Pacific Journal of Mathematics

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THEODORE WILLIAM GAMELIN

Vol. 61, No. 1 November 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

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T. W. GAMELIN

Summary. Fix $\alpha \leqq 0$. We will describe the polynomial hulls of compact subsets of C^2 that are invariant under the transformations T_θ, $0 \leqq \theta \leqq 2\pi$, defined by

$$T_\theta(z,w) = (e^{i\theta}z, e^{i\alpha\theta}w)\,, \qquad (z,w) \in C^2\,. \tag{0.1}$$

1. **Introduction.** Let X be a compact subset of C^2. In [10], J. Wermer describes the polynomial hull $\hat{X}$ of X, for a certain class of sets X which are invariant under the one-parameter group of transformations

$$(z,\, w) \longrightarrow (e^{i\theta}z,\, e^{-i\theta}w)\,, \qquad 0 \leqq \theta \leqq 2\pi\,. \tag{1.1}$$

He develops an idea for introducing analytic structure in $\hat{X}\backslash X$, and he shows in particular that every point of $\hat{X}\backslash X$ lies on an analytic disc in $\hat{X}$.

Our aim is to combine Wermer's ideas with some elementary results in potential theory, in order to describe the polynomial hull $\hat{X}$ of an arbitrary compact subset X of C^2 invariant under the group (1.1), or more generally under the transformations defined by (0.1).

If the number α is irrational, then the transformation group defined by (0.1) is dense in the two-parameter group of transformations

$$(z,\, w) \longrightarrow (e^{i\theta}z,\, e^{i\varphi}w)\,, \qquad 0 \leqq \theta, \varphi \leqq 2\pi\,. \tag{1.2}$$

If X is invariant under the group (0, 1), then X is also invariant under the groups (1.2). Such sets are said to be *circled*. The description of the polynomial hull of compact circled sets is classical ([6], [2], [3], [4, § III. 3], [7, § 2.4], [9, § 14]). We may confine our attention to the case in which α is rational.

Consider next the case $\alpha = 0$. The transformations (0.1) then assume the form

$$(z,\, w) \longrightarrow (e^{i\theta}z,\, w)\,, \qquad 0 \leqq \theta \leqq 2\pi\,. \tag{1.3}$$

If X is invariant under the transformations (1.3), its polynomial hull can be described as follows [9]. Let J be the projection of X into the w-plane, and define

$$r(w) = \sup\{|z|\colon (z,\, w) \in X\}\,, \qquad w \in J\,.$$

Define a function R on the polynomial hull $\hat{J}$ of J by requiring that $\log R$ be the lower envelope of the family of functions u which are

superharmonic in a neighborhood of $\hat{J}$ and which satisfy $u \geqq \log r$ on J. Then

$$\hat{X} = \{(z, w): w \in \hat{J}, |z| \leqq R(w)\} .$$

(This description is valid for subsets of C^n, providing w is interpreted as an $(n-1)$-tuple in C^{n-1}, and "superharmonic" is replaced by "pluri-superharmonic".)

In §5 we formulate our main result, Theorem 5.2, which describes $\hat{X}$ in the case that $\alpha = -1$, that is, in the case that X is invariant under the group (1.1). The description again involves the envelopes of certain families of superharmonic functions.

Sections 2 through 4 include some preliminary results, while the proof of Theorem 5.2 is given in §§6 and 8. In §9, we indicate how the discussion can be modified to cover the case in which α is an arbitrary negative rational number. In §10 we observe that every point of $\hat{X}\backslash X$ lies on an analytic disc in $\hat{X}$, and that $P(X) = C(X)$ whenever there are no analytic discs in $\hat{X}$.

A source for standard definitions and notation is [4]. The complex plane will be denoted by C, and the open disc with center z and radius δ will be denoted by $\Delta(z; \delta)$.

The space of all continuous complex-valued functions on X is denoted by $C(X)$. The uniform closure in $C(X)$ of the analytic polynomials is denoted, as usual, by $P(X)$. Then $\hat{X}$ can be identified with the maximal ideal space of $P(X)$, and $P(X)$ is isometrically isomorphic to $P(\hat{X})$.

2. **A result from potential theory.** Let K be a compact subset of C, such that $C\backslash K$ is connected. Let φ be an upper semicontinuous function from K to $[-\infty, \infty)$. Let $\mathscr{C}$ be the family of real-valued functions ψ which are continuous and superharmonic on a neighborhood of K, and which satisfy $\psi \geqq \varphi$ on K. Let Ψ be the lower envelope of $\mathscr{C}$:

$$\Psi(z) = \inf \{\psi(z): \psi \in \mathscr{C}\} .$$

Then Ψ is an upper semi-continuous function from K to $[-\infty, \infty)$ which satisfies

$$\varphi(z) \leqq \Psi(z) , \qquad z \in K .$$

THEOREM 2.1. *Let K, φ and ψ be as above. Let Ω be the subset of $z \in K$ such that $\varphi(z) < \Psi(z)$. Then Ω is an open subset of C, and Ψ is harmonic on Ω.*

Proof. Since φ is upper semi-continuous, there exists $c > 0$ such

that

$$\varphi(z) < c\,, \qquad z \in K\,.$$

Then also $\Psi(z) < c$ for all $z \in K$.

First we show that $\varphi(z_0) = \Psi(z_0)$ if $z_0 \in \partial K$. This depends only on the existence of a barrier at z_0, and the proof is classical. Indeed, let $\varepsilon > 0$, and let N be a neighborhood of z_0 such that $\varphi(z) < \varphi(z_0) + \varepsilon$ for $z \in N \cap K$. Since z_0 is a regular point for the "outer" Dirichlet problem (or since z_0 is a peak point for $P(K)$), there exists a function u harmonic in a neighborhood of K, such that $u > 0$ on K, $u(z_0) < \varepsilon$, and $u(z) > c - \varphi(z)$ for $z \in K \backslash N$. The function $\psi = u + \psi(z_0)$ is then harmonic in a neighborhood of K, $\psi > \varphi$ on K, and $\psi(z_0) < \varphi(z_0) + \varepsilon$. It follows that $\Psi(z_0) < \varphi(z_0) + \varepsilon$, this for all $\varepsilon > 0$, so that

$$\Psi(z_0) = \varphi(z_0)\,, \qquad z_0 \in \partial K\,.$$

We have shown that Ω is a subset the interior of K.

Next we wish to show that Ω is open. We will use the following standard estimate, which follows for instance from [8, Theorem III. 67].

LEMMA 2.2. *There is a positive function $\chi(t)$, $t > 0$, such that $\chi(t) \to 0$ as $t \to 0$, which has the following property. If $\Delta_0 = \Delta(\zeta_0; \delta)$ is any open disc, if W is a domain such that $\zeta_0 \in W \subset \Delta_0$, if E is a connected subset of $\bar{\Delta}_0 \backslash W$ which meets $\partial \Delta_0$, and if $d = \inf\{|\zeta - \zeta_0|: \zeta \in E\}$, then the harmonic measure μ_{ζ_0} on ∂W for ζ_0 satisfies*

$$\mu_{\zeta_0}(\partial \Delta_0 \cap \partial W) \leqq \chi(d/\delta)\,.$$

Now let $\zeta_0 \in \Omega$. Choose a so that $\varphi(\zeta_0) < a < \Psi(\zeta_0)$. On account of the upper semi-continuity of φ, it suffices to show that $\Psi(\zeta) \geqq a$ for ζ near ζ_0. For this, it suffices to find $\delta_0 > 0$ such that $\psi(\zeta) \geqq a$ for all $\zeta \in \Delta(\zeta_0; \delta_0)$ and for all functions ψ which are continuous and superharmonic in a neighborhood of K and satisfy $\psi \geqq \varphi$ on K.

So let ψ be such a function. Replacing ψ by $\min(\psi, c)$, we can assume that $\psi \leqq c$.

Let $\Delta_0 = \Delta(\zeta_0; \delta)$ be an open disc such that $\bar{\Delta}_0 \subset K^0$, and such that

$$\varphi(\zeta) < a\,, \qquad \zeta \in \bar{\Delta}_0\,.$$

The set $\{\zeta \in \Delta_0: \psi(\zeta) > a\}$ is open and includes ζ_0. Let W be the component of this set which includes ζ_0. Then $\psi = a$ on $\Delta_0 \cap \partial W$, while $\psi \geqq a$ on $\partial \Delta_0 \cap \partial W$.

We claim that every component of $\bar{\Delta}_0 \backslash W$ meets $\partial \Delta_0$. Indeed, sup-

pose not. Then there is a connected open subset U of Δ_0 such that $\partial U \subset W$, and U includes a portion of ∂W. Now $U \cup W$ is connected, and $\partial(U \cup W) \subset \partial W$, so that $\psi \geqq a$ on $\partial(U \cup W)$. Since ψ attains its minimum value a over $U \cup W$ on the portion of ∂W inside $U \cup W$, ψ is identically equal to a on $U \cup W$. This contradicts the definition of W, and the claim is established.

Let μ_ζ be the harmonic measure on ∂W for $\zeta \in W$. Define

$$\psi_1(\zeta) = \begin{cases} \psi(\zeta) , & \zeta \notin W . \\ \int_{\partial W} \psi d\mu_\zeta , \end{cases}$$

On account of the preceding paragraph, every point of ∂W is a regular point for the Dirichlet problem, so that ψ_1 is continuous in a neighborhood of K. Evidently ψ_1 is superharmonic. Since $\psi_1 \geqq a$ on ∂W, also $\psi_1 \geqq a$ on W. Consequently $\psi_1 \geqq \varphi$, and $\psi_1 \in \mathscr{C}$.

Now

$$\Psi(\zeta_0) - a \leqq \psi_1(\zeta_0) - a = \int_{\partial W} [\psi(\zeta) - a] d\mu_{\zeta_0}(\zeta) .$$

Since $\psi = a$ on $\Delta_0 \cap \partial W$, the integral is bounded above by $(c - a)\mu_{\zeta_0}(\partial\Delta_0 \cap \partial W)$. The estimate of Lemma 2.2 then yields

$$\Psi(\zeta_0) - a \leqq (c - a)\chi(d/\delta) ,$$

when d is the distance from ζ_0 to ∂W. Since $c > a$ and $\Psi(\zeta_0) > a$, the quantity $\chi(d/\delta)$ is bounded away from zero, independent of ψ. Consequently the quantity d is bounded away from zero, independent of ψ. Let $\delta_0 > 0$ be a lower bound for such d. From the definition of d, we obtain $\Delta(\zeta_0; \delta_0) \subset W$, hence $\psi > a$ on $\Delta(\zeta_0; \delta_0)$, this for all ψ, so that $\Psi \geqq a$ on $\Delta(\zeta_0; \delta_0)$. We conclude that $\Delta(\zeta_0; \delta_0) \subset \Omega$, and Ω is open.

Now the restrictions of the functions in $\mathscr{C}$ to Ω form a Perron family on Ω. The classical proof then shows that their lower envelope Ψ is either harmonic on Ω or is equal to $-\infty$ on Ω. Since $\Psi > \varphi$ on Ω, the latter alternative is excluded, and the theorem is proved.

3. Analytic discs. The maximum modulus principle allows us to conclude that if V is an analytic variety in $\boldsymbol{C}^n$ whose boundary lies in X, then $V \subset \hat{X}$. We will need a slightly more general assertion to the same effect.

A subset S of $\boldsymbol{C}^n$ is an *analytic disc* if there is a continuous one-to-one map φ of an open disc Δ_0 in the complex plane onto S, such that $f \circ \varphi$ is analytic on Δ_0 for every analytic polynomial f on $\boldsymbol{C}^n$.

LEMMA 3.1. *Let Y be a compact subset of $\boldsymbol{C}^n$. Let E be a bounded subset of $\boldsymbol{C}^n$ such that $\bar{E}\backslash E \subset Y$, and such that E is a union of analytic discs. Then $E \subset \hat{Y}$.*

Proof. Let $y \in Y \cup E$ be a peak point for $P(Y \cup E)$. Then y can lie on no analytic disc in $Y \cup E$; otherwise any function peaking at y would be constant on that disc, an absurdity. Consequently the peak points for $P(Y \cup E)$ all lie in Y. Since these are dense in the Shilov boundary, the Shilov boundary of $P(Y \cup E)$ is contained in Y. It follows that $E \subset \hat{Y}$.

4. **A lemma of B. Cole.** The next lemma is due to by B. Cole, who uses it in his work extending the classical inequalities of A. N. Kolmogoroff, M. Riesz and A. Zygmund to a function-algebra setting[1].

LEMMA 4.1. *Let J be a compact subset of $\boldsymbol{C}$. Let $\zeta_0 \in \hat{J}$, and let η be a measure on J which is a Jensen measure for ζ_0 with respect to the algebra $P(J)$. Then*

$$\int \psi d\eta \leqq \psi(\zeta_0)$$

for any real-valued function ψ which is superharmonic in a neighborhood of $\hat{J}$.

Proof. Recall that a Jensen measure for ζ_0 is a probability measure η on J which satisfies

$$\log |f(\zeta_0)| \leqq \int \log |f| \, d\eta$$

for all $f \in P(J)$. In particular,

$$\log |\zeta_0 - \xi| \leqq \int \log |\zeta - \xi| \, d\eta(\zeta) , \qquad \xi \in \boldsymbol{C} . \tag{4.1}$$

By a theorem of F. Riesz, ψ is the logarithmic potential of some positive measure λ defined in a neighborhood of $\hat{J}$:

$$\psi(\zeta) = - \int \log |\zeta - \xi| \, d\lambda(\xi) , \qquad \zeta \in \hat{J} .$$

Integrating with respect to η, interchanging the orders of integration, and using (4.1), we obtain

$$\int \psi(\zeta) \, d\eta(\zeta) \leqq - \int \log |\zeta_0 - \xi| \, d\lambda(\xi) = \psi(\zeta_0) ,$$

[1] Communicated to a Colloquium audience at Tulane University, December, 1970.

as required.

5. Formulation of the theorem.

In §§5 through 8 we assume that X is a compact subset of $\boldsymbol{C}^2$ that is invariant under the transformations

$$T_\theta\colon (z, w) \longrightarrow (e^{i\theta}z, e^{-i\theta}w)\,, \qquad 0 \leqq \theta \leqq 2\pi\,. \tag{5.1}$$

Define $F\colon \hat{X} \to \boldsymbol{C}$ by

$$F(z, w) = zw\,, \qquad (z, w) \in \hat{X}\,. \tag{5.2}$$

The fiber M_ζ of $\hat{X}$ over ζ is defined by

$$M_\zeta = F^{-1}(\zeta)\,, \qquad \zeta \in F(\hat{X})\,.$$

The various M_ζ partition $\hat{X}$ into disjoint closed subsets. Evidently $\hat{X}$ is invariant under the T_θ, and one checks easily that each M_ζ is invariant under the T_θ. *For* $\zeta \in F(\hat{X})$, define

$$Q(\zeta) = \sup\{|z|\colon (z, w) \in M_\zeta\}\,, \tag{5.3}$$

$$R(\zeta) = \sup\{|w|\colon (z, w) \in M_\zeta\}\,. \tag{5.4}$$

Since $zw = \zeta$ on M_ζ,

$$\inf\{|z|\colon (z, w) \in M_\zeta\} = |\zeta|/R(\zeta)\,, \tag{5.5}$$

for $\zeta \in F(\hat{X})\backslash\{0\}$. In particular; the infimum is strictly positive, for $\zeta \neq 0$.

LEMMA 5.1. *If* $\zeta \in F(\hat{X})$, $\zeta \neq 0$, *then* M_ζ *coincides with the circle or closed annulus described by*

$$\left\{(z, \zeta/z)\colon \frac{|\zeta|}{R(\zeta)} \leqq |z| \leqq Q(\zeta)\right\}\,. \tag{5.6}$$

Furthermore, if $0 \in F(\hat{X})$, *then* M_0 *coincides with the set*

$$\{(z, 0)\colon |z| \leqq Q(0)\} \cup \{(0, w)\colon |w| \leqq R(0)\}\,. \tag{5.7}$$

Proof. Suppose $\zeta \neq 0$. On account of (5.5) and the definitions of Q and R, M_ζ is included in the annulus (5.6). Furthermore, on account of the invariance of M_ζ under the T_θ, M_ζ includes the boundary circles of (5.6). Since the annulus described by (5.6) is "analytic", M_ζ includes the entire annulus. That proves the first statement of the lemma, and the proof of the second statement is similar.

To describe $\hat{X}$, it now suffices to determine $F(\hat{X})$, and to determine

the upper semi-continuous functions $Q(\zeta)$ and $R(\zeta)$. Since the problem at hand is invariant under an interchange of coordinates, any expression for $Q(\zeta)$ will lead to an expression for $R(\zeta)$. Consequently it suffices to describe $F(\hat{X})$ and Q.

Let q be the upper semi-continuous function on $F(X)$ defined by

$$\begin{aligned} q(\zeta) &= \sup\{|z|: (z, w) \in X \cap M_\zeta\} \\ &= \sup\{|z|: (z, \zeta/z) \in X\} . \end{aligned} \tag{5.8}$$

Let K be the polynomial hull of $F(X)$:

$$K = \widehat{F(X)} . \tag{5.9}$$

Let $\mathscr{F}$ be the family of real-valued functions ψ such that ψ is superharmonic in a neighborhood of K and $\psi \geqq \log q$ on $F(X)$. Define

$$\Psi(\zeta) = \inf\{\psi(\zeta): \psi \in \mathscr{F}\} , \qquad \zeta \in K . \tag{5.10}$$

The following theorem yields the description of $\hat{X}$ that we are aiming at.

THEOREM 5.2. *Let X be a compact subset of C^2 that is invariant under the T_θ given by* (5.1). *Let F, Q and ψ be as defined above. Then $F(\hat{X})$ coincides with the polynomial hull K of $F(X)$. Furthermore,*

$$Q(\zeta) = \exp(\Psi(\zeta)) , \qquad \zeta \in K . \tag{5.11}$$

The proof will be postponed to §8. Here we make two preliminary observations.

LEMMA 5.3. *$F(\hat{X})$ is included in $K = \widehat{F(X)}$.*

Proof. This follows from an elementary principle in Banach-algebra theory, since $F \in P(X)$.

LEMMA 5.4. *If $\zeta \in F(\hat{X})$, then*

$$Q(\zeta) \leqq \exp(\Psi(\zeta)) . \tag{5.12}$$

Proof. Let $\zeta_0 \in F(X)$, and let $(z_0, w_0) \in M_{\zeta_0}$. Let ν be a Jensen measure on X for (z_0, w_0) with respect to the algebra $P(X)$. Then the projected measure $F^*(\nu)$ is a Jensen measure on $F(X)$ for ζ_0 with respect to the algebra $P(K)$. For any $\psi \in \mathscr{F}$, $\log|z_0| \leqq \int \log|z| d\nu(z, w) \leqq \int \log q(\zeta) d(F^*\nu)(\zeta) \leqq \int \psi(\zeta) d(F^*\nu)(\zeta)$. By Cole's lemma, the latter integral is bounded by $\psi(\zeta_0)$. Taking the infimum over ψ and the supremum over $(z_0, w_0) \in M_{\zeta_0}$, we obtain (5.12).

6. **Proof of Theorem 5.2: A special case.** In this section, we assume that

$$F(X) = \partial K ,$$

that is, $F(X)$ forms the boundary of its polynomial hull. In this case, the lower envelope Ψ of $\mathscr{P}$ is given by

$$\Psi(\zeta) = \begin{cases} \log q(\zeta) , & \zeta \in \partial K , \\ \int \log q(\xi) d\mu_\zeta(\xi) , & \zeta \in K^0 , \end{cases}$$

where μ_ζ is the harmonic measure on ∂K for $\zeta \in K^0$. We will show shortly that the integral converges.

Define a lower semi-continuous function p on ∂K by

$$p(\zeta) = \inf \{|z| : (z, w) \in X \cap M_\zeta\} , \qquad \zeta \in \partial K .$$

Then $-\infty \leqq p \leqq q$. Let $c > 0$ satisfy

$$|z|, |w| \leqq c , \qquad (z, w) \in X .$$

The relation $zw = \zeta$ then yields the estimate

$$\log |z| \geqq \log |\zeta| - \log c , \qquad (z, w) \in X \cap M_\zeta .$$

Taking an infimum, we obtain

$$\log p(\zeta) \geqq \log |\zeta| - \log c , \qquad \zeta \in \partial K .$$

A simple application of Fatou's Lemma reveals that the function $\log |\zeta|$ is integrable with respect to harmonic measure on ∂K [1, p. 170]. Consequently $\log p$ and $\log q$ are both integrable with respect to μ_ζ. In particular, Ψ is finite.

Define

$$\Phi(\zeta) = \begin{cases} \log p(\zeta) , & \zeta \in \partial K , \\ \int \log p(\xi) d\mu_\zeta(\xi) , & \zeta \in K^0 . \end{cases}$$

Then $\Phi(\zeta) > -\infty$ on K^0, and

$$\Phi(\zeta) \leqq \Psi(\zeta) , \qquad \zeta \in K .$$

Since each component of K^0 is simply connected, every point of ∂K^0 is a regular point for the Dirichlet problem, and the mass of μ_ζ accumulates towards ξ as $\zeta \in K^0$ tends to $\xi \in \partial K$. On account of the semi-continuity of $\log p$ and $\log q$, we obtain

$$\log p(\xi) \leqq \liminf_{K^0 \ni \zeta \to \xi} \Phi(\zeta) , \qquad \xi \in \partial K , \tag{6.1}$$

$$\log q(\xi) \geqq \limsup_{K^0 \ni \zeta \to \xi} \Psi(\zeta) , \qquad \xi \in \partial K . \tag{6.2}$$

Let $^*\Psi$ be a conjugate harmonic function on K^0 for Ψ, and set

$$g(\zeta) = \exp [\Psi(\zeta) + i^*\Psi(\zeta)] , \qquad \zeta \in K^0 .$$

Then g is analytic and bounded on K^0, and $|g| = \exp(\Psi)$. From (6.1) and (6.2) we obtain

$$p(\xi) \leqq \liminf |g(\zeta)| \leqq \limsup |g(\zeta)| \leqq q(\xi) , \tag{6.3}$$

where $\zeta \in K^0$ tends to $\xi \in \partial K$.

Define

$$V = \{(g(\zeta), \zeta/g(\zeta)) : \zeta \in K^0\} .$$

Then V is a union of analytic discs in C^2. The relation (6.3) shows that if $\zeta_n \in K^0$ tends to $\xi \in \partial K$, then any cluster point (z_0, w_0) of the $(g(\zeta_n), \zeta_n/g(\zeta_n))$ satisfies $z_0 w_0 = \xi$ and $p(\xi) \leqq |z_0| \leqq q(\xi)$. In particular, any such cluster point (z_0, w_0) belongs to M_ζ. It follows that $\bar{V} \setminus V \subset \hat{X}$. By Lemma 3.1, $V \subset \hat{X}$.

For each $\zeta \in K^0$, $(g(\zeta), \zeta/g(\zeta))$ belongs to M_ζ, so that $F(\hat{X})$ includes K^0. From Lemma 5.3 and the hypothesis $F(X) = \partial K$, we conclude that $F(\hat{X}) = K$. Furthermore, if $\zeta \in K^0$, then $Q(\zeta) \geqq |g(\zeta)| = \exp(\Psi(\zeta))$. From Lemma 5.4 it follows that $Q = \exp(\Psi)$ on K^0, hence on K. This proves Theorem 5.2, in the case at hand. In this case, though, we can give a more explicit description of $\hat{X}$.

THEOREM 6.1. *Let X be a compact subset of C^2 that is invariant under the transformations* (5.1). *Define F, Φ, ψ and M_ζ as in Sections* 5 *and* 6. *Suppose that $F(X)$ forms the boundary ∂K of a polynomially convex subset K of C. Let Ω be a component of K^0, and let $\zeta \in \Omega$. If $0 \notin \Omega$, then*

$$M_\zeta = \{(z, \zeta/z) : \Phi(\zeta) \leqq \log |z| \leqq \Psi(\zeta)\} .$$

If $0 \in \Omega$, then

$$M_\zeta = \{(z, \zeta/z) : \Phi(\zeta) + G(\zeta, 0) \leqq \log |z| \leqq \Psi(\zeta)\} ,$$

where $G(\zeta, 0)$ is the Green's function for Ω with singularity at 0.

Proof. In view of Lemma 5.1 and the special case of Theorem 5.2 proved above, we need show only that

$$\inf \{\log |z| : (z, w) \in M_\zeta\} = \begin{cases} \Phi(\zeta) , & \zeta \in \Omega, 0 \notin \Omega , \\ \Phi(\zeta) + G(\zeta, 0) , & \zeta \in \Omega, 0 \in \Omega . \end{cases} \tag{6.4}$$

The definition of p and the relation $zw = \zeta$ show that

$$\sup\{\log|w|: (z, w) \in X \cap M_\zeta\} = \log|\zeta| - \log p(\zeta),\ \zeta \in \partial K,\ \zeta \neq 0 .$$

Applying the version of Theorem 5.2 already proved, with the variables interchanged, we obtain

$$\begin{aligned}\sup\{\log|w|: (z, w) \in M_\zeta\} &= \int [\log|\xi| - \log p(\xi)] d\mu_\zeta(\xi) \\ &= \int \log|\xi|\, d\mu_\zeta(\xi) - \Phi(\zeta),\ \zeta \in \Omega .\end{aligned}$$

The relation $zw = \zeta$ then yields

$$\inf\{\log|z|: (z, w) \in M_\zeta\} = \Phi(\zeta) + \log|\zeta| - \int \log|\xi|\, d\mu_\zeta(\xi) ,$$

for $\zeta \in \Omega$. Now $\int \log|\xi|\, d\mu_\zeta(\xi)$ is equal to either $\log|\zeta|$ or $\log|\zeta| - G(\zeta, 0)$, depending on whether $0 \notin \Omega$ or $0 \in \Omega$. That establishes (6.4), and the proof of Theorem 6.1 is complete.

7. **An example.** As an example, we consider a class of tori which are close to those treated by Wermer [10].

Let Γ be any simple closed Jordan curve in C which does not pass through the origin, let q be a positive continuous function on Γ, and let

$$X = \{(z, \zeta/z): \zeta \in \Gamma,\ |z| = q(\zeta)\} .$$

Then X is a torus that is invariant under the transformations (5.1). In this case, $F(X) = \Gamma$, while K is the union of Γ and the bounded component Ω of $C\backslash\Gamma$. Furthermore, $p = q = Q$ on Γ. There are two cases that occur.

Suppose $0 \notin \Omega$. Let Ψ be the harmonic extension of $\log q$ to Ω, let $^*\Psi$ be a conjugate harmonic function for Ψ on Ω, and set $g(\zeta) = \exp(\Psi(\zeta) + i^*\Psi(\zeta))$, $\zeta \in \Omega$, as before. Define

$$V_\alpha = \{(e^{i\alpha} g(\zeta), e^{-i\alpha}\zeta/g(\zeta)): \zeta \in \Omega\},\ 0 \leqq \alpha < 2\pi .$$

Then the V_α are disjoint analytic discs whose boundaries lie on X, and $\hat{X}$ is the union of X and the V_α's.

On the other hand, suppose $0 \in \Omega$. Define g and the V_α's as above. Also, define an analytic function h on Ω so that

$$\log|h(\zeta)| = \log|\zeta| - G(\zeta, 0) - \Psi(\zeta),\ \zeta \in \Omega ,$$

and set

$$W_\alpha = \{(e^{-i\alpha}\zeta/h(\zeta), e^{i\alpha} h(\zeta)): \zeta \in \Omega\},\ 0 \leqq \alpha < 2\pi .$$

The W_α are disjoint analytic discs whose boundaries are included in

X. The topological boundary of $\hat{X}$ is the disjoint union of X, the V_α's, and the W_α's. In some sense, $\hat{X}$ is a deformed bidisc.

If the curve Γ passes through the origin, one must assume that $q > 0$ on $\Gamma\backslash\{0\}$, while

$$\limsup_{\Gamma \ni \zeta \to 0} \frac{|q(\zeta)|}{|\zeta|} < \infty \,. \tag{7.1}$$

In this case, define X to be the closure of the set of pairs $(z, \zeta/z)$, such that $\zeta \in \Gamma$, $\zeta \neq 0$, and $|z| = q(\zeta)$. The condition (7.1) guarantees that X is bounded. The polynomial hull $\hat{X}$ of X is the union of X, and the analytic discs V_α, $0 \leqq \alpha < 2\pi$, together with an analytic disc or two in the fiber M_0.

8. **Proof of Theorem 5.2: The general case.** We return to an arbitrary compact subset X of C^2 that is invariant under the transformations (5.1). Recall that K is the polynomial hull of $F(X)$. The compact set $Y = F^{-1}(\partial K)$ is also invariant, and $F(Y) = \partial K$. Applying the results of §6, we find that $F(\hat{Y}) = K$. Consequently $F(\hat{X}) = K$. To complete the proof of Theorem 5.2, it suffices to establish the identity (5.11).

Set $\log q(\zeta) = -\infty$ on $K\backslash F(X)$, and let $\Omega = \{\zeta \in K: \log q(\zeta) < \Psi(\zeta)\}$. By Theorem 2.1, Ω is an open subset of K^0 on which Ψ is harmonic. Let E be the set of $(z, w) \in C^2$ such that $\zeta = zw$ belongs to Ω and $\log|z| = \Psi(\zeta)$. To complete the proof, it will suffice to show that $E \subset \hat{X}$.

Let $(z_0, w_0) \in E$, and let ${}^*\Psi$ be the harmonic conjugate function for Ψ defined near $\zeta_0 = z_0 w_0$ and satisfying ${}^*\Psi(\zeta_0) = \arg z_0$. Define $g(\zeta) = \exp(\Psi(\zeta) + i{}^*\Psi(\zeta))$ as before. Then as ζ varies near ζ_0, the points $(g(\zeta), \zeta/g(\zeta))$ describe an analytic disc in E which passes through (z_0, w_0). Consequently E is a union of analytic discs.

In view of Lemma 3.2, it suffices now to prove that $\bar{E}\backslash E \subset \hat{X}$.

Suppose $(z_0, w_0) \in \bar{E}\backslash E$. Choose $(z_n, w_n) \in E$ such that (z_n, w_n) converges to (z_0, w_0). Then $\zeta_n = z_n w_n \in \Omega$ tends to $\zeta_0 = z_0 w_0$, and evidently $\zeta_0 \in \partial\Omega$. Since Ψ is upper semi-continuous, $\log|z_0| = \lim \log|z_n| = \lim \Psi(\zeta_n) \leqq \Psi(\zeta_0) = \log Q(\zeta_0)$. On the other hand, let u be the harmonic extension of $\log q|_{\partial K}$ to K. Then $u \leqq \Psi$. From (5.5) and the results in the special case already treated,

$$\log|\zeta| - \log R(\zeta) \leqq u(\zeta), \quad \zeta \in K^0 \,.$$

Using the upper semi-continuity of R, we obtain $\log|z_0| = \lim \log|z_n| = \lim \Psi(\zeta_n) \geqq \lim u(\zeta_n) \geqq \log|\zeta_0| - \lim \log R(\zeta_n) \geqq \log|\zeta_0| - \log R(\zeta_0)$. Consequently

$$|\zeta_0|/R(\zeta_0) \leqq |z_0| \leqq Q(\zeta_0) \,.$$

From (5.5) and the description of M_{ζ_0} given in §5, we conclude that $(z_0, w_0) \in M_{\zeta_0}$. Hence $\bar{E}\backslash E \subset \hat{X}$. The proof is complete.

9. The case in which $\alpha = -m/n$. Let m and n be positive integers that are relatively prime. In this Section, we suppose that X is invariant under the transformations $(z, w) \to (e^{i\theta}z, e^{-i(m/n)\theta}w)$. Equivalently, X is invariant under the transformations

$$(z, w) \longrightarrow (e^{in\theta}z, e^{-im\theta}w) , \qquad 0 \leqq \theta \leqq 2\pi . \tag{9.1}$$

The polynomial hull $\hat{X}$ is also invariant under the transformations (9.1).

The description of $\hat{X}$ can be obtained by modifying slightly the discussion of §§5 through 8. In the case at hand, the polynomial map F from $\hat{X}$ to $\boldsymbol{C}$ and the fibers M_ζ are defined by

$$F(z, w) = z^m w^n , \qquad (z, w) \in \hat{X} , \tag{9.2}$$

$$M_\zeta = F^{-1}(\{\zeta\}) , \qquad \zeta \in \boldsymbol{C} . \tag{9.3}$$

The fibers M_ζ are invariant under the transformations (9.1). Again define $Q(\zeta)$ by (5.3) and $R(\zeta)$ by (5.4). If $\zeta \neq 0$, the fiber M_ζ is the annulus or circle given by

$$M_\zeta = \{(\lambda^n, \zeta^{1/n}/\lambda^m) : |\zeta|^{1/(mn)} R(\zeta)^{-1/m} \leqq |\lambda| \leqq Q(\zeta)^{1/n}\} .$$

Note that the various choices of the nth root of ζ lead to the same set above. The description of M_0 is idential to (5.7). To describe $\hat{X}$, it again suffices to describe $F(\hat{X})$, and to describe the function Q. The final result is almost identical to Theorem 5.2, and so is the proof. We state the result formally, but omit the proof.

THEOREM 9.1. *Let X be a compact subset of $\boldsymbol{C}^2$ that is invariant under the transformations* (9.1). *Define F as in* (9.2). *Then $F(\hat{X})$ coincides with the polynomial hull of $F(X)$. Furthermore, if Q, q and Ψ are as defined in* (5.3), (5.8) *and* (5.10) *respectively, then*

$$Q(\zeta) = \exp(\Psi(\zeta)) , \qquad \zeta \in \widehat{F(X)} .$$

10. Analytic structure in $\hat{X}$. As a consequence of the description of $\hat{X}$ given above, one can extend Wermer's results on analytic structure in $\hat{X}$ to the compacts sets X treated here.

THEOREM 10.1. *Fix $\alpha \leqq 0$, let X be a compact subset of $\boldsymbol{C}^2$ that is invariant under the T_θ, defined in* (0.1). *Then every point of $\hat{X}\backslash X$ lies on an analytic disc in $\hat{X}$. Furthermore, $P(X) = C(X)$ if and only if there are no analytic discs in $\hat{X}$.*

Proof. We consider only the case $\alpha = -1$. The other cases are similar.

Let X be a compact subset of C^2 that is invariant under the transformations (5.1). We will employ the notation of §5.

Suppose $(z, w) \in \hat{X}$ does not lie on an analytic disc in $\hat{X}$. Let $\zeta = zw$, so that $(z, w) \in M_\zeta$. Since (z, w) does not lie in an analytic annulus in M_ζ, either $|z| = Q(\zeta)$ or $|w| = R(\zeta)$. Suppose for the sake of definiteness, that $|z| = Q(\zeta)$. If $\zeta \in \Omega$, then (z, w) lies on one of the analytic discs in $\hat{X}$ constructed in the proof of Theorem 5.2 (cf. the third paragraph of §8). We conclude that $\zeta \notin \Omega$. Hence $q(\zeta) = Q(\zeta)$. From the definition of q and the invariance of X, we conclude that $(z, w) \in X$. That proves the first assertion of the theorem.

If $P(X) = C(X)$, then $\hat{X} = X$, and evidently there are no analytic discs in X.

Conversely, suppose that there are no analytic discs in $\hat{X}$. Let $Y = F^{-1}(\partial K)$, as in §8. Then Y is invariant, $F(Y) = \partial K$, and $F(\hat{Y}) = K$. Furthermore, according to the description given in §6, $\hat{Y} \cap F^{-1}(K^0)$ is a union of analytic discs. We conclude that K^0 is empty. Hence $K = F(X)$. By Lavrentieff's Theorem, $P(K) = C(K)$. By the theory of sets of antisymmetry [4, Theorem I.13.1], each restriction algebra $P(\hat{X})|_{M_\zeta}$ is a closed subalgebra of $C(M_\zeta)$, and $P(\hat{X})$ consists of all functions in $C(\hat{X})$ which belong to $P(\hat{X})|_{M_\zeta}$ for all $\zeta \in K$. Now our hypothesis on $\hat{X}$ shows that each M_ζ is a circle (or a point, if $\zeta = 0$). Furthermore, polynomials in z^2w and zw^2 are dense in $C(M_\zeta)$. Consequently $P(\hat{X})|_{M_\zeta} = C(M_\zeta)$, and $P(\hat{X}) = C(\hat{X})$. In particular, $\hat{X} = X$. The proof is complete.

The following question is open (cf. [5]): Does $P(X)$ consist of precisely those functions $F \in C(\hat{X})$ such that F is analytic on each analytic disc in $\hat{X}$?

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Received May 13, 1975. This work was supported in part under NSF grant No. GP-42009.

UNIVERSITY OF CALIFORNIA, LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 61, No. 1 November, 1975