A NOTE ON STARSHAPED SETS

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If $S$ is a compact subset of $\mathbb{R}^d$, it is shown that $S$ is starshaped if and only if $S$ is nonseparating and the intersection of the stars of the $(d-2)$-extreme points of $S$ is non-empty.

Let $S \subset \mathbb{R}^d$. The $(d-2)$-extreme points of $S$ are by definition those points of $S$ such that if $D \subset S$ is a $(d-1)$-dimensional simplex then $x \in \text{relint } D$ (the relative interior of $D$). The totality of $(d-2)$-extreme points of $S$ is denoted by $E(S)$. For each $y \in S$ we define $S(y)$, the star of $y$ by $S(y) = \{z: [y, z] \subset S\}$, where $[y, z]$ denotes the closed line segment from $y$ to $z$. $S$ is said to be starshaped if $\ker S \neq \emptyset$ where $\ker S = \{S(y): y \in S\}$. In [2] it is shown that if $S$ is a compact starshaped set in $\mathbb{R}^d$ then $\ker S = \bigcap \{S(y): y \in E(S)\}$. Thus the following question arises: if $S$ is such that $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, under what hypothesis is $S$ starshaped? It is clearly desirable that the hypothesis should be as weak as possible in order to indicate to what extent $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$ implies that $S$ is starshaped. In [3] it is shown that one suitable hypothesis is that $S$ should have the half-ray property, that is, for any point $x$ in $\mathbb{R}^d \setminus S$ there is a half-line $l$ with vertex $x$ such that $l \cap S = \emptyset$. Now we note that this hypothesis is a rather strong one especially as it is being used to deduce the fact that a certain set is starshaped. Thus one suspects that a much weaker hypothesis might suffice. This suspicion is further strengthened by the example given in [3] to show that, in fact, some hypothesis is necessary. More precisely, the example given is a separating set that is, its complement is not connected. The purpose of this note is to prove the following

**Theorem.** If $S \subset \mathbb{R}^d$ is a nonseparating compact set and $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, then $S$ is starshaped.

**Proof.** Let $z \in \bigcap \{S(y): y \in E(S)\}$. We shall show that for any $x$ in $\mathbb{R}^d \setminus S$, $l(x, z) \cap S = \emptyset$ where $l(x, z)$ is the half-line with vertex $x$ which does not contain $z$ but is such that the line containing $l(x, z)$ does contain $z$. Clearly this suffices to show that $S$ is starshaped.

Choose $x_0$ in the complement of the convex hull of $S$, then $l(x_0, z) \cap S = \emptyset$. Now since $S$ is a nonseparating compact set, its complement is a path-connected unbounded open set (see [1, p. 356]). Thus any point in $\mathbb{R}^d \setminus S$ can be “joined” to $x_0$ by a finite polygonal path in $\mathbb{R}^d \setminus S$ such that if $t$ is any segment of the path then the line
containing \( t \) does not contain \( z \).

Now we assume \( l(x, z) \cap S \neq \emptyset \) for some point \( x \) in \( \mathbb{R}^d \setminus S \) and seek a contradiction. Let \( P \) be a polygonal path as described above with consecutive vertices \( v_1 = x, v_2, v_3, \ldots, v_n = x_0 \). Put \( i = \max \{ j : l(v_j, z) \cap S \neq \emptyset \} \) then \( 1 \leq i < n \). Let the closed segment \([v_i, v_{i+1}]\) be the image under the continuous function \( f \) of the unit interval, with \( f(0) = v_i \) and \( f(1) = v_{i+1} \). Note that if \( p \neq q \) then \( l(f(p), z) \cap l(f(q), z) = \emptyset \). Now \( l(f(1), z) \cap S = \emptyset \) and so, since \( S \) is compact we can put \( p = \max \{ q : l(f(q), z) \cap S \neq \emptyset \} \) and then \( 0 \leq p < 1 \). Let \( y \) be the point of \( S \) on \( l(f(p), z) \) which is furthest from \( z \). Now suppose \( D \) is a \((d-1)\)-simplex with \( D \subset S \) and \( y \in \text{relint } D \).

Then \( y \) must be the mid-point of a segment which is contained in \( S \cap Q \) where \( Q \) is the plane through \( z, v_i, v_{i+1} \). But this is impossible because of the definition of \( y \) and the fact that \( l(f(q), z) \cap S = \emptyset \) for \( p < q \leq 1 \). Hence \( y \in E(S) \) and so \( f(p) \in S \). This contradiction shows that \( l(x, z) \cap S = \emptyset \) and thus completes the proof.

Finally, as a result of the above theorem and the comments made in [2] we are led to ask: if \( S \) has the half-ray property and has a point which "sees" just the extreme points of the convex hull of \( S \) and not all the \((d-2)\)-extreme points, is \( S \) necessarily starshaped? The following example shows that the answer is negative:

\[
S = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\} \setminus \{(x, y) \in \mathbb{R}^2 : |x| < \frac{1}{2}, |y| > \frac{1}{2}\}.
\]

Similarly we observe that if we rotate \( S \) about the \( y \)-axis we obtain a three dimensional set with the required properties.

**References**


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