Pacific Journal of Mathematics

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Vol. 61, No. 1 November 1975

PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

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Let R be a commutative ring with identity, and consider the power series ring $R[[X]]$ in one analytic indeterminate over R. Is the coefficient ring R unique in the sence that if $R[[X]]$ is isomorphic to $S[[Y]]$ with Y an analytic indeterminate over S, need S be isomorphic to R? Whenever this is the case, R will be called power-invariant. It will be shown that if R is a quasi-local or a complete semi-local ring then R is power-invariant.

The answer to the *general* question was not known by the author until the Commutative Algebra Conference in June of 1974 at the University of Nebraska where a counterexample was produced by Andy Magid. He has graciously requested that it be reproduced in this paper. The fact that rings with nilpotent Jacobson radical are power-invariant is known [6]. The paper will also show that if $R[[X]] = S[[Y]]$, under the assumption that certain elements are not zero-divisors that there exist one-to-one maps from R into S and S into R. In particular, this is the case if R is a domain. Finally, the paper generalizes the power-invariant results to an arbitrary number of variables.

The following notational conventions will be observed and referred to throughout the paper. $J(R)$ will denote the Jacobson radical of a ring R. W, R, S, X, Y, j, u, v will be such that $W = R[[X]] = S[[Y]]$ where X and Y are analytic indeterminates over R and S respectively; $Y = j + uX$, and $X = k + vY$ where $J \in J(R)$, $u \in W$, $k \in J(S)$, and $v \in W$.

Note that since W is complete with respect to both (X) and (Y), W is also complete with respect to (X, Y) and thus also complete with respect to the ideals jW and kW contained in (X, Y). (If $\{w_n\}$ is a Cauchy sequence in the (X, Y)-adic topology, it can be written as a sum of two Cauchy sequences $\{s_n\}$ and $\{t_n\}$ which are also Cauchy with respect to the (X)-adic and (Y)-adic topologies respectively. Let s be the limit of $\{s_n\}$ in the (X)-adic topology and t the limit of $\{t_n\}$ in the (Y)-adic topology. Since s and t are then limit points of $\{s_n\}$ and $\{t_n\}$ respectively in the (X, Y)-adic topology, $s + t$ is a limit point of $\{w_n\}$. Conceivably the (X, Y)-adic topology may not be Hausdorff, so that limits aren't necessarily unique.) Further, R is certainly a closed subset of W in the jW-adic topology so R is complete with respect to (j), the ideal of R generated by j, but perhaps not Hausdorff. Similarly, S is complete but perhaps not

Hausdorff with respect to (k), the ideal of S generated by (k).

1. Andy Magid's counterexample. The example is the completion with respect to a certain ideal of Melvin Hochster's counterexample to the question whether $R[X]$ isomorphic to $S[Y]$ implies S is isomorphic to R, where X and Y are ordinary indeterminates over R and S respectively. Specifically, there exists a Noetherian ring R with zero Jacobson radical which has a finitely generated nonfree module P such that $P \oplus R \cong R^3$. Taking symmetric algebras of both sides yields $A[T]$ isomorphic to $R[X, Y, Z]$ where T, $\{X, Y, Z\}$ are indeterminates over A and R respectively and A is the symmetric algebra of P, and is not isomorphic to $R[X, Y]$. See either [3] or [4] for more details. If M is a B-module, let $\hat{S}_B(M)$ denote the complete symmetric algebra of M over B, i.e., the completion of the symmetric algebra $S_B(M)$ with respect to the ideal generated by M. To get the counterexample for the power series case, take the complete symmetric algebras of $P \oplus R$ and R^3 over R. Then $\hat{S}_R(P \oplus R) \cong \hat{S}_R(P)[[T]]$ and $\hat{S}_R(R^3) \cong R[[X, Y, Z]]$ with T an analytic indeterminate over $\hat{S}_R(P)$ and X, Y, Z independent analytic indeterminates over R. It remains to show that $\hat{S}_R(P)$ is not isomorphic to $R[[X, Y]]$. It suffices to show that $\hat{S}_R(P) \cong R[[X, Y]]$ implies $S_R(P) \cong R[X, Y]$. Let M be any finitely generated R-module. $J(\hat{S}_R(M))$ is the ideal generated by M since $M\hat{S}(M)$ is certainly contained in $J(\hat{S}_R(M))$ and $\hat{S}(M)/M\hat{S}(M) \cong R$ whose Jacobson radical is zero. Thus, the associated graded ring of $\hat{S}(M)$ with respect to $J(\hat{S}(M))$ is $\hat{S}(M)/M\hat{S}(M) \oplus M\hat{S}(M)/]M\hat{S}(M)]^2 \oplus \cdots$ which is isomorphic to $\hat{S}(M)$. Thus, $\hat{S}(M)$ determines $S(M)$ and the result follows.

2. Some power-invariant rings. The following theorem from [8] and [9] will be needed for the results of this section.

THEOREM 1. Let $B = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$, *and suppose that* ϕ *is an* R*-endomorphism of* $R[[X]]$ *such that* $\phi(X) = B$. *Then*:

(a) ϕ *is onto if and only if* a_1 *is a unit of* R.

(b) *If* ϕ *is onto, then* ϕ *is one to one.*

(c) ϕ *is an automorphism if and only if* a_1 *is a unit of* R.

THEOREM 2. *Let* $B = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$. *If* $\bigcap_{n=1}^{\infty} (b_0^n) = 0$ *in* R (or $\bigcap_{n=1}^{\infty} (B^n) = 0$ in $R[[X]]$) *and* R *is complete with respect to* (b_0) (or $R[[X]]$ is complete with respect to (B)) *then* ϕ_B *which maps* $\sum_{i=0}^{\infty} a_i X^i$ *to* $\sum_{i=0}^{\infty} a_i B^i$ *is an* R*-endomorphism.*

THEOREM 3. *Let* $B = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$. *Let* A *be an ideal of* R. *If* $b_0 A = A$, *then* $A \subseteq \bigcap_{k=1}^{\infty} (B^k)$.

Also recall that an element of $R[[X]]$ is invertible if its constant term is invertible [7].

We are ready for our first result.

THEOREM 4. *If R is quasi-local, R is power-invariant.*

Proof. Let the notation be as in the next-to-last paragraph of the introduction. If u is an invertible element of W, then Theorem 1 implies $R[[Y]] = R[[X]] = S[[Y]]$ and Y is an analytic indeterminate over R. Thus, $S \cong W/(Y) \cong R$. By symmetry we can reduce to the case where both u and v are in the maximal ideal of W. For $w \in W$, let $w_i \in S$ be such that $w = w_0 + w_1 Y + \cdots$. Taking the Y-coefficient of both sides of $Y = j + uX$ we get $1 = j_1 + u_0 X_1 + u_1 X_0 = j_1 + u_0 X_1 + u_1 k$. u_0 is in the maximal ideal since u is, so j_1 is invertible. Similarly k^1, the X-coefficient of k is invertible. Suppose for the moment that there is an R-endomorphism of $R[[X]]$ which takes X to k, and an S-endomorphism which takes Y to j. In this case Theorem 1 would imply $R[[k]] = R[[X]] = S[[Y]]$ with k analytically independent over R, and $R \cong S/(k)[[Y]]$ with Y an analytic indeterminate over $S/(k)$. Similarly, $S \cong R/(j)[[X]]$ with X analytically independent over $R/(j)$. However, $(X, j) = (X, Y) = (k, Y)$ so that $S/(k) \cong W/(X, Y) \cong R/(j)$. This yield $R \cong S$. It remains only to show the desired endomorphisms exist. W is certainly complete with respect to (j) (or (k)), so by Theorem 2 and symmetry, the result will follow from the following proposition.

PROPOSITION. *Let $R[[X]] = S[[Y]] = W$ where $Y = j + uX$ and $X = k + vY$ as above. If u is in $J(W)$, then $\bigcap_{n=1}^{\infty} (j^n) = 0$ as an ideal in R.*

Proof. Since $\bigcap_{n=1}^{\infty} (Y^n) = 0$, by Theorem 3 it suffices to show that $j[\bigcap_{n=1}^{\infty} (j^n)] = \bigcap_{n=1}^{\infty} (j^n)$. Let $A = \bigcap_{n=1}^{\infty} (j^n)$. First assume that $\mathrm{Ann}_R\, j \subseteqq A$. Let $f \in A$ so that $f = jt$, for some t_1, and $f = j^n t_n$ for some t_n given n. $j(t_1 - j^{n-1} t_n) = 0$ implies $t_1 - j^{n-1} t_n \in A$, which further implies $t_1 \in (j^{n-1})$. Since n was arbitrary $t_1 \in A$. Thus, $f \in jA$ and $jA = A$. Now let us show that $\mathrm{Ann}_R\, j \subseteqq A$. It suffices to show that $fj^i = 0$ implies $f \in (j)$. Let $fj^i = 0$, in which case $fY^i = \sum_{h=1}^{i} \binom{i}{h} f j^{i-h} u^h X^h$. If we take the Y^i coefficient of both sides, we get f_0 expressed as a sum of terms of the form $f_a j_{b_1} j_{b_{i-h}} u_{c_1} u_{c_h} X_{d_1} X_{d_h}$ where a, b_α, c_α, d_α, are all integers, and $1 \leqq h \leqq i$. (Recall w_i denotes the Y^i coefficient in S of w in $S[[Y]]$.) Further, $a + \sum_{\alpha=1}^{i-h} b_\alpha + \sum_{\alpha=1}^{h} c_\alpha + \sum_{\alpha=1}^{h} d_\alpha = i$. If such a term involves f_0, i.e., $a = 0$, then not all of the b_α, c_α, and d_α can be $\geqq 1$ since this would give $\sum_{\alpha=1}^{i-h} b_\alpha + \sum_{\alpha=1}^{h} c_\alpha +$

$\sum_{\alpha=1}^{h} d_\alpha \geqq (i-h)+h+h>i$. Thus, any term involving f_0 involves either j_0, u_0, or X_0 each of which is in $J(W)$. If a term does not involve f_0, then it must involve X_0 or j_0. (If not then $a+\sum_{\alpha=1}^{i-h} b_\alpha + \sum_{\alpha=1}^{h} c_\alpha + \sum_{\alpha=1}^{h} d_\alpha \geqq 1+(i-h)+h>i$.) Since $X_0=k$ and $j_0 \in (k)$, we have $f_0 = af_0 + bk$ where $a \in J(W)$. Since $1-a$ is invertible, $f_0 \in (k) \subseteqq (k, Y) = (j, X)$. However, $f \in R$ so $f \in (j)$ as required.

COROLLARY. *Let $R[[X]] = S[[Y]]$, $y = j+uX$, and $X = k+vY$. Let P be a maximal ideal of R, and Q the maximal ideal of S such that $(P, X) = (Q, Y)$, then $\hat{R}_P \cong \hat{S}_Q$ where the completion can be taken with respect to the maximal ideals in question or with respect to $(j)R_P$ and $(k)S_Q$.*

Proof. Let $M = (P, X) = (Q, Y)$, then W_M can be thought of as a subring of $R_P[[X]]$ or $S_Q[[Y]]$. In either case the completion of W_M with respect to M is all of $\hat{R}_P[[X]]$ or $\hat{S}_Q[[Y]]$ where $\hat{R}_P$ is completion with respect to PR_P and $\hat{S}_Q$ is completion with respect to QS_Q. Thus, $\hat{R}_P[[X]] \cong \hat{S}_Q[[Y]]$ and the result follows from Theorem 4. The proof of the other completion is similar, this time complete W_M with respect to $(X, Y) = (j, X) = (k, Y)$.

THEOREM 5. *If R is a complete semi-local ring, R is power-invariant.*

Proof. Let $R[[X]] = S[[Y]] = W$, then W and S are also complete semilocal. Since a complete semilocal ring is a direct sum of complete local rings, the result follows from the corollary.

3. Existence of one to one maps. The following lemmas are needed for our next result. With notation as in the introduction, we keep the convention that if $w \in W = R[[X]] = S[[Y]]$ that w_i is the Y^i coefficient in S of W, and we let w^i denote the X^i coefficient in R of w. Since $i=0$ and $i=1$ are the only cases of interest, there should be no confusion with exponents.

Define $\phi: R \to R$ by $\phi(r) = (r_0)^0$.

LEMMA 1. *With $\varnothing$, R, S, W, X, Y as above and $Y = j+uX$, $X = k+vY$, we have $(Y) \cap R \subseteqq \operatorname{Ker}\phi \subseteqq (j)$.* ($(j)$ denotes the ideal of R generated by j).

Proof. Since $Y_0 = 0$ the first containment is trivial. Now suppose $r \in R_0$ and $(r_0)^0 = 0$, then $r_0 = tX$ and $r = tX + fY$. Since $r = r^0$ $r = f^0 y^0 = f^0 j$.

LEMMA 2. *With everything as in Lemma* 1, *assume* $\phi(j)$ *is either zero or not a zero-divisor. Then* $\operatorname{Ker}\phi = 0$, *or* $\operatorname{Ker}\phi = (j)$.

Proof. If $\phi(j) = 0$, $\operatorname{Ker}\phi = (j)$ by Lemma 1. If $\phi(j) \neq 0$, let $f \in \operatorname{Ker}\phi$. $f = aj$ also by Lemma 1. $0 = \phi(f)\phi(a)\phi(j)$, which implies $\phi(a) = 0$. Thus, $a = bj$ and it is clear that $f \in \bigcap_{n=1}^{\infty}(j^n)$. Since $\phi(j) = (u_0)^0 v^0 j$ is not a zero-divisor, j is not a zero-divisor. Thus $j[\bigcap_{n=1}^{\infty}(j^n)] = \bigcap_{n=1}^{\infty}(j^n) = 0$ by Theorem 3. Thus, $f = 0$ and $\operatorname{Ker}\phi = 0$.

LEMMA 3. *With notation as above, j not a zero-divisor implies k is not a zero-divisor.*

Proof. j not a zero-divisor implies $\{j, X\}$ is a W-sequence. Suppose k is a zero-divisor, then some element of S kills k, say $sk = 0$. We get $sX = svY$. Now $sv \notin (x)$ since then s would be a multiple of Y. However, $sX = svj + svuX$ makes svj a multiple of X which is a contradiction. Thus, k is not a zero-divisor.

THEOREM 6. *Let* $R[[X]] = S[[Y]]$, $Y = j + uX$, $X = k + vY$. *If j and* $\phi(j) = (j_0)^0$ *are not zero-divisors* (unless 0), *then there exist* 1 *to* 1 *maps from R into S and from S into R.*

Proof. We need only consider the two cases of Lemma 2.

Case 1. $\operatorname{Ker}\phi = (j) \neq 0$.

In this case R is actually isomorphic to S. Let $A = Im\phi$. If $r \in R$, $r = r_0 + wY$ for some $w \in W$ and $r = r^0 = (r_0)^0 + w^0 Y^0 = \phi(r) + w^0 j$. Thus, $R = A + (j)$, $\phi = \phi^2$, and $A \cap (j) = 0$ all follow. $\mathrm{R} = A + (j) = A + jR$ implies $R = A + jA + \cdots + j^n A + j^{n+1}R$ for any n. R is complete with respect to (j) and also Hausdorff as in the proof of Lemma 2. Thus, $R = A[[j]]$. j not a zero-divisor and $A \cap (j) = 0$ together imply that j is an analytic indeterminate over A so $R \cong A[[X]]$. We have $W = A[[k, Y]]$. We next show that $(Y) \cap A[[k]] = 0$. Let $t \in (Y) \cap A[[k]]$. $t = ly = \sum_{i=0}^{\infty} a_i k^i$. (Here $a_i \in A$ and i is an ordinary subscript.) $a_0 \in (j) \cap A = 0$ so $ly = \sum_{i=1}^{\infty} a_i k^i$. Taking constant terms in S we get $0 = \sum_{i=1}^{\infty}(a_i)_0 k^i$. Since k is not a zero-divisor $\sum_{i=1}^{\infty}(a_i)_0 k^{i-1} = 0$ and $(a_1)_0 \in (k)$. Thus, $a_1 \in (k, Y) \cap A = (j) \cap A = 0$. By induction $a_i = 0$ for all i and $(Y) \cap A[[k]] = 0$. But then $A[[k]] \cong W/(Y) \cong S$. k not a zero-divisor and $(k) \cap A \subseteqq (j) \cap A = 0$ imply k is also an analytic indeterminate over A and $S \cong A[[X]] \cong R$.

Case 2. $\operatorname{Ker}\phi = 0$.

Clearly the map from R to S which takes r to r_0 is 1-1. Let

ϕ denote the map from S to S which takes s to $(s^0)_0$. Since k is not a zero-divisor, $\phi(k)$ is zero or not a zero-divisor. Thus, by Lemma 2 $\operatorname{Ker}\phi = 0$ or $\operatorname{Ker}\phi = (k)$. If $\operatorname{Ker}\phi = (k)$ the above argument with the role of R and S reversed yields $R \cong S$. If $\operatorname{Ker}\phi = 0$, the map from S to R which takes s to s^0 is 1-1.

Note. There is the following analogue of Theorem 6. The proof is due to Nagata and appears in [1].

THEOREM 7. *If R is an integral domain and $R[X] = S[Y]$, then there exist injective homomorphisms of R into S and S into R.*

4. *n*-**Variable case.** I believe the work on power-invariance to data has involved only one variable. It is natural to consider the following question. When can one conclude $S \cong R$, if there exists some n such that $R[[X_1, \cdots, X_n]] = S[[Y_1, \cdots, Y_n]]$ where the X_i and Y_i are independent analytic indeterminates over R and S respectively. To wit, we give the following definition:

DEFINITION. R will be called "forever power-invariant" provided $S \cong R$ whenever there is a positive integer n such that $S[[X_1, \cdots, X_n]] \cong R[[X_1, \cdots, X_n]]$ where the X_i are independent analytic indeterminates over R and S.

Induction readily yields the following:

THEOREM 8. *If R is a quasi-local or a complete semi-local ring, then R is forever power-invariant.*

The next theorem generalizes the result that a ring with nilpotent Jacobson radical is power-invariant under the 1-variable definition. It also relaxes the nilpotent condition to a nil Jacobson radical.

THEOREM 9. *If every element of $J(R)$ is nilpotent, then R is forever power-invariant.*

Proof. Let $R[[X_1, \cdots, X_n]] = S[[Y_1, \cdots, Y_n]] = W$. Let X, Y denote the vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$; J, K vectors in R^n and S^n respectively such that $Y = J + XU$, $X = K + YV$ where U and V are nxn matrices with entries in W. Since $J(R)$ is nil, each component of the vector J is nilpotent. The key point is that if j is a nilpotent element of R, every coefficient in its expression as an element of $S[[Y]]$ is also nilpotent. Now, careful examination of $Y = J + KU + YVU$ yields $(1, \cdots, 1) = J_1 + KU_1 + \operatorname{diag} V_0U_0$ by taking the Y_i coefficient of the ith component of each side. Here J_1 denotes the result of this application to J; U_1 is the nxn matrix whose ith column consists of the Y_i coefficients of the ith column of U; and $U_0(V_0)$ is the nxn matrix of constant terms of $U(V)$. Since J_1 and K have entries in $J(S)$, (the elements of J_1 being nilpotent) the elements of the diagonal of V_0U_0 are invertible. The same is, of course, then true for VU. By a similar scrutiny of the same, i.e., $Y = J + KU + YVU$, this time taking the Y_j coefficient of the ith component of both sides with $i \neq j$, we get the entries of VU which are off the diagonal to be in $J(W)$. Thus, VU has an invertible determinant, whence both V and U do, and both are invertible matrices. Thus, if the maps from $R[X]$ to $R[[X]]$ which take R to R and X to $J + XU$ or $(X - J)U^{-1}$ can be extended to $R[X]]$ in the natural way, they are clearly inverse maps, and thus R-automorphisms. In order for the extensions to be made $R[[X]]$ needs to be complete and Hausdorff with respect to the ideal generated by the images of the $\{X_i\}$. This is clear in the case that X maps to $J + XU = Y$. $R[[X]]$ is certainly complete with respect to $(j_1, \cdots, j_n, X_1, \cdots, X_n)$. Since the j_i are nilpotent and $\bigcap_{k=1}^{\infty} (X_1, \cdots, X_n)^k = 0$, $\bigcap_{k=1}^{\infty} (j_1, \cdots, j_n, X_1, \cdots, X_n)^k = 0$ which is equivalent to $R[[X]]$ being Hausdorff with respect to $(j_1, \cdots, j_n, X_1, \cdots, X_n)$. Thus, the map taking X to $(X - J)U^{-1}$ can also be extended. Thus, $R[[X]] = R[[Y]]$ and $R \cong U/(Y) \cong S$.

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Received December 27, 1974. The authors wishes to give special thanks to Andy Magid for the counterexample in the paper. She also wishes to thank Melvin Hochster for helpful conversations.

NORTH CENTRAL COLLEGE

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 61, No. 1 November, 1975