Pacific Journal of Mathematics

LINEAR OPERATORS FOR WHICH T^*T AND $T + T^*$ COMMUTE

STEPHEN LAVERN CAMPBELL

Vol. 61, No. 1 November 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

LINEAR OPERATORS FOR WHICH T^*T AND $T+T^*$ COMMUTE

STEPHEN L. CAMPBELL

This paper is about the bounded linear operators T acting in a separable Hilbert space $\mathscr{H}$ such that T^*T and $T+T^*$ commute. It will be shown that such operators are normal if they are either compact or quasinilpotent. It is conjectured that if T^*T and $T+T^*$ commute, then $T=A+Q$ where $A=A^*$, $AQ=QA$, and Q is quasinormal. This conjecture is shown to be equivalent to $[T^*T-TT^*]T[T^*T-TT^*]$ being hermitian.

For bounded linear operators X, Y, let $[X, Y]=XY-YX$. Let $\theta=\{T: [T^*T, T+T^*]=0\}$. The defining condition for θ appears in the work of Embry. She has shown that if $\sigma(T^*)\cap\sigma(T)=\emptyset$ and T or T^* are in θ, then T is normal [9, p. 236]. She has also shown that if $T\in\theta$ and $[T^*T, TT^*]=0$, then T is quasinormal [8, p. 459]. On the other hand if Q is quasinormal, $A=A^*$, and $[A, Q]=0$, then $A+Q\in\theta$. Thus Embry's result shows that the intersection of the class (BN) $=\{T: [T^*T, TT^*]=0\}$ (see [4] and [5]) and θ is trivial, i.e., the quasinormals. In particular, there are no nonquasinormal centered [11] operators in θ. These last observations are helpful when trying to construct examples of nonquasinormal operators in θ since (BN) includes all weighted shifts and most weighted translation operators. Using [13] it is also easy to see that if T^2 is normal and $T\in\theta$, then T is normal.

It seems reasonable to make the following conjecture:

$$\text{(C)}\qquad \theta=\{A+Q: [Q, Q^*Q]=0, [Q, A]=0, A^*=A\}.$$

If (C) is true, then using the canonical form for quasinormals given in [1], it is easy to see that every operator in θ is subnormal. While we have not been able to resolve (C) we shall present several results which show that the operators in θ behave much as if they were hyponormal. In particular, we shall show that if $T\in\theta$ is compact or quasinilpotent, then it is normal. This will strengthen the result in [6] which asserts that if $T\in\theta$ and T is trace class, then T is normal.

Finally, let $B(\lambda)=(\lambda-T^*)(\lambda-T)=\lambda^2-\lambda(T^*+T)+T^*T$. Note that if $T\in\theta$, then the values of $B(\lambda)$ form a commutative family of normal operators.

2. Main results. Recall from [6] that if $T\in\theta$, then $\lambda+T\in\theta$ for real λ. Also if $T\in\theta$, then the null space of T, $N(T)$, is reducing.

Finally, $T \in \theta$ if and only if $T^*[T^*, T] = [T^*, T]T$.

THEOREM 1. *Suppose that $T \in \theta$ and λ is an eigenvalue of T. Then the eigenspace of T associated with λ is reducing.*

Proof. Suppose that $T \in \theta$ and λ is an eigenvalue. If λ is real, we are done. Suppose that λ is not real. Since $N(T)$ is reducing we may also assume that T is one-to-one. Let ϕ be such that $T\phi = \lambda\phi$. Then $[T^*, T]\phi = (\lambda - T)T^*\phi$. Thus $T^*[T^*, T]\phi = [T^*, T]T\phi$ becomes $B(\lambda)T^*\phi = 0$. Since $B(\lambda)$ is normal, and $B(\lambda)^* = B(\bar{\lambda})$, we have $B(\bar{\lambda})T^*\phi = 0$. Thus

$$0 = \lambda B(\bar{\lambda})T^*\phi = B(\bar{\lambda})T^*T\phi = T^*TB(\bar{\lambda})\phi ,$$

so that $B(\bar{\lambda})\phi = 0$. But then

$$0 = B(\bar{\lambda})\phi = (\bar{\lambda} - T^*)(\bar{\lambda} - T)\phi = (\bar{\lambda} - \lambda)(\bar{\lambda} - T^*)\phi .$$

Hence $T^*\phi = \bar{\lambda}\phi$ and the eigenspace is reducing.

That the eigenspaces of a hyponormal operator are reducing is well known. See, for example, [12, p. 420].

THEOREM 2. *If $T \in \theta$ and T is quasinilpotent, then $T = 0$.*

Proof. Suppose that $T \in \theta$ and $\sigma(T) = \{0\}$. We may assume that T is one-to-one if T is not zero. If $T^*T(T + T^*) = 0$, we are done. Suppose then that $T^*T(T + T^*) \neq 0$. Since $\sigma(T) = \{0\}$, $B(\lambda)$ is invertible for all $\lambda \neq 0$. Let $E(\cdot)$ be the spectral measure associated with the commutative Banach $*$-algebra generated by T^*T and $T + T^*$. Then there exist E measurable functions g, h such that

$$T^*T = \int_\Delta g(s)E(ds),\ T^* + T = \int_\Delta h(s)E(ds)$$

and Δ is a compact subset of the plane. (In fact $\Delta \subseteqq \sigma(T^*T) \times \sigma(T^* + T)$.) Since $(T^*T)(T + T^*) \neq 0$, there exists $s_0 \in \Delta$, s_0 in the support of E, such that $g(s_0)$, $h(s_0)$ are in the E-essential ranges of g, h, respectively, and both $g(s_0)$, $h(s_0)$ are nonzero. The polynomial $\lambda^2 + h(s_0)\lambda + g(s_0)$ has at least one nonzero root. Call it λ_0. Then

$$B(\lambda_0) = \int_\Delta \Big(\lambda_0^2 + h(s)\lambda_0 + g(s)\Big)E(ds)$$

is not invertible which is a contradiction. Hence $T = 0$.

As an immediate consequence of Theorems 1 and 2 we get:

COROLLARY 1. *If $T \in \theta$ and T is compact, then T is normal.*

Our next result has two interesting corollaries.

THEOREM 3. *Suppose that N is normal, $B \in \theta$, and $[N, B] = 0$. Then $N + B \in \theta$ if and only if, relative to the same orthogonal decomposition of the underlying Hilbert space, $N = N_1 \oplus N_2$, $B = B_1 \oplus B_2$, $N_1 = N_1^*$ and B_2 is normal.*

Proof. The only if part is clear. Suppose then that $T = N + B \in \theta$ where N is normal, $[N, B] = 0$, and $B \in \theta$. Note that $[N, B^*] = 0$ by Fuglede's theorem. Then $[T^*, T] = [B^*, B]$, so that $T^*[T^*, T] = [T^*, T]T$ becomes $(N^* - N)[B^*, B] = 0$. Let P be the orthogonal projection onto the null space of $N^* - N$. Then $PN = NP$ and $PB = BP$ since P is a measurable function of N. Thus the range of P reduces both N and B, so that $N = N_1 \oplus N_2$, $B = B_1 \oplus B_2$ relative to $R(P) \oplus R(I - P)$. But $N_1^* = N_1$ by definition of P and B_2 is normal since $P[B^*, B] = [B^*, B]$.

COROLLARY 2. *If $T \in \theta$, $\lambda + T \in \theta$, and λ is not real, then T is normal.*

COROLLARY 3. *If $T \in \theta$ and T is completely nonnormal, then there does not exist any nonhermitian normal operator N such that $[T, N] = 0$ and $T + N \in \theta$.*

3. Block matrix representation. If Conjecture (C) is true, then if $T \in \theta$ and T is completely nonnormal, T must have a lower triangular block matrix representation with all zero entries except on the diagonal and first subdiagonal. All diagonal entries are the same self-adjoint operator A, and all subdiagonal entries are the same positive operator P. This decomposition follows easily from the work of Brown on quasinormal operators [1].

It is easy to compute what subspace the first block corresponds to. It is the closure of the range of $T^*T - TT^*$. Morrel has developed a decomposition for operators T which have a subspace of $N[T^*T - TT^*]$ invariant [10]. Applying this to $T \in \theta$ yields a lower triangular block representation for T provided that $T^*T - TT^*$ is not one-to-one. If this approach is to verify Conjecture (C) then it will be necessary and sufficient to show that $[T^*T - TT^*]T[T^*T - TT^*]$ is hermitian.

THEOREM 4. *Suppose that $T \in \theta$ is completely nonnormal. If $[T^*, T]T[T^*, T]$ is hermitian, then $T = A + Q$ where $[A, Q] = 0$, $A =$*

A^*, $[Q, Q^*Q] = 0$.

Proof. Suppose that $T \in \theta$ is completely nonnormal and $[T^*, T]T[T^*, T]$ is hermitian. If $[T^*, T]$ is one-to-one we have $T = T^*$ and are done. Assume then that $[T^*, T]$ is not one-to-one. Since T is nonnormal we have $[T^*, T] \neq 0$. Thus from [10] we get that

$$(1) \qquad \begin{bmatrix} A_1 & 0 & 0 & \cdot \\ B_1 & A_2 & 0 & \cdot \\ 0 & B_2 & A_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

on $\mathscr{H} = \sum_{i=0}^{l} \oplus H_i$, $H_0 = \overline{R([T^*, T])}$, $l \leqq \infty$, $\dim H_i \geqq \dim H_{i+1}$. By assumption $A_1 = A_1^*$. But then $[T^*, T] = B_1^* B_1$ so that B_1 is one-to-one. Using the fact that $H_0 = \overline{R([T^*, T])}$ one gets by direct computation from (1) that

$$(2) \qquad B_i^* A_{i+1} = A_i B_i^* \ , \quad A_{i+1}^* A_{i+1} + B_{i+1}^* B_{i+1} = B_i B_i^* + A_{i+1} A_{i+1}^*$$

for $i = 1, 2, \cdots$ where $A_{l+1} = B_{l+1} = 0$ if $l < \infty$. Furthermore, by definition of the H_i we have B_i has dense range so that B_i^* is one-to-one. Now since $T^*[T^*, T] = [T^*, T]T$ we have that $A_1 B_1^* B_1 = B_1^* B_1 A_1$, or $B_1^* A_2 B_1 = B_1^* A_2^* B_1$ Since B_1 is one-to-one with dense range we get that $A_2 = A_2^*$. But then from (2), we see that $B_2^* B_2 = B_1 B_1^*$ and B_2 is one-to-one. Thus from $B_2^* A_3 = A_2 B_2^*$ we get that $B_2^* A_3 B_2 = A_2 B_2^* B_2 = A_2 B_1 B_1^* = B_1 A_1 B_1^* = B_1 B_1^* A_2$. Hence $A_3 = A_3^*$ and $[A_2, B_2^* B_2] = 0$. Suppose now that $A_i = A_i^*$, $[A_i, B_i^* B_i] = 0$, $B_{i+1}^* B_{i+1} = B_i B_i^*$, and B_i is one-to-one with dense range for $i \leqq k$. Then B_{k+1} is one-to-one with dense range. Also $B_k^* A_{k+1} B_k = A_k B_k^* B_k$ and hence $A_{k+1}^* = A_{k+1}$. Thus $B_{k+2}^* B_{k+2} = B_{k+1} B_{k+1}^*$ so that B_{k+2} is one-to-one with dense range. But then $A_{k+1} B_{k+1}^* B_{k+1} = A_{k+1} B_k B_k^* = B_k A_k B_k^* = B_k B_k^* A_{k+1}$. Hence $[A_{k+1}, B_{k+1}^* B_{k+1}] = 0$.

If $l < \infty$, then the lth equation is $A_{l+1}^* A_{l+1} = B_l B_l^* + A_{l+1} A_{l+1}^*$. As before we get $A_{l+1}^* = A_{l+1}$ and hence $B_l = 0$. But then $B_i = 0$ for all i which is a contradiction of the nonnormality of T. Thus $l = \infty$. Now let

$$A = \begin{bmatrix} A_1 & 0 & & \\ 0 & A_2 & & \\ & & \cdot & \\ & & & \cdot \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdot \\ B_1 & 0 & 0 & \cdot \\ 0 & B_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Then $B^*A = AB^*$ from (2). But $A = A^*$ so that $[B, A] = 0$. Hence $B = T - A \in \theta$. However $B^*[B^*, B] = 0$ so that $B^*(B^*B) = (B^*B)B^*$ and B is quasinormal as desired.

3. Comments. The conclusion of Theorem 1, that eigenspaces are reducing, appears in the work of Berberian. Using Theorem 1, it follows immediately from [3, p. 276] that if $T \in \theta$, $\sigma(T)$ is countable, and T is reduction-isoloid [3, p. 277], then T is normal.

In studying nonnormal operators one usually picks off a normal summand and studies the completely nonnormal operator that is left. Theorem 1 tells us that any condition which provides for eigenvalues is incompatible with the complete nonnormality of a $T \in \theta$. Thus one can prove results such as [2, p. 190], [3, p. 277].

THEOREM 5. *If $T \in \theta$ is completely nonnormal and T is also (G_1) or restriction convexoid, then $\sigma(T)$ has no isolated points.*

Finally, we note that the restriction of a $T \in \theta$ to an invariant subspace is not necessarily in θ. The quasinormal operator in [7] whose restriction to an invariant subspace is not quasinormal is an example.

REFERENCES

1. Arlen Brown, *On a class of operators*, **4** (1953), 723-728.
2. S. K. Berberian, *Some conditions on an operator implying normality*, Math. Ann., **184** (1970), 188-192.
3. S. K. Berberian, *Some conditions on an operator implying normality, II*, Proc. Amer. Math. Soc., **26** (1970), 277-281.
4. Stephen L. Campbell, *Linear operators for which T^*T and TT^* commute*, Proc. Amer. Math. Soc., **34** (1972), 177-180.
5. ———, *Linear operators for which T^*T and TT^* commute*, II, Pacific J. Math., **53** (1974), 355-361.
6. ———, *Operator-valued inner functions analytic on the closed disc*, II, Pacific J. Math., (to appear).
7. ———, *Subnormal operators with non-trivial quasinormal extensions*, Acta Sci. Math. Szeged. (to appear).
8. Mary R. Embry, *Conditions implying normality in Hilbert space*, Pacific J. Math., **18** (1966), 457-460.
9. ———, *A connection between commutativity and separation of spectra of operators*, Acta Sci. Math. Szeged., **32** (1971), 235-237.
10. Bernard B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J., **23** (1973), 497-511.
11. Bernard B. Morrel and P. S. Muhly, *Centered operators*, Studia Math., **LI** (1974), 251-263.
12. C. R. Putnam, *Spectra of polar factors of hyponormal operators*, Trans. Amer. Math. Soc., **188** (1974), 419-428.
13. H. Radjavi and P. Rosenthal, *On roots of normal operators*, J. Math. Anal. and Appl., **34** (1971), 653-664.

Received May 28, 1975 and in revised form July 28, 1975.

NORTH CAROLINA STATE UNIVERSITY—RALEIGH

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 61, No. 1 November, 1975