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Let A and B be $n \times n$ matrices with elements in a field $\mathscr{F}$ and let $\Delta_A B = AB - BA$. Let $f_k(x) = x^{2K+1} - c_1 x^{2K-1} + c_2 x^{2K-3} + \cdots + (-1)^K c_K x$, where the c_i are in $\mathscr{F}$ and $K = k(k-1)/2$. In this paper we examine the consequences of the relation $f_k(\Delta_A)B = 0$, where $1 \leqq k < n$, and show how the replacement of A by $xA + yB$, when $k = 2$, leads to a splitting of the characteristic curve, $\det(xA + yB - zI) = 0$, into lines and conics.

1. Introduction. We open with some notation and some definitions. Let $\mathscr{F}$ be a field and let $\mathscr{F}_n$ denote the $n \times n$ matrices with elements in $\mathscr{F}$. If A and $B \in \mathscr{F}_n$, the *characteristic curve* of the pencil $xA + yB$ is the curve in the projective x, y, z-plane whose equation is $\det(xA + yB - zI) = 0$. If $A \in \mathscr{F}_n$, the operator Δ_A is given by $\Delta_A X = AX - XA$, for all X in $\mathscr{F}_n$. If $k \geqq 1$ is an integer and $K = k(k-1)/2$ and if $c_1, c_2, \cdots, c_K \in \mathscr{F}$, we let $f_k(x) = x^{2K+1} - c_1 x^{2K-3} + \cdots + (-1)^K c_K x$. Next we need some ideas usually associated with the Perron-Frobenius theory of nonnegative matrices. If $X = (x_{ij}) \in \mathscr{F}_n$, the *digraph* $\mathscr{G}(X)$ consists of vertices labeled $1, 2, 3, \cdots, n$ and there is an edge from i to j, i.e. $i \rightarrow j$, if and only if $x_{ij} \neq 0$. The matrix $X \in \mathscr{F}_n$ is *permutation-irreducible* if and only if it can not be transformed by a permutation similarity to the form $\begin{pmatrix} Y & Z \\ 0 & W \end{pmatrix}$ where Y and W are square matrices.

In [4] Taussky and Wielandt proved.

THEOREM 1. *If A and $B \in \mathscr{F}_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the eigenvalues* (in some extension field of $\mathscr{F}$) *of A, then $f_n(\Delta_A)B = 0$, where c_i is the ith elementary symmetric function of the $N = n(n-1)/2$ quantities $(\alpha_r - \alpha_s)^2$, $1 \leqq r < s \leqq n$; $i = 1, 2, \cdots, N$.*

Since $f_1(\Delta_A)B = AB - BA$, the relation $f_k(\Delta_A)B = 0$, with $1 < k < n$, for some $c_1, c_2, \cdots, c_K \in \mathscr{F}$, is a generalization of commutativity. As a generalization of matrix commutativity it is, however, quite weak, since it is still possible for A and B to satisfy such an identity (when $n = 3$) and to generate $\mathscr{F}_3$ (see the examples in § 4). However, it will be shown that the relation $f_k(\Delta_A)B = 0$ imposes a restriction on the eigenvalues of A, when A and B generate $\mathscr{F}_n$. We call an expression of the form $f_k(\Delta_A)B$ a *Kato-Taussky-Wielandt commutator*.

We shall need one well-known result from graph theory which can be found, for example, in [5].

THEOREM 2. *$X \in \mathscr{F}_n$ is permutation-irreducible if and only if $\mathscr{G}(X)$ is strongly connected.*

2. The main theorem. Our principal result is

THEOREM 3. *Let A and $B \in \mathscr{F}_n$ and suppose A and B generate $\mathscr{F}_n$,* i.e. *every matrix in $\mathscr{F}_n$ has the form $P(A, B)$, where $P(x, y)$ is a polynomial over $\mathscr{F}$ in the noncommuting indeterminates x and y. If the characteristic of $\mathscr{F}$ does not divide n and if, for some fixed integer k with $1 \leqq k < n$, there exist $c_1, c_2, \cdots, c_K$ in $\mathscr{F}$ so that $f_k(\Delta_A)B = 0$, then the eigenvalues of A belong to the splitting field of $f_k(x)$ over $\mathscr{F}$.*

Proof. If $k = 1$ then $AB = BA$ and, since A and B generate $\mathscr{F}_n$, we get $n = 1$. The theorem is then obvious.

If A has only one eigenvalue α, then $n\alpha =$ trace A is in $\mathscr{F}$ and, consequently, $\alpha \in \mathscr{F}$, since the characteristic of $\mathscr{F}$ does not divide n.

So assume that $1 < k < n$, that A has at least two distinct eigenvalues and that there exist $c_1, c_2, \cdots, c_K \in \mathscr{F}$ so that $f_k(\Delta_A)B = 0$. By extending $\mathscr{F}$ to its algebraic closure $\bar{\mathscr{F}}$, we may assume (via similarity) that $A = \sum_{i=1}^{r} \oplus A_i$ is in Jordan canonical form, where A_i is a direct sum of Jordan blocks, all of which have the same eigenvalue α_i and $\alpha_i \neq \alpha_j$ when $i \neq j$. We then view B as a matrix over $\bar{\mathscr{F}}$ (via the similarity above) and let $B = (B_{ij})$ be the partition of B into blocks corresponding to that of $A = \sum_{i=1}^{r} \oplus A_i$. We shall prove

LEMMA 1. *If $B_{ij} \neq 0$, then $\alpha_i - \alpha_j$ satisfies $f_k(x) = 0$.*

LEMMA 2. *$B = (B_{ij})$ is permutation-irreducible as a block matrix.*

If we assume these lemmas we can complete the proof of the theorem in a few lines. Let $\mathscr{G}(B)$ be the digraph of B viewed as a block matrix, i.e. $i \rightarrow j$ if and only if $B_{ij} \neq 0$. Then Lemma 2 and a modification of Theorem 2 (for block matrices) imply that $\mathscr{G}(B)$ is strongly connected. Thus, if α_i, α_j are distinct eigenvalues of A, there exists a sequence $i, i_1, i_2, \cdots, i_m, j$ so that $\alpha_i - \alpha_{i_1}, \alpha_{i_1} - \alpha_{i_2}, \cdots, \alpha_{i_m} - \alpha_j$ are roots of $f_k(x) = 0$. Let $\mathscr{L}$ be the splitting field of $f_k(x)$ over $\mathscr{F}$. Then

$$\alpha_i - \alpha_j = (\alpha_i - \alpha_{i_1}) + (\alpha_{i_1} - \alpha_{i_2}) + \cdots + (\alpha_{i_m} - \alpha_j) \in \mathscr{L} .$$

Let n_j be the multiplicity of α_j as an eigenvalue of A. Then

$$\sum_{j=1}^{r} n_j \alpha_i - \sum_{j=1}^{r} n_j \alpha_j \in \mathscr{L} ,$$

i.e. $n\alpha_i$ − trace $A \in \mathscr{L}$. Thus $\alpha_i \in \mathscr{L}$, $i = 1, 2, \cdots, r$, since the characteristic of $\mathscr{L}$ does not divide n.

It remains to prove the lemmas to complete the proof of the theorem.

Proof of Lemma 1. We use the relation $f_k(\Delta_A)B = 0$. Suppose $B_{ij} \neq 0$. Let b_{st} be the "first" nonzero element of B_{ij} in the following sense: if the lower left-hand corner element of B_{ij} is nonzero let it be b_{st}; otherwise let b_{st} be a nonzero element of B_{ij} so that $b_{uv} = 0$ if $u \geqq s$ and $v \leqq t$, and $(u, v) \neq (s, t)$, where, of course, we only consider those elements b_{uv} of B_{ij}. Thus

$$B_{ij} = \begin{bmatrix} & & * & & & & * \\ 0 & \cdot & \cdot & \cdot & 0 & b_{st} & \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \\ \cdot & & & & \cdot & \cdot & * \\ 0 & \cdot & \cdot & \cdot & 0 & 0 & \end{bmatrix} .$$

If b_{st} is the (s, t) element of B_{ij} we calculate the (s, t) element of the (i, j) block of $f_k(\Delta_A)B$. To simplify calculations assume that A_j has eigenvalue zero and A_i has eigenvalue $\alpha_{ij} = \alpha_i - \alpha_j$ (Subtract $\alpha_j I$ from A. Since we take commutators, this operation does not affect the end result of the calculations). The matrix $f_k(\Delta_A)B$ is a linear combination of matrices of the type $\Delta_A^m B$. The (i, j) block of $\Delta_A B$ is $A_i B_{ij} - B_{ij} A_j$. The (i, j) block of $\Delta_A^m B$ only involves B_{ij}, A_i and A_j; it consists of a linear combination of matrices of the type $A_i^c B_{ij} A_j^d$ where $c + d = m$. The (s, t) element of $A_i^c B_{ij} A_j^d$ is obtained by multiplying the tth column of $B_{ij} A_j^d$ by the tth row of A_i^c. Those elements in the tth column of $B_{ij} A_j^d$ from the sth row down are all that matter here. But these elements are zero, except when $d = 0$, since A_j has zeros on and below the main diagonal. Thus the (s, t) element of the (i, j) block of $\Delta_A^m B$ is $\alpha_{ij}^m b_{st}$. So the equation $f_k(\Delta_A)B = 0$ gives $f_k(\alpha_{ij}) = 0$, since $b_{st} \neq 0$.

Proof of Lemma 2. Suppose there exists a block permutation matrix Q, partitioned conformally with $B = (B_{ij})$, so that $Q^{-1}BQ$ has the form (†) where $m < r$. Then A and B are reduced by Q, since Q simply permutes the blocks on the diagonal of A. Thus the algebra generated by A and B over $\overline{\mathscr{F}}$ is reducible. But A and B generate $\overline{\mathscr{F}}_n$. This contradiction proves that $B = (B_{ij})$ is permutation-irreducible as a block matrix.

$$(\dagger) \qquad \begin{bmatrix} B_{11} & \cdots & B_{1m} & B_{1\,m+1} & \cdots & B_{1r} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ B_{ml} & \cdots & B_{mm} & B_{m\,m+1} & \cdots & B_{mr} \\ 0 & \cdots & 0 & B_{m+1\,m+1} & \cdots & B_{m+1\,r} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & B_{r\,m+1} & \cdots & B_{rr} \end{bmatrix}$$

THEOREM 4. *Let A and B satisfy the conditions of Theorem 3 with $k = 2$ and let A have at least two distinct eigenvalues. Then there exists an ordering $\alpha_1, \alpha_2, \cdots, \alpha_r$ of the distinct eigenvalues of A so that $\alpha_1 - \alpha_2 = \alpha_2 - \alpha_3 = \cdots = \alpha_{r-1} - \alpha_r$ satisfies $x^2 - c_1 = 0$.*

Proof. As in the proof of Theorem 3, let $B = (B_{ij})$ be the block form of B (over $\mathscr{F}$) corresponding to the Jordan canonical form $\sum_{i=1}^{r} \oplus A_i$ of A. We have $f_2(\varDelta_A)B = 0$, for some $c_1 \in \mathscr{F}$.

We claim that B can not have more than two nonzero off-diagonal blocks in each row or column. For if B_{ij}, B_{il} and B_{im} are nonzero off-diagonal blocks, where j, l, and m are distinct, then $\alpha_i - \alpha_j$, $\alpha_i - \alpha_l$ and $\alpha_i - \alpha_m$ satisfy the equation $x^2 - c_1 = 0$. Thus two of these α's, at least, are equal, contradicting the fact that the α's are distinct. If B_{ij} and B_{il} (resp. B_{ji} and B_{li}) are nonzero off-diagonal blocks with $j \neq l$, and B_{mi}(resp. B_{im}) is also a nonzero off-diagonal block, a similar argument proves $m = j$ or $m = l$.

Let $\mathscr{G}(B)$ be the digraph of B viewed as a block matrix. We write $i \sim j$ if $i \rightarrow j$ or $j \rightarrow i$. So if $i \sim j$ and $i \sim l$, where i, j and l are distinct, then $i \sim m$ implies either j or l is m. We claim that, by relabeling the vertices of $\mathscr{G}(B)$, we get the subgraph

$$\underset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{3}{\bullet} \quad \cdots \quad \underset{r-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{r}{\bullet}$$

where $i \sim i + 1$, for $i = 1, 2, \cdots, r - 1$. For let

$$\mu = \underset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{2}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{3}{\bullet} \quad \cdots \quad \underset{s-1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\underset{s}{\bullet}$$

be a maximal "path" in $\mathscr{G}(B)$ (on relabeling vertices), where $i \sim i + 1$ for $i = 1, 2, \cdots, s - 1$, and suppose $s \neq r$. If j is a vertex not in μ then neither $j \sim 1$ nor $j \sim s$ can hold, since μ is maximal. Since $\mathscr{G}(B)$ is strongly connected, there exists an internal vertex $i \in \mu$

and a vertex $j \notin \mu$ so that $i \sim j$. But $i \sim i+1$ and $i \sim i-1$ and, since neither or these is j, we get a contradiction. Thus $\mathscr{G}(B)$ contains the required subgraph. By Lemma 1, this means the distinct eigenvalues α_i of A can be relabeled so that $\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \cdots, \alpha_{r-1} - \alpha_r$ satisfy the equation $x^2 - c_1 = 0$. Since the α's are distinct, we get

$$\alpha_i - \alpha_{i+1} = \alpha_{i+1} - \alpha_{i+2}\,, \qquad 1 = 1, 2, \cdots .$$

This completes the proof of the theorem.

3. **Generalized L-property.** Let A and B be $n \times n$ matrices with elements in $\mathscr{F}$ and suppose the eigenvalues of A and B are also in $\mathscr{F}$. If there exist fixed orderings $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$ of the eigenvalues of A and B, respectively, so that the eigenvalues of $xA + yB$ are $x\alpha_i + y\beta_i$, for all x and y in $\mathscr{F}$, where $i = 1, 2, \cdots, n$, then A and B have *property L*. Property L has been discussed by Motzkin and Taussky [3]; it is clearly equivalent to the assertion that the characteristic curve of the pencil $xA + yB$ is the union of n lines (if $\mathscr{F}$ is big enough). In this section we discuss a condition which forces the characteristic curve to decompose into lines and conices.

Let $\mathscr{F}[x, y]$ be the integral domain of polynomials over $\mathscr{F}$ in the (commuting) indeterminates x and y. Let $\mathscr{F}(x, y)$ be its quotient field.

LEMMA 3. *Let $p(x, y, z)$ be a homogeneous polynomial in x, y and z, with coefficients in $\mathscr{F}$. Suppose*

$$p(x, y, z) = \prod_{i=1}^{r} p_i^{k_i}\,,$$

where each p_i is an irreducible polynomial in z over $\mathscr{F}(x, y)$. Then each p_i is a homogeneous polynomial in x, y, and z, with ooefficients in $\mathscr{F}$.

Proof. $\mathscr{F}[x, y]$ is a unique factorization domain (UFD). Since a UFD is integrally closed ([2], p. 84), the coefficients of the powers of z in p_i must be polynomials in x and y.

Suppose p_i is not homogeneous in x, y and z. Let $M(q)$ (resp. $m(q)$) be the maximum (resp. minimum) degree of the monomials in a polynomial q. Then $M(p_i) > m(p_i)$ and M (resp. m) has the property that $M(q_1q_2) = M(q_1) + M(q_2)(m(q_1q_2) = m(q_1) + m(q_2))$ for polynomials q_1 and q_2. Hence $M(p) > m(p)$, which is false. The Lemma is proved.

We now apply the results of §2 to $xA + yB$ and B.

THEOREM 5. *Let A and $B \in \mathscr{F}_n$, where $\mathscr{F}$ is an infinite field* *whose characteristic does not divide n. If A and B generate $\mathscr{F}_n$ and if, for each x and y in $\mathscr{F}$, there exists c_1 in $\mathscr{F}$ so that*

$$f_2(\Delta_{xA+yB})B = 0 ,$$

then the characteristic polynomial $p(x, y, z)$ of $xA + yB$ splits into linear and quadratic homogeneous factors with coefficients in $\mathscr{F}$.

Proof. Without loss, we may assume that $n \geqq 3$. Let $X = xA + yB$. If $\Delta_X B = 0$, for $x \neq 0$, then $AB = BA$, which implies $n = 1$. So $\Delta_X B \neq 0$ (for $x \neq 0$) and the relation $f_2(\Delta_X)B = 0$ imply that c_1 is a rational function of x and y. Since $\mathscr{F}$ is infinite, we may replace x and y by two algebraically independent indeterminates and the relation $f_2(\Delta_X)B = 0$ still holds. X and B clearly generate $\mathscr{F}(x, y)_n$, and thus we may apply Theorem 3. Since $f_2(w) = w^3 - c_1 w$, each eigenvalue of $X = xA + yB$ satisfies an equation of degree at most 2 over $\mathscr{F}(x, y)$. Lemma 3 is now used to complete the proof.

COROLLARY. *Let $\mathscr{F}$ be an algebraically closed field of characteristic zero or greater than n. If A and $B \in \mathscr{F}_n$ and if, for each x and $y \in \mathscr{F}$, there exists $c_1 \in \mathscr{F}$ so that*

$$f_2(\Delta_{xA+yB}) \in \mathscr{J} ,$$

where $\mathscr{J}$ is the radical of the algebra generated by A and B over $\mathscr{F}$, then the characteristic polynomial $p(x, y, z)$ of $xA + yB$ splits into linear and quadratic homogeneous factors in x, y and z with coefficients in $\mathscr{F}$.

Proof. If A_1 and B_1 are the representatives of A and B respectively, in an irreducible representation of the algebra generated by A and B over $\mathscr{F}$, then $f_2(\Delta_{X_1})B_1 = 0$, where $X_1 = xA_1 + yB_1$. Also A_1 and B_1 generate a complete matrix algebra, since $\mathscr{F}$ is algebraically closed. A and B may be transformed by a similarity into block upper triangular form, where the corresponding diagonal blocks generate irreducible matrix algebras. The conclusion follows.

REMARK. Nothing we have said so far forces the characteristic curve of $xA + yB$ to contain a line. If $\mathscr{F}$ has characteristic zero or greater than n and $f_2(\Delta_{xA+yB})B = 0$ for some $c_1 \in \mathscr{F}(x, y)$, where x and y are algebraically independent over $\mathscr{F}$, and if $xA + yB$ has an *odd* number of distinct eigenvalues, then at least one of the eigenvalues has the form $x\alpha + y\beta$, where $\alpha, \beta \in \mathscr{F}$. For let $z_1, z_2, \cdots, z_r$ be the distinct eigenvalues of $xA + yB$, with r odd. If $r = 1$ the result is trivial. Let $r \geqq 3$; then, by Theorem 4, we may assume the eigenvalues are ordered so that $z_1 - z_2 = z_2 - z_3 = \cdots z_{r-1} - z_r$ satisfies $w^2 - c_1 = 0$. By the condition on the characteristic of $\mathscr{F}$, the irreducible factors of $p(x, y, z)$ — the characteristic polynomial of $xA + yB$ — are separable. Hence $\sum_{i=1}^{r} z_i \in \mathscr{F}(x, y)$. Now

$$(1/r)\sum_{i=1}^{r} z_i = z_r + ((r-1)/2)c_1^{1/2},$$

since $z_i - z_{i+1} = c_1^{1/2}$, $i = 1, 2, \cdots, r-1$. Since r is odd, $(r-1)/2 = s$ is an integer and $z_{r-s} = z_r + ((r-1)/2)c_1^{1/2}$ is in $\mathscr{F}(x, y)$. By Lemma 3, $z_{r-s} = x\alpha + y\beta$, where α and $\beta \in \mathscr{F}$.

4. Examples.

EXAMPLE 1. This example illustrates the main results of the paper. Let $\mathscr{F}$ be a field of characteristic not 2 or 3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Then A and B generate $\mathscr{F}_3$. If $X = xA + yB$, where x and y are algebraically independent over $\mathscr{F}$, then X and B generate $\mathscr{F}(x, y)_3$ and, if $c_1 = \sqrt{2x^2 + y^2}$, then $f_2(\Delta_X)B = 0$. The characteristic polynomial of $xA + yB$ is

$$(x + 2y - z)(z^2 - (2x + 4y)z - x^2 + 4xy + 3y^2)$$

(cf. Theorem 5). The eigenvalues of $xA + yB$ are

$$z_2 = x + 2y$$

$$z_1 = x + 2y + \sqrt{2x^2 + y^2}, \quad z_3 = x + 2y - \sqrt{2x^2 + y^2}.$$

Clearly

$$z_1 - z_2 = z_2 - z_3 = \sqrt{2x^2 + y^2}$$

(cf. Theorem 4). We see that

$$z_1, z_2, z_3 \in \mathscr{F}(x, y, \sqrt{2x^2 + y^2})$$

(cf. Theorem 3).

EXAMPLE 2. The example we give here is a counterexample to Theorem 3 and Theorem 5, when the condition on the characteristic of $\mathscr{F}$ is not satisfied. Let $\mathscr{F}$ have characteristic 3 and let

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then A and B generate $\mathscr{F}_3$ and, if $X = xA + yB$ and $c_1 = 0$, then $f_2(\Delta_X)B = 0$. Now $f_2(w) = w^3$ and the characteristic polynomial of $xA + yB$ is $xy^2 - z^3 = ((xy^2)^{1/3} - z)^3$. Theorems 3 and 5 clearly fail here.

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