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**KATO-TAUSKY-WIELANDT COMMUTATOR RELATIONS AND  
CHARACTERISTIC CURVES**

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## KATO-TAUSSKY-WIELANDT COMMUTATOR RELATIONS AND CHARACTERISTIC CURVES

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**Let  $A$  and  $B$  be  $n \times n$  matrices with elements in a field  $\mathcal{F}$  and let  $\Delta_A B = AB - BA$ . Let  $f_k(x) = x^{2K+1} - c_1 x^{2K-1} + c_2 x^{2K-3} + \dots + (-1)^K c_K x$ , where the  $c_i$  are in  $\mathcal{F}$  and  $K = k(k-1)/2$ . In this paper we examine the consequences of the relation  $f_k(\Delta_A)B = 0$ , where  $1 \leq k < n$ , and show how the replacement of  $A$  by  $xA + yB$ , when  $k = 2$ , leads to a splitting of the characteristic curve,  $\det(xA + yB - zI) = 0$ , into lines and conics.**

1. Introduction. We open with some notation and some definitions. Let  $\mathcal{F}$  be a field and let  $\mathcal{F}_n$  denote the  $n \times n$  matrices with elements in  $\mathcal{F}$ . If  $A$  and  $B \in \mathcal{F}_n$ , the *characteristic curve* of the pencil  $xA + yB$  is the curve in the projective  $x, y, z$ -plane whose equation is  $\det(xA + yB - zI) = 0$ . If  $A \in \mathcal{F}_n$ , the operator  $\Delta_A$  is given by  $\Delta_A X = AX - XA$ , for all  $X$  in  $\mathcal{F}_n$ . If  $k \geq 1$  is an integer and  $K = k(k-1)/2$  and if  $c_1, c_2, \dots, c_K \in \mathcal{F}$ , we let  $f_k(x) = x^{2K+1} - c_1 x^{2K-1} + \dots + (-1)^K c_K x$ . Next we need some ideas usually associated with the Perron-Frobenius theory of nonnegative matrices. If  $X = (x_{ij}) \in \mathcal{F}_n$ , the *digraph*  $\mathcal{G}(X)$  consists of vertices labeled  $1, 2, 3, \dots, n$  and there is an edge from  $i$  to  $j$ , i.e.  $i \rightarrow j$ , if and only if  $x_{ij} \neq 0$ . The matrix  $X \in \mathcal{F}_n$  is *permutation-irreducible* if and only if it can not be transformed by a permutation similarity to the form  $\begin{pmatrix} Y & Z \\ 0 & W \end{pmatrix}$  where  $Y$  and  $W$  are square matrices.

In [4] Taussky and Wielandt proved.

**THEOREM 1.** *If  $A$  and  $B \in \mathcal{F}_n$  and  $\alpha_1, \alpha_2, \dots, \alpha_n$  are the eigenvalues (in some extension field of  $\mathcal{F}$ ) of  $A$ , then  $f_n(\Delta_A)B = 0$ , where  $c_i$  is the  $i$ th elementary symmetric function of the  $N = n(n-1)/2$  quantities  $(\alpha_r - \alpha_s)^2$ ,  $1 \leq r < s \leq n$ ;  $i = 1, 2, \dots, N$ .*

Since  $f_1(\Delta_A)B = AB - BA$ , the relation  $f_k(\Delta_A)B = 0$ , with  $1 < k < n$ , for some  $c_1, c_2, \dots, c_K \in \mathcal{F}$ , is a generalization of commutativity. As a generalization of matrix commutativity it is, however, quite weak, since it is still possible for  $A$  and  $B$  to satisfy such an identity (when  $n = 3$ ) and to generate  $\mathcal{F}_3$  (see the examples in §4). However, it will be shown that the relation  $f_k(\Delta_A)B = 0$  imposes a restriction on the eigenvalues of  $A$ , when  $A$  and  $B$  generate  $\mathcal{F}_n$ . We call an expression of the form  $f_k(\Delta_A)B$  a *Kato-Taussky-Wielandt commutator*.

We shall need one well-known result from graph theory which

can be found, for example, in [5].

**THEOREM 2.**  $X \in \mathcal{F}_n$  is permutation-irreducible if and only if  $\mathcal{G}(X)$  is strongly connected.

2. The main theorem. Our principal result is

**THEOREM 3.** Let  $A$  and  $B \in \mathcal{F}_n$  and suppose  $A$  and  $B$  generate  $\mathcal{F}_n$ , i.e. every matrix in  $\mathcal{F}_n$  has the form  $P(A, B)$ , where  $P(x, y)$  is a polynomial over  $\mathcal{F}$  in the noncommuting indeterminates  $x$  and  $y$ . If the characteristic of  $\mathcal{F}$  does not divide  $n$  and if, for some fixed integer  $k$  with  $1 \leq k < n$ , there exist  $c_1, c_2, \dots, c_K$  in  $\mathcal{F}$  so that  $f_k(\Delta_A)B = 0$ , then the eigenvalues of  $A$  belong to the splitting field of  $f_k(x)$  over  $\mathcal{F}$ .

*Proof.* If  $k = 1$  then  $AB = BA$  and, since  $A$  and  $B$  generate  $\mathcal{F}_n$ , we get  $n = 1$ . The theorem is then obvious.

If  $A$  has only one eigenvalue  $\alpha$ , then  $n\alpha = \text{trace } A$  is in  $\mathcal{F}$  and, consequently,  $\alpha \in \mathcal{F}$ , since the characteristic of  $\mathcal{F}$  does not divide  $n$ .

So assume that  $1 < k < n$ , that  $A$  has at least two distinct eigenvalues and that there exist  $c_1, c_2, \dots, c_K \in \mathcal{F}$  so that  $f_k(\Delta_A)B = 0$ . By extending  $\mathcal{F}$  to its algebraic closure  $\overline{\mathcal{F}}$ , we may assume (via similarity) that  $A = \sum_{i=1}^r \oplus A_i$  is in Jordan canonical form, where  $A_i$  is a direct sum of Jordan blocks, all of which have the same eigenvalue  $\alpha_i$  and  $\alpha_i \neq \alpha_j$  when  $i \neq j$ . We then view  $B$  as a matrix over  $\overline{\mathcal{F}}$  (via the similarity above) and let  $B = (B_{ij})$  be the partition of  $B$  into blocks corresponding to that of  $A = \sum_{i=1}^r \oplus A_i$ . We shall prove

**LEMMA 1.** If  $B_{ij} \neq 0$ , then  $\alpha_i - \alpha_j$  satisfies  $f_k(x) = 0$ .

**LEMMA 2.**  $B = (B_{ij})$  is permutation-irreducible as a block matrix.

If we assume these lemmas we can complete the proof of the theorem in a few lines. Let  $\mathcal{G}(B)$  be the digraph of  $B$  viewed as a block matrix, i.e.  $i \rightarrow j$  if and only if  $B_{ij} \neq 0$ . Then Lemma 2 and a modification of Theorem 2 (for block matrices) imply that  $\mathcal{G}(B)$  is strongly connected. Thus, if  $\alpha_i, \alpha_j$  are distinct eigenvalues of  $A$ , there exists a sequence  $i, i_1, i_2, \dots, i_m, j$  so that  $\alpha_i - \alpha_{i_1}, \alpha_{i_1} - \alpha_{i_2}, \dots, \alpha_{i_m} - \alpha_j$  are roots of  $f_k(x) = 0$ . Let  $\mathcal{L}$  be the splitting field of  $f_k(x)$  over  $\mathcal{F}$ . Then

$$\alpha_i - \alpha_j = (\alpha_i - \alpha_{i_1}) + (\alpha_{i_1} - \alpha_{i_2}) + \dots + (\alpha_{i_m} - \alpha_j) \in \mathcal{L}.$$

Let  $n_j$  be the multiplicity of  $\alpha_j$  as an eigenvalue of  $A$ . Then

$$\sum_{j=1}^r n_j \alpha_i - \sum_{j=1}^r n_j \alpha_j \in \mathcal{L},$$

i.e.  $n\alpha_i - \text{trace } A \in \mathcal{L}$ . Thus  $\alpha_i \in \mathcal{L}, i = 1, 2, \dots, r$ , since the characteristic of  $\mathcal{L}$  does not divide  $n$ .

It remains to prove the lemmas to complete the proof of the theorem.

*Proof of Lemma 1.* We use the relation  $f_k(\Delta_A)B = 0$ . Suppose  $B_{ij} \neq 0$ . Let  $b_{st}$  be the "first" nonzero element of  $B_{ij}$  in the following sense: if the lower left-hand corner element of  $B_{ij}$  is nonzero let it be  $b_{st}$ ; otherwise let  $b_{st}$  be a nonzero element of  $B_{ij}$  so that  $b_{uv} = 0$  if  $u \geq s$  and  $v \leq t$ , and  $(u, v) \neq (s, t)$ , where, of course, we only consider those elements  $b_{uv}$  of  $B_{ij}$ . Thus

$$B_{ij} = \begin{bmatrix} & & & & * & & \\ & & & & & & * \\ 0 & \cdot & \cdot & \cdot & 0 & b_{st} & \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \\ \cdot & & & & \cdot & \cdot & * \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \end{bmatrix}.$$

If  $b_{st}$  is the  $(s, t)$  element of  $B_{ij}$  we calculate the  $(s, t)$  element of the  $(i, j)$  block of  $f_k(\Delta_A)B$ . To simplify calculations assume that  $A_j$  has eigenvalue zero and  $A_i$  has eigenvalue  $\alpha_{ij} = \alpha_i - \alpha_j$  (Subtract  $\alpha_j I$  from  $A$ . Since we take commutators, this operation does not affect the end result of the calculations). The matrix  $f_k(\Delta_A)B$  is a linear combination of matrices of the type  $\Delta_A^m B$ . The  $(i, j)$  block of  $\Delta_A B$  is  $A_i B_{ij} - B_{ij} A_j$ . The  $(i, j)$  block of  $\Delta_A^m B$  only involves  $B_{ij}, A_i$  and  $A_j$ ; it consists of a linear combination of matrices of the type  $A_i^c B_{ij} A_j^d$  where  $c + d = m$ . The  $(s, t)$  element of  $A_i^c B_{ij} A_j^d$  is obtained by multiplying the  $t$ th column of  $B_{ij} A_j^d$  by the  $t$ th row of  $A_i^c$ . Those elements in the  $t$ th column of  $B_{ij} A_j^d$  from the  $s$ th row down are all that matter here. But these elements are zero, except when  $d = 0$ , since  $A_j$  has zeros on and below the main diagonal. Thus the  $(s, t)$  element of the  $(i, j)$  block of  $\Delta_A^m B$  is  $\alpha_{ij}^m b_{st}$ . So the equation  $f_k(\Delta_A)B = 0$  gives  $f_k(\alpha_{ij}) = 0$ , since  $b_{st} \neq 0$ .

*Proof of Lemma 2.* Suppose there exists a block permutation matrix  $Q$ , partitioned conformally with  $B = (B_{ij})$ , so that  $Q^{-1}BQ$  has the form  $(\dagger)$  where  $m < r$ . Then  $A$  and  $B$  are reduced by  $Q$ , since  $Q$  simply permutes the blocks on the diagonal of  $A$ . Thus the algebra generated by  $A$  and  $B$  over  $\overline{\mathcal{F}}$  is reducible. But  $A$  and  $B$  generate  $\overline{\mathcal{F}}_n$ . This

contradiction proves that  $B = (B_{ij})$  is permutation-irreducible as a block matrix.

$$(\dagger) \quad \begin{bmatrix} B_{11} & \cdot & \cdot & \cdot & B_{1m} & B_{1\ m+1} & \cdot & \cdot & \cdot & B_{1r} \\ \cdot & & & & \cdot & \cdot & & & & \cdot \\ \cdot & & & & \cdot & \cdot & & & & \cdot \\ B_{m1} & \cdot & \cdot & \cdot & B_{mm} & B_{m\ m+1} & \cdot & \cdot & \cdot & B_{mr} \\ 0 & \cdot & \cdot & \cdot & 0 & B_{m+1\ m+1} & \cdot & \cdot & \cdot & B_{m+1\ r} \\ \cdot & & & & \cdot & \cdot & & & & \cdot \\ \cdot & & & & \cdot & \cdot & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & B_{r\ m+1} & \cdot & \cdot & \cdot & B_{rr} \end{bmatrix}$$

**THEOREM 4.** *Let  $A$  and  $B$  satisfy the conditions of Theorem 3 with  $k = 2$  and let  $A$  have at least two distinct eigenvalues. Then there exists an ordering  $\alpha_1, \alpha_2, \dots, \alpha_r$  of the distinct eigenvalues of  $A$  so that  $\alpha_1 - \alpha_2 = \alpha_2 - \alpha_3 = \dots = \alpha_{r-1} - \alpha_r$  satisfies  $x^2 - c_1 = 0$ .*

*Proof.* As in the proof of Theorem 3, let  $B = (B_{ij})$  be the block form of  $B$  (over  $\mathcal{F}$ ) corresponding to the Jordan canonical form  $\sum_{i=1}^r \oplus A_i$  of  $A$ . We have  $f_2(\mathcal{A}_A)B = 0$ , for some  $c_1 \in \mathcal{F}$ .

We claim that  $B$  can not have more than two nonzero off-diagonal blocks in each row or column. For if  $B_{ij}, B_{il}$  and  $B_{im}$  are nonzero off-diagonal blocks, where  $j, l$ , and  $m$  are distinct, then  $\alpha_i - \alpha_j, \alpha_i - \alpha_l$  and  $\alpha_i - \alpha_m$  satisfy the equation  $x^2 - c_1 = 0$ . Thus two of these  $\alpha$ 's, at least, are equal, contradicting the fact that the  $\alpha$ 's are distinct. If  $B_{ij}$  and  $B_{il}$  (resp.  $B_{ji}$  and  $B_{li}$ ) are nonzero off-diagonal blocks with  $j \neq l$ , and  $B_{mi}$  (resp.  $B_{im}$ ) is also a nonzero off-diagonal block, a similar argument proves  $m = j$  or  $m = l$ .

Let  $\mathcal{G}(B)$  be the digraph of  $B$  viewed as a block matrix. We write  $i \sim j$  if  $i \rightarrow j$  or  $j \rightarrow i$ . So if  $i \sim j$  and  $i \sim l$ , where  $i, j$  and  $l$  are distinct, then  $i \sim m$  implies either  $j$  or  $l$  is  $m$ . We claim that, by relabeling the vertices of  $\mathcal{G}(B)$ , we get the subgraph

$$\begin{array}{ccccccc}
 \cdot & \text{---} & \cdot & \text{---} & \cdot & \cdots & \cdot & \text{---} & \cdot \\
 1 & & 2 & & 3 & & r-1 & & r
 \end{array}$$

where  $i \sim i + 1$ , for  $i = 1, 2, \dots, r - 1$ . For let

$$\mu = \begin{array}{ccccccc}
 \cdot & \text{---} & \cdot & \text{---} & \cdot & \cdots & \cdot & \text{---} & \cdot \\
 1 & & 2 & & 3 & & s-1 & & s
 \end{array}$$

be a maximal "path" in  $\mathcal{G}(B)$  (on relabeling vertices), where  $i \sim i + 1$  for  $i = 1, 2, \dots, s - 1$ , and suppose  $s \neq r$ . If  $j$  is a vertex not in  $\mu$  then neither  $j \sim 1$  nor  $j \sim s$  can hold, since  $\mu$  is maximal. Since  $\mathcal{G}(B)$  is strongly connected, there exists an internal vertex  $i \in \mu$

and a vertex  $j \notin \mu$  so that  $i \sim j$ . But  $i \sim i + 1$  and  $i \sim i - 1$  and, since neither of these is  $j$ , we get a contradiction. Thus  $\mathcal{G}(B)$  contains the required subgraph. By Lemma 1, this means the distinct eigenvalues  $\alpha_i$  of  $A$  can be relabeled so that  $\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \dots, \alpha_{r-1} - \alpha_r$  satisfy the equation  $x^2 - c_1 = 0$ . Since the  $\alpha$ 's are distinct, we get

$$\alpha_i - \alpha_{i+1} = \alpha_{i+1} - \alpha_{i+2}, \quad 1 = 1, 2, \dots .$$

This completes the proof of the theorem.

**3. Generalized  $L$ -property.** Let  $A$  and  $B$  be  $n \times n$  matrices with elements in  $\mathcal{F}$  and suppose the eigenvalues of  $A$  and  $B$  are also in  $\mathcal{F}$ . If there exist fixed orderings  $\alpha_1, \alpha_2, \dots, \alpha_n$  and  $\beta_1, \beta_2, \dots, \beta_n$  of the eigenvalues of  $A$  and  $B$ , respectively, so that the eigenvalues of  $xA + yB$  are  $x\alpha_i + y\beta_i$ , for all  $x$  and  $y$  in  $\mathcal{F}$ , where  $i = 1, 2, \dots, n$ , then  $A$  and  $B$  have *property  $L$* . Property  $L$  has been discussed by Motzkin and Taussky [3]; it is clearly equivalent to the assertion that the characteristic curve of the pencil  $xA + yB$  is the union of  $n$  lines (if  $\mathcal{F}$  is big enough). In this section we discuss a condition which forces the characteristic curve to decompose into lines and conics.

Let  $\mathcal{F}[x, y]$  be the integral domain of polynomials over  $\mathcal{F}$  in the (commuting) indeterminates  $x$  and  $y$ . Let  $\mathcal{F}(x, y)$  be its quotient field.

**LEMMA 3.** *Let  $p(x, y, z)$  be a homogeneous polynomial in  $x, y$  and  $z$ , with coefficients in  $\mathcal{F}$ . Suppose*

$$p(x, y, z) = \prod_{i=1}^r p_i^i,$$

*where each  $p_i$  is an irreducible polynomial in  $z$  over  $\mathcal{F}(x, y)$ . Then each  $p_i$  is a homogeneous polynomial in  $x, y$ , and  $z$ , with coefficients in  $\mathcal{F}$ .*

*Proof.*  $\mathcal{F}[x, y]$  is a unique factorization domain (UFD). Since a UFD is integrally closed ([2], p. 84), the coefficients of the powers of  $z$  in  $p_i$  must be polynomials in  $x$  and  $y$ .

Suppose  $p_i$  is not homogeneous in  $x, y$  and  $z$ . Let  $M(q)$  (resp.  $m(q)$ ) be the maximum (resp. minimum) degree of the monomials in a polynomial  $q$ . Then  $M(p_i) > m(p_i)$  and  $M$  (resp.  $m$ ) has the property that  $M(q_1 q_2) = M(q_1) + M(q_2)$  ( $m(q_1 q_2) = m(q_1) + m(q_2)$ ) for polynomials  $q_1$  and  $q_2$ . Hence  $M(p) > m(p)$ , which is false. The Lemma is proved.

We now apply the results of §2 to  $xA + yB$  and  $B$ .

**THEOREM 5.** *Let  $A$  and  $B \in \mathcal{F}_n$ , where  $\mathcal{F}$  is an infinite field*

whose characteristic does not divide  $n$ . If  $A$  and  $B$  generate  $\mathcal{F}_n$  and if, for each  $x$  and  $y$  in  $\mathcal{F}$ , there exists  $c_1$  in  $\mathcal{F}$  so that

$$f_2(\Delta_{xA+yB})B = 0 ,$$

then the characteristic polynomial  $p(x, y, z)$  of  $xA + yB$  splits into linear and quadratic homogeneous factors with coefficients in  $\mathcal{F}$ .

*Proof.* Without loss, we may assume that  $n \geq 3$ . Let  $X = xA + yB$ . If  $\Delta_x B = 0$ , for  $x \neq 0$ , then  $AB = BA$ , which implies  $n = 1$ . So  $\Delta_x B \neq 0$  (for  $x \neq 0$ ) and the relation  $f_2(\Delta_x)B = 0$  imply that  $c_1$  is a rational function of  $x$  and  $y$ . Since  $\mathcal{F}$  is infinite, we may replace  $x$  and  $y$  by two algebraically independent indeterminates and the relation  $f_2(\Delta_x)B = 0$  still holds.  $X$  and  $B$  clearly generate  $\mathcal{F}(x, y)_n$ , and thus we may apply Theorem 3. Since  $f_2(w) = w^3 - c_1 w$ , each eigenvalue of  $X = xA + yB$  satisfies an equation of degree at most 2 over  $\mathcal{F}(x, y)$ . Lemma 3 is now used to complete the proof.

**COROLLARY.** Let  $\mathcal{F}$  be an algebraically closed field of characteristic zero or greater than  $n$ . If  $A$  and  $B \in \mathcal{F}_n$  and if, for each  $x$  and  $y \in \mathcal{F}$ , there exists  $c_1 \in \mathcal{F}$  so that

$$f_2(\Delta_{xA+yB}) \in \mathcal{L} ,$$

where  $\mathcal{L}$  is the radical of the algebra generated by  $A$  and  $B$  over  $\mathcal{F}$ , then the characteristic polynomial  $p(x, y, z)$  of  $xA + yB$  splits into linear and quadratic homogeneous factors in  $x, y$  and  $z$  with coefficients in  $\mathcal{F}$ .

*Proof.* If  $A_1$  and  $B_1$  are the representatives of  $A$  and  $B$  respectively, in an irreducible representation of the algebra generated by  $A$  and  $B$  over  $\mathcal{F}$ , then  $f_2(\Delta_{X_1})B_1 = 0$ , where  $X_1 = xA_1 + yB_1$ . Also  $A_1$  and  $B_1$  generate a complete matrix algebra, since  $\mathcal{F}$  is algebraically closed.  $A$  and  $B$  may be transformed by a similarity into block upper triangular form, where the corresponding diagonal blocks generate irreducible matrix algebras. The conclusion follows.

**REMARK.** Nothing we have said so far forces the characteristic curve of  $xA + yB$  to contain a line. If  $\mathcal{F}$  has characteristic zero or greater than  $n$  and  $f_2(\Delta_{xA+yB})B = 0$  for some  $c_1 \in \mathcal{F}(x, y)$ , where  $x$  and  $y$  are algebraically independent over  $\mathcal{F}$ , and if  $xA + yB$  has an odd number of distinct eigenvalues, then at least one of the eigenvalues has the form  $x\alpha + y\beta$ , where  $\alpha, \beta \in \mathcal{F}$ . For let  $z_1, z_2, \dots, z_r$  be the distinct eigenvalues of  $xA + yB$ , with  $r$  odd. If  $r = 1$  the result is trivial. Let  $r \geq 3$ ; then, by Theorem 4, we may assume

the eigenvalues are ordered so that  $z_1 - z_2 = z_2 - z_3 = \dots z_{r-1} - z_r$  satisfies  $w^2 - c_1 = 0$ . By the condition on the characteristic of  $\mathcal{F}$ , the irreducible factors of  $p(x, y, z)$  — the characteristic polynomial of  $xA + yB$  — are separable. Hence  $\sum_{i=1}^r z_i \in \mathcal{F}(x, y)$ . Now

$$(1/r) \sum_{i=1}^r z_i = z_r + ((r - 1)/2)c_1^{1/2},$$

since  $z_i - z_{i+1} = c_1^{1/2}$ ,  $i = 1, 2, \dots, r - 1$ . Since  $r$  is odd,  $(r - 1)/2 = s$  is an integer and  $z_{r-s} = z_r + ((r - 1)/2)c_1^{1/2}$  is in  $\mathcal{F}(x, y)$ . By Lemma 3,  $z_{r-s} = x\alpha + y\beta$ , where  $\alpha$  and  $\beta \in \mathcal{F}$ .

4. Examples.

EXAMPLE 1. This example illustrates the main results of the paper. Let  $\mathcal{F}$  be a field of characteristic not 2 or 3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then  $A$  and  $B$  generate  $\mathcal{F}_3$ . If  $X = xA + yB$ , where  $x$  and  $y$  are algebraically independent over  $\mathcal{F}$ , then  $X$  and  $B$  generate  $\mathcal{F}(x, y)_3$  and, if  $c_1 = \sqrt{2x^2 + y^2}$ , then  $f_2(\mathcal{A}_X)B = 0$ . The characteristic polynomial of  $xA + yB$  is

$$(x + 2y - z)(z^2 - (2x + 4y)z - x^2 + 4xy + 3y^2)$$

(cf. Theorem 5). The eigenvalues of  $xA + yB$  are

$$z_2 = x + 2y$$

$$z_1 = x + 2y + \sqrt{2x^2 + y^2}, \quad z_3 = x + 2y - \sqrt{2x^2 + y^2}.$$

Clearly

$$z_1 - z_2 = z_2 - z_3 = \sqrt{2x^2 + y^2}$$

(cf. Theorem 4). We see that

$$z_1, z_2, z_3 \in \mathcal{F}(x, y, \sqrt{2x^2 + y^2})$$

(cf. Theorem 3).

EXAMPLE 2. The example we give here is a counterexample to Theorem 3 and Theorem 5, when the condition on the characteristic of  $\mathcal{F}$  is not satisfied. Let  $\mathcal{F}$  have characteristic 3 and let



$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then  $A$  and  $B$  generate  $\mathcal{F}_3$  and, if  $X = xA + yB$  and  $c_1 = 0$ , then  $f_2(\Delta_x)B = 0$ . Now  $f_2(w) = w^3$  and the characteristic polynomial of  $xA + yB$  is  $xy^2 - z^3 = ((xy^2)^{1/3} - z)^3$ . Theorems 3 and 5 clearly fail here.

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