. By Lemma 4.5, $|H| \leqq \max(|\boldsymbol{R}|, |P(\mathscr{F}(H))|)$. Since $L(g)$ is closed in $\mathrm{Cn}(G, g)$, Corollary 2.7 gives $|\mathscr{F}(H)| \leqq |\mathscr{F}(\mathrm{Cn}(G, g))|$. By Lemma 4.4, $|\mathscr{F}(\mathrm{Cn}(G, g)| \leqq |\mathscr{F}(G))|$. Therefore $|H| \leqq \max(|\boldsymbol{R}|, |P(\mathscr{F}(G))|)$. Consider now a positive k in $\mathrm{Cl}(G, g)$. We know $k = \sup\{k \wedge g^n \mid n = 1, 2, \cdots\}$, which implies $L(g)k = \sup\{L(g)(k \wedge g^n)\}$ since $L(g)$ is closed in G. By associating with each coset $L(g)k$ the countable subset $\{L(g)(k \wedge g^n) \mid n = = 1, 2, \cdots\}$ of H, the result follows.

The next lemma, due to McCleary, is proved in [1].

LEMMA 4.7. *Let X be a set of ordered pairs of subgroups of a group G such that $A \subsetneqq B$ for each pair $(A, B) \in X$, and such that for all $g \in G$ there is a pair $(A, B) \in X$ with $g \in B - A$. Then there* *is a one-to-one function taking G into the set theoretic cartesian product $\Pi\{A/B \mid (A, B) \in X\}$, where A/B is the set of cosets of B in A.*

THEOREM 4.8. *For any l-group G, $|G| \leqq \max(|\boldsymbol{R}|, |P(\mathscr{F}(G))|)$.*

Proof. Take $\{(\mathrm{Cl}(G, g), L(g)) \mid 1 < g \in G\}$ to be X in McCleary's lemma. Since the number of such pairs is at most $|\mathscr{K}(G)|^2$, the theorem follows from Lemma 4.6.

COROLLARY 4.9. *For any l-group G, $|G| \leqq \max(|\boldsymbol{R}|, |P^2(G))|)$.*

Proof. Since every full convex l-subgroup is a union of closed convex l-subgroups, $|\mathscr{K}(G)| \leqq |\mathscr{F}(G)| \leqq |P(\mathscr{K}(G))|$.

By a standard induction argument we arrive at the main result:

THEOREM 4.10. *Suppose U is a class of l-groups with the property that the union of any set of members of U totally ordered by* $\prec$ (a^*-extension) *is itself a member of U. Then every l-group of U has an f-closure* (a^*-closure) *relative to U.*

Some examples of important classes to which the preceding theorem applies are: the class of all l-groups, the class of abelian l-groups, the class of archimedean l-groups, the class of normal-valued l-groups, and the class of representable l-groups.

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