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A NOTE ON STARSHAPED SETS

PAUL R. GOODEY

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If S is a compact subset of R^d , it is shown that S is starshaped if and only if S is nonseparating and the intersection of the stars of the $(d-2)$ -extreme points of S is nonempty.

Let $S \subset R^d$. The $(d-2)$ -extreme points of S are by definition those points of S such that if $D \subset S$ is a $(d-1)$ -dimensional simplex then $x \notin \text{relint } D$ (the relative interior of D). The totality of $(d-2)$ -extreme points of S is denoted by $E(S)$. For each $y \in S$ we define $S(y)$, the star of y by $S(y) = \{z: [y, z] \subset S\}$, where $[y, z]$ denotes the closed line segment from y to z . S is said to be starshaped if $\text{Ker } S \neq \emptyset$ where $\text{Ker } S = \{S(y): y \in S\}$. In [2] it is shown that if S is a compact starshaped set in R^d then $\text{Ker } S = \bigcap \{S(y): y \in E(S)\}$. Thus the following question arises: if S is such that $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, under what hypothesis is S starshaped? It is clearly desirable that the hypothesis should be as weak as possible in order to indicate to what extent $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$ implies that S is starshaped. In [3] it is shown that one suitable hypothesis is that S should have the *half-ray property*, that is, for any point x in $R^d \setminus S$ there is a half-line l with vertex x such that $l \cap S = \emptyset$. Now we note that this hypothesis is a rather strong one especially as it is being used to deduce the fact that a certain set is starshaped. Thus one suspects that a much weaker hypothesis might suffice. This suspicion is further strengthened by the example given in [3] to show that, in fact, some hypothesis is necessary. More precisely, the example given is a *separating set* that is, its complement is not connected. The purpose of this note is to prove the following

THEOREM. *If $S \subset R^d$ is a nonseparating compact set and $\bigcap \{S(y): y \in E(S)\} \neq \emptyset$, then S is starshaped.*

Proof. Let $z \in \bigcap \{S(y): y \in E(S)\}$. We shall show that for any x in $R^d \setminus S$, $l(x, z) \cap S = \emptyset$ where $l(x, z)$ is the half-line with vertex x which does not contain z but is such that the line containing $l(x, z)$ does contain z . Clearly this suffices to show that S is starshaped.

Choose x_0 in the complement of the convex hull of S , then $l(x_0, z) \cap S = \emptyset$. Now since S is a nonseparating compact set, its complement is a path-connected unbounded open set (see [1, p. 356]). Thus any point in $R^d \setminus S$ can be "joined" to x_0 by a finite polygonal path in $R^d \setminus S$ such that if t is any segment of the path then the line

containing t does not contain z .

Now we assume $l(x, z) \cap S \neq \emptyset$ for some point x in $R^d \setminus S$ and seek a contradiction. Let P be a polygonal path as described above with consecutive vertices $v_1 = x, v_2, v_3, \dots, v_n = x_0$. Put $i = \max \{j: l(v_j, z) \cap S \neq \emptyset\}$ then $1 \leq i < n$. Let the closed segment $[v_i, v_{i+1}]$ be the image under the continuous function f of the unit interval, with $f(0) = v_i$ and $f(1) = v_{i+1}$. Note that if $p \neq q$ then $l(f(p), z) \cap l(f(q), z) = \emptyset$. Now $l(f(1), z) \cap S = \emptyset$ and so, since S is compact we can put $p = \max \{q: l(f(q), z) \cap S \neq \emptyset\}$ and then $0 \leq p < 1$. Let y be the point of S on $l(f(p), z)$ which is furthest from z . Now suppose D is a $(d-1)$ -simplex with $D \subset S$ and $y \in \text{relint } D$.

Then y must be the mid-point of a segment which is contained in $S \cap Q$ where Q is the plane through z, v_i, v_{i+1} . But this is impossible because of the definition of y and the fact that $l(f(q), z) \cap S = \emptyset$ for $p < q \leq 1$. Hence $y \in E(S)$ and so $f(p) \in S$. This contradiction shows that $l(x, z) \cap S = \emptyset$ and thus completes the proof.

Finally, as a result of the above theorem and the comments made in [2] we are led to ask: if S has the half-ray property and has a point which "sees" just the extreme points of the convex hull of S and not all the $(d-2)$ -extreme points, is S necessarily starshaped? The following example shows that the answer is negative:

$$S = \{(x, y) \in R^2: |x| \leq 1, |y| \leq 1\} \setminus \left\{ (x, y) \in R^2: |x| < \frac{1}{2}, |y| > \frac{1}{2} \right\}.$$

Similarly we observe that if we rotate S about the y -axis we obtain a three dimensional set with the required properties.

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