SCATTERED COMPACTIFICATION FOR $\mathbb{N} \cup \{p\}$

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In this paper, it is shown that the scattered space $N \cup \{p\}$ admits a scattered Hausdorff compactification for a large class of points $p$ in $\beta N - N$. This gives a partial solution to the following problem raised by Z. Semadeni in 1959: "Is there a scattered Hausdorff compactification for the space $N \cup \{p\}$ where $p$ is any point of $\beta N - N$?" (See "Sur les ensembles clairsemés," Rozprawy Matematyczne, 19 (1959).) The proofs are purely topological and the compactifications are easy to visualize.

In 1970, C. Ryll-Nardzewski and R. Telgarsky [5], using deep results from Boolean Algebras, have proved that $N \cup \{p\}$ has a scattered compactification if $p$ is a $P$-point of $\beta N - N$. In the first section of this paper, it is shown that the space $\gamma N$ constructed by S. P. Franklin and M. Rajagopalan [1] serves as a scattered compactification for $N \cup \{p\}$ when $p$ is a $P$-point of $\beta N - N$. In the second section, a scattered Hausdorff compactification for $N \cup \{p\}$ is provided, when $p$ is a $P$-point of order 2 for $\beta N - N$ (definition follows). In this case, it is also shown that the compactification of $N \cup \{p\}$ is a space $Y$ such that $Y - N$ is a homeomorph of $[1, \Omega] \times \gamma N$.

**Definition 1.1.** A $P$-point of $\beta N - N$ is said to be a $P$-point of order 1 for $\beta N - N$. Suppose that for $n \in N$, we have defined a $P$-point of order $n$. Then we define a $P$-point of order $n + 1$ to be a $P$-point of the derived set of a countable set of $P$-points each being of order $n$ in $\beta N - N$.

We will now proceed to get a scattered compactification for $N \cup \{p\}$ where $p$ is a $P$-point of order 1 for $\beta N - N$, by constructing a suitable quotient space of $\beta N$ which is scattered and Hausdorff and which contains $N \cup \{p\}$ as a dense subspace. The following two lemmas are easy to prove and their proofs are omitted.

**Lemma 1.2.** Let $p$ be a $P$-point of order 1 for $\beta N - N$. Then using continuum hypothesis $\beta N - N - \{p\}$ can be written as the union of a collection $\{F_\alpha\}_{\alpha \in [1, \Omega]}$ of clopen sets in $\beta N - N$ such that $F_\alpha \subseteq F_\beta$ for all $\alpha, \beta \in [1, \Omega]$ such that $\alpha < \beta$.

**Lemma 1.3.** Let $\pi$ be a partition of $\beta N - N$ such that the quotient space $(\beta N - N)/\pi$ is Hausdorff in its quotient topology. Let $\bar{\pi}$ be the partition of $\beta N$ where each member of $N$ is a member of $\bar{\pi}$ and each member of $\pi$ is also a member of $\bar{\pi}$. Then $Y = \beta N/\bar{\pi}$
is compact and Hausdorff and the image of \( N \) in \( Y \) is an open
discrete dense subspace of \( Y \).

Further, if \((\beta N - N)/\pi \) is scattered in quotient topology, \( Y \) is
also scattered in quotient topology.

**Lemma 1.4.** Let \( p \in \beta N - N \). Let \( \pi \) be a partition of \( \beta N - N \)
such that \( \{p\} \in \pi \) and \((\beta N - N)/\pi \) is Hausdorff. Let \( \bar{\pi} \) be the
partition of \( \beta N \) as described in Lemma 1.3. Let \( \bar{q}: \beta N \rightarrow \beta N/\bar{\pi} = Y \) be
the canonical map. Then \( \bar{q} \) is a homeomorphism when restricted to
\( N \cup \{p\} \).

**Proof.** Clearly \( \bar{q}|(N \cup \{p\}): N \cup \{p\} \rightarrow N \cup \{p\} \) is continuous, one-
to-one and onto. Also \( \bar{q}: \beta N \rightarrow \beta N/\bar{\pi} \) is continuous, \( \beta N \) is compact
and by Lemma 1.3, \( Y \) is \( T_2 \). Therefore \( \bar{q} \) is a closed map and hence
upper semi-continuous. Let \( O \subset N \cup \{p\} \) be open relative to \( N \cup \{p\} \).
Then \( O = (N \cup \{p\}) \cap U \) where \( U \) is open in \( \beta N \). Let \( W \) be the union
of all partition classes with respect to \( \pi \) within \( U \). Then, by the
upper semicontinuity of \( \bar{q}, W \) is open in \( \beta N \). Since \( W \) is also saturated
under \( \bar{\pi}, \bar{q}(W) \) is open in \( \beta N/\bar{\pi} \). Also \( W \cap (N \cup \{p\}) = O \) and hence
\( \bar{q}(W) \cap \bar{q}(N \cup \{p\}) = \bar{q}(O) \). Therefore, \( \bar{q}(N \cup \{p\}) \) is open relative to \( \bar{q}(N \cup \{p\}) \).
Thus, \( \bar{q}|(N \cup \{p\}) \) is an open map. Therefore, \( \bar{q}|(N \cup \{p\}) \) is a
homeomorphism.

**Lemma 1.5.** Let \( p \) be a P-point of \( \beta N - N \). Then there exists
a partition \( \pi \) for \( \beta N - N \) such that (i) \( \{p\} \in \pi \) and (ii) the induced
quotient space \( X = (\beta N - N)/\pi \) is homeomorphic to \([1, \Omega]\).

**Proof.** By Lemma 1.2, \( \beta N - N - \{p\} \) can be written as \( \bigcup_{\alpha \in [1, \Omega)} F_{\alpha} \)
such that \( F_{\alpha} \) is clopen in \( \beta N - N \) for each \( \alpha \) and \( F_{\alpha} \subset F_{\beta} \) for \( \alpha, \beta \in [1, \Omega) \)
such that \( \alpha < \beta \). Put \( H_{\alpha} = F_{\alpha} \) and for each \( \alpha \) such that \( 1 < \alpha < \Omega \),
put \( H_{\alpha} = F_{\alpha} - \bigcup_{\beta < \alpha} F_{\beta} \), and put \( H_0 = \{p\} \). Then the collection \( \{H_{\alpha}\}_{\alpha \in [1, \Omega]} \)
forms a partition \( \pi \) of \( \beta N - N \) by closed sets in \( \beta N - N \). Let \( q: \beta N - N \rightarrow (\beta N - N)/\pi \) be the induced quotient map. Let \( q(H_{\alpha}) = b_{\alpha} \)
for all \( \alpha \in [1, \Omega] \). Let \( \tau_1 \) be the usual order topology induced on
\( \{b_{\alpha} | 1 \leq \alpha \leq \Omega\} \) by the bijection \( b_{\alpha} \rightarrow \alpha \) from \( \{b_{\alpha} | 1 \leq \alpha \leq \Omega\} \) onto \([1, \Omega]\) and let \( \tau_2 \) be the quotient topology on \( \{b_{\alpha} | 1 \leq \alpha \leq \Omega\} \) induced on it
by the partition \( \pi \) of \( \beta N - N \). Then the topologies \( \tau_1 \) and \( \tau_2 \) on
\( \{b_{\alpha} | 1 \leq \alpha \leq \Omega\} \) are both compact and Hausdorff and comparable and
hence they are homeomorphic.

**Theorem 1.6.** Let \( p \) be a P-point of order 1 for \( \beta N - N \). Then
\( N \cup \{p\} \) has a scattered compactification.

**Proof.** Let \( \pi \) be the partition of \( \beta N - N \) obtained as in Lemma
1.4. Then \( \{p\} \in \pi \) and the quotient space \((\beta N - N)/\pi = X\) is homeomorphic to \([1, 0]\). Hence \(X\) is a compact, scattered and Hausdorff space. Let \(\tilde{\pi}\) be the partition of \(\beta N\) as in Lemma 1.3. Then, by Lemma 4, \(\beta N/\tilde{\pi}\) contains a homeomorphic copy of \(N \cup \{p\}\). Since \(N\) is dense in \(\beta N\), \(N \cup \{p\}\) is dense in \(\beta N/\tilde{\pi}\). Thus, \(\beta N/\tilde{\pi}\) is a scattered, Hausdorff compactification for \(N \cup \{p\}\).

**Remark 1.6a.** The above scattered Hausdorff compactification of \(N \cup \{p\}\) is a space \(X\) such that the remainder \(X - N\) is homeomorphic to \([1, 0]\). This compact Hausdorff space \(X\) is called \(\gamma N\) by S. P. Franklin and M. Rajagopalan in [1].

2. Scattered Hausdorff compactification for \(N \cup \{p\}\) where \(p\) is \(P\)-point of order 2 in \(\beta N - N\):

**Notations.** Let \(p \in \beta N - N\). Let \(p\) be a \(P\)-point of order 2 in \(\beta N - N\). Then there exists a countable set \(\{p_1, p_2, \ldots, p_n, \ldots\}\) of distinct \(P\)-points in \(\beta N - N\) such that \(P\) is a \(P\)-point of the set

\[ B = \text{cl}_{\beta N - N} \{p_1, p_2, p_3, \ldots, p_n, \ldots\} - \{p_1, p_2, \ldots, p_n, \ldots\} \]

**Lemma 2.7.** There exists a countable collection \(\{O_n\}_{n \in N}\) of clopen sets in \(\beta N - N\) such that (i) \(O_n \cap O_m = \emptyset\) for \(n, m \in N\) such that \(n \neq m\) and (ii) \(p_n \in O_n \forall n = 1, 2, 3, \ldots\)

**Proof.** Using the zero dimensionality of \(\beta N - N\) and the fact that \(p_1\), is a \(P\)-point for \(\beta N - N\), we can get a clopen set \(O_1\) in \(\beta N - N\) containing \(p_1\) and disjoint with \(\{p_2, p_3, \ldots, p_n, \ldots\} \cup \{p\}\). Since, \(p_1\) is a \(P\)-point of \(\beta N - N\), we get a clopen set \(F_1\) in \(\beta N - N\) containing \(p_2\) and disjoint with \(p_1, p_3, p_4, \ldots, p_n, \ldots, p\). Put \(O_2 = F_1 - O_1\). Proceeding like this, by induction, for each \(n \in N\), we can get a clopen set \(O_n\) in \(\beta N - N\) satisfying the conditions (i) and (ii) of the Lemma 2.7.

**Lemma 2.8.** Let \(O\) be any \(\sigma\)-compact subset of \(\beta N - N\). Then \(\text{cl}_{\beta N - N}(O) = \beta O\).

**Proof.** This follows from the fact that \(O\) is a dense subset of the compact set \(\text{cl}_{\beta N - N}(O)\) and any continuous function \(f: O \rightarrow [0, 1]\) admits a continuous extension to \(\beta N\).

**Corollary 2.9.** Let the collection \(\{O_n\}_{n \in N}\) be as in Lemma 2.7. Let \(\text{cl}_{\beta N - N}(\bigcup_{n=1}^{\infty} O_n) = M\). Then \(\bigcup_{n=1}^{\infty} O_n\) is a \(\sigma\)-compact subset of
\[ \beta N - N \text{ and } M = \beta(\bigcup_{n=1}^{\infty} O_n). \]

**Corollary 2.10.** Let \( \{p_1, p_2, \ldots, p_n, \ldots\} \) be a countable collection of P-points of \( \beta N - N \). Let \( B = \text{cl}_{\beta N - N} \{p_1, p_2, \ldots, p_n, \ldots\} - \{p_1, p_2, \ldots, p_n, \ldots\}. \) Then \( B \cup \{p_1, p_2, \ldots, p_n, \ldots\} = \beta(\{p_1, \ldots, p_n, \ldots\}) \).

**Note 2.11.** Let \( X \) be any Tychonoff space. Let \( A \subset X \) be clopen in \( X \). Then \( \text{cl}_{\beta X} A \) is clopen in \( \beta X \).

**Proof.** The function \( f: X \to [0, 1] \) given by

\[
\begin{align*}
    f(x) &= 0, \text{ for all } x \in A \\
    &= 1, \text{ for all } x \in X - A
\end{align*}
\]

is continuous on \( X \). Therefore, \( f \) admits a continuous extension \( \tilde{f}: \beta X \to [0, 1] \). Then, it is clear that \( \tilde{f}(x) = 0 \) for all \( x \in \text{cl}_{\beta X} A \) and \( \tilde{f}(x) = 1 \) for all \( x \in \beta X - \text{cl}_{\beta X} A \). Hence, the result follows.

**Lemma 2.12.** Let the collection \( \{O_n\}_{n \in N} \) be as in Lemma 2.7. Let \( B \) be as in Corollary 2.10. Let \( \text{cl}_{\beta N - N} \left( \bigcup_{n=1}^{\infty} O_n \right) = M \). Let \( M - \bigcup_{n=1}^{\infty} O_n = K \). Then, there exists an increasing collection \( \{A_\alpha\}_{\alpha \in [1, \Omega]} \) of clopen sets relative to \( K \) such that \( \bigcup_{\alpha \in [1, \Omega]} A_\alpha = K - B \).

**Proof.** For each \( n \in N, \) \( p_n \) is a P-point of \( \beta N - N \) and \( p_n \in O_n \). Hence, \( p_n \) is a P-point of \( O_n \) for all \( n = 1, 2, 3, \ldots \). Therefore, as in Lemma 1.2, using continuum hypothesis, for each \( n \in N, \) \( O_n - \{p_n\} \) can be expressed as the union of an increasing collection \( \{A_\alpha\}_{\alpha \in [1, \Omega]} \) of clopen sets relative to \( O_n \) (and hence relative to \( \beta N - N \) also). For each \( n \in N, \) put \( A_\alpha = \left[ \text{cl}_{\beta N - N} \left( \bigcup_{n=1}^{\infty} A_{\alpha_n} \right) \right] \cap K \). Then, by Corollary 2.9 and Note 2.11 above, \( A_\alpha \) is clopen relative to \( K \) for all \( \alpha \in [1, \Omega] \).

Since \( A_{\alpha_n} \subset A_{\beta_n} \) for \( \alpha < \beta, \alpha, \beta \in [1, \Omega], \) it follows that \( A_\alpha \subset A_\beta \) for all \( \alpha, \beta \in [1, \Omega] \) such that \( \alpha < \beta \).

Now it remains to show that \( \bigcup_{\alpha \in [1, \Omega]} A_\alpha = K - B \). Clearly \( A_\alpha \cap B = \emptyset \) for all \( \alpha \in [1, \Omega] \) and hence \( \bigcup_{\alpha} A_\alpha \subset K - B \). To get the other inclusion, let \( x_0 \in K - B \). Now, \( K - B \) is open relative to \( K \) and \( K \) is zero-dimensional. Therefore, there exists a clopen set \( V \) relative to \( K \) such that \( x_0 \in V \subset K - B \). Since \( V \subset K \) is clopen in \( K \) and \( \beta N - N \) is zero dimensional, there exists a clopen set \( W \) in \( \beta N - N \) such that \( V = W \cap K \). Put \( W \cap O_n = W_n \) for all \( n = 1, 2, 3, \ldots \). We note that \( p_n \) can belong to \( W_n \) for at most a finite number of \( n \)'s. Therefore, \( \exists k_o \in N \) such that \( p_n \notin W_n \forall n > k_o \). Hence, for each \( n > k_o \), there exists a countable ordinal \( \alpha_n \) such that \( A_{\alpha_n} \supset W_n \). Let the supremum of \( \alpha_n \) for \( n > k_o \), be \( \gamma \). Then \( A_{\gamma_n} \supset W_n \forall n > k_o \). Therefore,
\[ \bigcup_{n=k_0+1}^{\infty} A_{\gamma_n} \supset \bigcup_{n=k_0+1}^{\infty} W_n. \]

Hence,

\[
\bigcup_{n=1}^{\infty} A_{\gamma_n} \cap K = A_\tau = \bigcup_{n=k_0+1}^{\infty} A_{\lambda_n} \cap K \\
= \bigcup_{n=k_0+1}^{\infty} W_n \cap K \\
= \bigcup_{n=1}^{\infty} W_n \cap K \\
= \bigcup_{n=1}^{\infty} (W \cap O_n) \cap K \\
= W \cap M \cap K \\
= W \cap K \\
= V .
\]

Also \( x_0 \in V \). Therefore, \( \bigcup_{\alpha \in \{1, \Omega\}} A_\alpha = K - B \).

**Lemma 2.13.** Let \( B \) be as defined in Corollary 2.10 and let \( K \) be as in Lemma 2.12. Then, there exists a collection \( \{X_\alpha\}_{\alpha \in \{1, \Omega\}} \) of clopen sets relative to \( K \) such that \( X_\alpha \subseteq X_\beta \forall \alpha, \beta \in \{1, \Omega\} \) such that \( \alpha < \beta \) and \( \bigcup_{\alpha \in \{1, \Omega\}} X_\alpha \cap B = B - \{p\} \).

**Proof.** Now, \( p \) is a \( P \)-point of \( B \) and hence, using continuum hypothesis, \( B - \{p\} \) can be written as the union of an ascending collection \( \{B_\alpha\}_{\alpha \in \{1, \Omega\}} \) of clopen sets relative to \( B \). Since, by Corollary 2.10, \( B \cup \{p_1, p_2, \ldots, p_n, \ldots\} = \beta(\{p_1, \ldots, p_n, \ldots\}) \), each \( B_\alpha \) gives a subset \( N_\alpha = \{p_{\alpha_1}, \ldots, p_{\alpha_n}, \ldots\} \) of \( \{p_1, p_2, \ldots, p_n, \ldots\} \) such that

\[ \text{cl}_{\beta N - N}(N_\alpha) \cap B_\alpha = B . \]

Since \( B_\alpha \subseteq B_\beta \) for \( \alpha < \beta \), we have \( N_\alpha \) is almost contained in \( N_\beta \) for \( \alpha < \beta \). Put \( \text{cl}_{\beta N - N}(\bigcup_{\alpha=1}^{\infty} O_\alpha) \cap K = X_\alpha \forall \alpha \in \{1, \Omega\} \). Then \( X_\alpha \) is closed in \( K \forall \alpha \in \{1, \Omega\} \), \( X_\alpha \subseteq X_\beta \) for \( \alpha < \beta \), \( X_\alpha \cap B = B_\alpha \forall \alpha \in \{1, \Omega\} \) and also \( (\bigcup_{\alpha} X_\alpha) \cap B = \bigcup_{\alpha} (X_\alpha \cap B) = \bigcup_{\alpha} B_\alpha = B - \{p\} \).

**Lemma 2.14.** Let the collection \( \{O_\alpha\}_{\alpha \in \mathbb{N}} \), \( M \) and \( K \) be as in Lemma 2.12. Let \( \beta N - N - M = T \). Let \( \{C_\alpha\}_{\alpha \in \{1, \Omega\}} \) be an ascending collection of clopen sets relative to \( K \). Then, there exists an ascending collection \( \{I_\alpha\}_{\alpha \in \{1, \Omega\}} \) of subsets of \( T \cup K \) such that \( I_\alpha \) is closed in \( T_\alpha \cup K \), \( I_\alpha \cap K = C_\alpha \forall \alpha \in \{1, \Omega\} \) and \( \bigcup_{\alpha} I_\alpha = \bigcup_{\alpha} C_\alpha = T \).

**Proof.** Using the fact that \( \beta N - N \) is zero-dimensional and is of weight \( c \) and also using the fact that the clopen sets of \( \beta N - N \)
satisfy the Dubois-Reymond separability condition, we can write $T$ as the union of an ascending collection $\{G_\alpha\}_{\alpha \in (1, \Omega)}$ of clopen sets in $\beta N - N$ such that $G_\alpha \cap M = \emptyset \forall \alpha \in [1, \Omega)$.

Now, $C_i$ is clopen in $K$. Since $\beta N - N$ is zero-dimensional, there exists a clopen set $J_i$ in $\beta N - N$ such that $J_i \cap K = C_i$. Put $I_i = [J_i \cap (T \cup K)] \cup I_a$. Then $I_i$ is clopen in $T \cup K$ and $I_i \cap K = C_i$. Suppose that we have constructed clopen sets $I_1, I_2, \ldots, I_n$ in $T \cup K$ for $n \in N$ such that $I_1 \subset I_2 \subset \cdots \subset I_n$ and $I_j \cap K = C_j$ for $j = 1, 2, \ldots, n$. Then we construct $I_{n+1}$ as follows: Since $C_{n+1}$ is clopen in $K$ and $\beta N - N$ is zero-dimensional, there exists a clopen set $J_{n+1}$ in $\beta N - N$ such that $J_{n+1} \cap K = C_{n+1}$. Put $I_{n+1} = [J_{n+1} \cap (T \cup K)] \cup I_n \cup G_{n+1}$. Then $I_{n+1}$ is clopen in $T \cup K$, $I_{n+1} \supset I_n$ and $I_{n+1} \cap K = C_{n+1}$. Having constructed $I_1, I_2, \ldots, I_n$, we now proceed to construct $I_n$. First, we claim that $\bigcap_{j=1}^n (I_j \cup \overline{I_j}) = \emptyset$. For, let $x_0 \in k - C_0$, which is clopen in $K$. Since $\beta N - N$ is zero-dimensional, there exists a clopen set $J_n$ in $\beta N - N$ such that $J_n \cap K = K - C_n$. Let $H_n \cap I_n = H_{n+1} \forall_n = 1, 2, 3, \ldots$. Then $H_{n+1}$ is closed in $\beta N - N$. We will now prove that $H_{n+1}$ is also open in $\beta N - N$. Since, $I_n$ is clopen in $T \cup K$ and $\beta N - N$ is zero dimensional, there exists a clopen set $G_n$ in $\beta N - N$ such that $G_n \cap (T \cup K) = I_n$. Then $G_n \cap [(T \cup K) \cap K] = I_n \cap K = C_n$. Now

$$H_{n+1} = (H_n \cap I_n) = H_n \cap [G_n \cap (T \cup K)] = H_n \cap [(G_n \cap T) \cup (G_n \cap K)] = (H_n \cap G_n \cap T) \cup (H_n \cap G_n \cap K) = (H_n \cap G_n \cap T) \cup [K \cap (K - C_n) \cap G_n] = (H_n \cap G_n \cap T) \cup (C_n \cap (K - C_n)) = H_n \cap G_n \cap T \text{ which is open in } \beta N - N.$$
Now $\beta N - N$ is zero dimensional, $C_ω \cup \text{cl}_{\beta N - N} (\bigcup_{n=1}^{\infty} I_n)$ is a compact subset of $\beta N - N$ and $D_i$ is an open set in $\beta N - N$ containing $C_ω \cup \text{cl}_{\beta N - N} (\bigcup_{n=1}^{\infty} I_n)$. Hence, there exists a clopen set $J_ω$ in $\beta N - N$ such that $D_i \supset J_ω \supset C_ω \cup \text{cl}_{\beta N - N} (\bigcup_{n=1}^{\infty} I_n)$. Now, $J_ω \cap D_i = \emptyset$ and hence $(K - C_ω) \cap J_ω = \emptyset$. Therefore, $J_ω \cap K = C_ω$. Take $L_a = [J_ω \cap (T \cup K)] \cup H_ω$. Then $L_a$ is clopen in $T \cup K$, $L_a \supset \bigcup_{n=1}^{\infty} I_a$ and $L_a \cap K = C_ω$. Continuing this process, we get an increasing collection $\{I_a\}_{\alpha \in [1, \Omega)}$ of clopen sets in $T \cup K$ such that $I_\alpha \cap K = C_\alpha \forall \alpha \in [1, \Omega)$. It can also be seen that $\bigcup_a I_a - \bigcup_a C_\alpha = T$.

**Corollary 2.15.** Let the collection $\{A_\alpha\}_{\alpha \in [1, \Omega)}$ be as in Lemma 2.12. Then, there exists a collection $\{S_\alpha\}_{\alpha \in [1, \Omega)}$ of clopen sets in $T \cup K$ such that $S_\alpha \subset S_\beta \forall \alpha, \beta \in [1, \Omega)$ such that $\alpha < \beta$, $S_\alpha \cap K = A_\alpha \forall \alpha \in [1, \Omega)$ and $\bigcup_a S_\alpha - \bigcup_a A_\alpha = T$.

**Corollary 2.16.** Let the collection $\{x_\alpha\}_{\alpha \in [1, \Omega)}$ be as in Lemma 2.13. Then, there exists an increasing collection $\{L_\alpha\}_{\alpha \in [1, \Omega)}$ of clopen sets in $T \cup K$ such that $L_\alpha \cap K = X_\alpha \forall \alpha \in [1, \Omega)$ and $\bigcup_a L_\alpha - \bigcup_a X_\alpha = T$.

**Definition 2.17.** Let $\sigma_1$ and $\sigma_2$ be two partitions of a nonempty set $X$. Then we define $\sigma_1 \cap \sigma_2$ to be the partition of $X$ given by the collection $\{A \cap B \mid A \in \sigma_1, B \in \sigma_2, A \cap B \neq \emptyset\}$ of nonempty subsets of $X$.

**Lemma 2.18.** Let $X$ be a compact Hausdorff space. Let $\sigma_1$, $\sigma_2$ be two Hausdorff partitions for $X$. Then $\sigma_1 \cap \sigma_2$ is also a Hausdorff partition for $X$.

**Proof.** Let $X/\sigma_1 = Y_1$ and $X/\sigma_2 = Y_2$. Let $q_1: X \to Y_1$ and $q_2: X \to Y_2$ be the corresponding quotient maps. Define $(q_1, q_2): X \to Y_1 \times Y_2$ by $(q_1, q_2)(x) = (q_1(x), q_2(x)) \forall x \in X$. This is a continuous function form $X$ into $Y_1 \times Y_2$. Now $Y_1 \times Y_2$ is Hausdorff. Consider $q_1, q_2$ as a map from $X$ onto $(q_1, q_2)(X)$. Let the partition induced on $X$ by this map be $\sigma$. Then $\sigma = \sigma_1 \cap \sigma_2$. Let $q: X \to X/\sigma$ be the corresponding quotient map. Let $g: X/\sigma \to (q_1, q_2)(X)$ be the natural fill-up map making the following diagram commutative.
Now $X/\sigma$ is compact, $(q_1, q_2)(X)$ is Hausdorff and $g$ is one-to-one, onto and continuous. Hence $g$ is a homeomorphism. Since $(q_1, q_2)(X)$ is Hausdorff, it follows that $X/\sigma$ is Hausdorff. Therefore $\sigma_1 \cap \sigma_2$ is a Hausdorff partition for $X$.

In the above proof, we also note that the quotient space induced by $\sigma_1 \cap \sigma_2$ is homeomorphic to the range of the function $(q_1, q_2)$ in $Y_1 \times Y_2$.

**Lemma 2.19.** Let $T$ and $K$ be as in Lemma 2.14. Let $B$ and $p$ be as in Lemma 2.13. Then, there exists a Hausdorff partition for $T \cup K$ with \{p\} as a separate partition class.

**Proof.** Let the collection \{\(S_a\)\}_{a \in [1, \Omega]} be as in Corollary 2.15 and let the collection \{\(L_a\)\}_{a \in [1, \Omega]} be as in Corollary 2.16. Put $H_1 = S_1$ and for each $\alpha \in [2, \Omega)$, $H_\alpha = S_\alpha - \bigcup_{1 \leq \gamma < \alpha} S_\gamma$ and $H_\Omega = K - \bigcup a A_a = B$.

Also, let $M_1 = L_1$; for each $\alpha \in [2, \Omega)$, $M_\alpha = L_\alpha - \bigcup_{1 \leq \gamma < \alpha} L_\gamma$ and $M_\Omega = K - \bigcup_{\alpha \in [1, \Omega)} X_\alpha$. Then, the collection \{\(H_a\)\}_{a \in [1, \Omega]} gives a partition $\pi_1$ for $T \cup K$ such that the quotient space $(T \cup K)/\pi_1$ is homeomorphic to $[1, \Omega]$. Therefore, $\pi_1$ is a Hausdorff partition for $T \cup K$. Similarly, the collection \{\(M_a\)\}_{a \in [1, \Omega]} gives a Hausdorff partition $\pi_2$ for $T \cup K$. Let $\pi_1 \cap \pi_2 = \pi_3$. Then, by Lemma 2.18, $\pi_3$ is a Hausdorff partition for $T \cup K$. Also

$$H_\Omega \cap M_\Omega = B \cap \left( K - \bigcup_\alpha X_\alpha \right)$$

$$= B - \bigcup_\alpha (B \cap X_\alpha)$$

$$= B - \bigcup_\alpha B_\alpha = \{p\}.$$

**Lemma 2.20.** Let $X$ be a topological space. Let $A_1$ and $A_2$ be closed in $X$. Let $A_1 \cup A_2 = X$. Let $A \subset X$ be such that $A \cap A_1$ is open relative to $A_1$ and $A \cap A_2$ is open relative to $A_2$. Then $A$ is open in $X$.

**Proof.** This follows from the fact that

$$A = (O_1 - A_2) \cup (O_2 - A_1) \cup (O_1 \cap O_2).$$

**Lemma 2.21.** Let $\pi_3$ be the partition of $T \cup K$ as obtained in the proof of Lemma 2.19. Let the collection of sets \{\(A_{ak}\)\}_{k \in N} be as obtained in the proof of Lemma 2.12. Let \{\(p_1, p_2, \cdots, p_n, \cdots\)\} be as in Corollary 2.10. For each $k \in N$, let $D_\alpha = A_{ak} - \bigcup_{1 \leq \gamma < \alpha} A_{\gamma k}$. Then the collection of sets \{\(D_{ak}\)\}_{k \in N} and \{\(p_\alpha\)\}_{\alpha \in N} together with the members of $\pi_3$ form a Hausdorff partition $\pi_4$ for $\beta N - N$. 


Proof. Clearly $\pi_4$ is a partition for $\beta N - N$. We will now prove that $(\beta N - N)/\pi_4$ is Hausdorff. Given any two partition classes $C_1$ and $C_2$ of $\beta N - N$ with respect to $\pi_4$, we must prove that there exists a clopen set $Y_1$ in $\beta N - N$ containing $C_1$, disjoint with $C_2$ and saturated under $\pi_4$. The cases where either $C_1$ or $C_2$ is a $D_{\alpha}$ or a $p_{\alpha}$ are easy to handle and we consider the following cases:

Case 1. Let $C_1 = H_{\alpha} \cap M_\beta$ and $C_2 = H_{\gamma} \cap M_\beta$ where $\alpha, \beta, \gamma \in [1, 2]$ and $\beta \neq \gamma$. Without loss of generality, we can assume that $\beta < \gamma$. Now, by definition $X_\beta = \text{cl}_{\beta N - N}(\bigcup_{n=1}^{N} O_n) \cap K$ where $\text{cl}_{\beta N - N}((p_\beta, \ldots, p_n, \ldots)) \cap B = B_\beta$ (see the proof of Lemma 2.13). Also $L_\beta \cap K = X_\beta$ where $L_\beta$ is clopen in $T \cup K$ (see Corollary 2.16). Now, $Y_1 = L_\beta \cup \text{cl}_{\beta N - N}(\bigcup_{n=1}^{N} O_n)$ is closed in $\beta N - N$ and using Lemma 2.20, we can see that it is also open in $\beta N - N$. Further $Y_1 \supset C_1$ and $Y_1 \cap C_2 = \emptyset$. Also, $Y_1$ is saturated under $\pi_4$. Therefore, $\pi_4$ is a Hausdorff partition for $\beta N - N$.

Case 2. Let $C_1 = H_{\alpha} \cap M_\beta$ and $C_2 = H_{\gamma} \cap M_\beta$ where $\alpha, \beta, \gamma, \delta \in [1, 2]$ and $\alpha \neq \gamma$. Without loss of generality, we can assume that $\alpha < \gamma$. In this case, using Lemma 2.20, we can verify that the set $Y_1 = \text{cl}_{\beta N - N}(\bigcup_{n=1}^{N} A_n) \cup S_\alpha$ is clopen in $\beta N - N$. Further, $Y_1 \supset C_1$ and $Y_1 \cap C_2 = \emptyset$. Also $Y_1$ is saturated under $\pi_4$. Therefore, $\pi_4$ is a Hausdorff partition for $\beta N - N$.

**Lemma 2.22.** Let $\pi_4$ be the Hausdorff partition of $\beta N - N$ as given in Lemma 2.21. Let $\pi_5$ be the partition of $M$ given by $\pi_5 = \pi_4|M = \{X \cap M | X \in \pi_4\}$. Then $\pi_5$ is a Hausdorff partition for $M$.

**Proof.** Let $D_{\alpha}$, $p_\alpha$, $B$, and $O_\alpha$ be as in above lemmas. Let $E_\alpha = A_\alpha$ and $E_\alpha = A_\alpha - \bigcap_{1 < \alpha < \beta} A_\alpha, \forall \alpha \in [2, 2]$. Then, it is easy to see that the partition $\pi_5$ of $M$ given by the collection $\{D_{\alpha}\alpha \in [1, 2]\}k \in N, \{p_{\alpha}\alpha \in [1, 2]\}$ and $B$ is a Hausdorff partition for $M$. Let $K_1 = X_1$ and $K_\alpha = X_\alpha - \bigcup_{1 < \alpha < \beta} X_\alpha, \forall \alpha \in [1, 2]$. Also, let $K_\alpha = K - \bigcup_{\alpha \in [1, 2]} X_\alpha$. Then, the partition $\pi_5$ of $M$ given by the collection $\{O_\alpha\alpha \in N\}$ and $\{K_\alpha\alpha \in [1, 2]\}$ is also a Hausdorff partition for $M$. Further $\pi_5 = \pi_4 \cap \pi_5$. Hence, by Lemma 2.18, $\pi_5$ is a Hausdorff partition for $M$.

**Lemma 2.23.** Let $M$, $\pi_4$ and $\pi_5$ be as in previous lemmas. Then $M/\pi_5$ is homeomorphic to $(\beta N - N)/\pi_4$.

**Proof.** Let $(\beta N - N)/\pi_4 = Y$ and let $q_4 : \beta N - N \to Y$ be the quotient map induced by the partition $\pi_4$ of $\beta N - N$. Then, by Lemma 2.21, $Y$ is Hausdorff. Now, the map $q_4 : M \to Y$ is a continuous function from $M$ onto $Y$ where $M$ is compact and $Y$ is Hausdorff. Hence, the topology of $Y$ is the quotient topology of $M$ induced on
it by the function \( q_4/M \). But \( q_4 \) induces the partition \( \pi_\delta \) on \( M \). Therefore, \( M/\pi_\delta \) is homeomorphic to \( Y = (\beta N - N)/\pi_\delta \).

**Lemma 2.24.** Let all notations be as in previous lemmas. Then \( M/\pi_\delta \) is homeomorphic to \( \gamma N \times [1, \Omega] \) where \( \gamma N \) is the compactification of \( N \) constructed by S. P. Frankline and M. Rajagopalan in [1]. (See also remark 1.6a).

**Proof.** Now \( \pi_\delta = \pi_6 \cap \pi_7 \) where \( \pi_6 \) and \( \pi_7 \) are Hausdorff partitions of \( M \) as given in the proof of Lemma 2.22. Let \( q_6: M \to M/\pi_6 \) and \( q_7: M \to M/\pi_7 \) be the corresponding quotient maps. Consider the function \( (q_6, q_7): M \to M/\pi_6 \times M/\pi_7 \) given by \( (q_6, q_7)(x) = (q_6(x), q_7(x)) \forall x \in M \). Since \( \pi_6 \cap \pi_7 = \pi_4 \), it follows from Lemma 2.18 that \( M/\pi_\delta \) is homeomorphic to the range of the function \( (q_6, q_7) \) from \( M \) into \( M/\pi_6 \times M/\pi_7 \). But it can be seen that \( M/\pi_\delta \) is homeomorphic to \( [1, \Omega] \times [1, \omega] \) with its usual product topology and \( M/\pi_\gamma \) is homeomorphic to \( \gamma N \) and that the range of the map \( (q_6, q_7) \) is homeomorphic to \( [1, \Omega] \times \gamma N \). Hence, \( M/\pi_\delta \) is homeomorphic to \( [1, \Omega] \times \gamma N \).

**Theorem 2.25.** \( N \cup \{p\} \) has a scattered Hausdorff compactification, when \( p \) is a \( P \)-point of order 2 for \( \beta N - N \).

**Proof.** Consider the partition \( \pi_4 \) of \( \beta N - N \) given in Lemma 2.21. Let \( \bar{\pi}_4 \) be the partition of \( \beta N \) whose members are the members of \( \pi_4 \) and the singletons in \( N \). Since, \( (\beta N - N)/\pi_4 \) is Hausdorff, by Lemma 1.3, it follows that \( \beta N/\bar{\pi}_4 \) is Hausdorff. Since \( \beta N \) is compact, we have \( \beta N/\bar{\pi}_4 \) is compact. Since \( (\beta N - N)/\pi_4 \) is homeomorphic to \( [1, \Omega] \times \gamma N \) which is scattered, we have that \( \beta N/\bar{\pi}_4 \) is also scattered. Since \( N \) is dense in \( \beta N \) and \( N \cup \{p\} \) maps homeomorphically onto itself under the quotient map from \( \beta N \) onto \( \beta N/\bar{\pi}_4 \) (Lemma 1.4), it follows that \( N \cup \{p\} \) is dense in \( \beta N/\bar{\pi}_4 \). Thus, \( \beta N/\bar{\pi}_4 \) is a scattered Hausdorff compactification for \( N \cup \{p\} \). Hence the theorem.

**References**


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