SCATTERED COMPACTIFICATION FOR $N \cup \{p\}$

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In this paper, it is shown that the scattered space \( N \cup \{ p \} \) admits a scattered Hausdorff compactification for a large class of points \( p \) in \( \beta N - N \). This gives a partial solution to the following problem raised by Z. Semadeni in 1959: "Is there a scattered Hausdorff compactification for the space \( N \cup \{ p \} \) where \( p \) is any point of \( \beta N - N \)?" (See "Sur les ensembles clairsemés," Rozprawy Matematyczne, 19 (1959).) The proofs are purely topological and the compactifications are easy to visualize.

In 1970, G. Ryll-Nardzewski and R. Telgarsky [5], using deep results from Boolean Algebras, have proved that \( N \cup \{ p \} \) has a scattered compactification if \( p \) is a P-point of \( \beta N - N \). In the first section of this paper, it is shown that the space \( \gamma N \) constructed by S. P. Franklin and M. Rajagopalan [1] serves as a scattered compactification for \( N \cup \{ p \} \) when \( p \) is a P-point of \( \beta N - N \). In the second section, a scattered Hausdorff compactification for \( N \cup \{ p \} \) is provided, when \( p \) is a P-point of order 2 for \( \beta N - N \) (definition follows). In this case, it is also shown that the compactification of \( N \cup \{ p \} \) is a space \( Y \) such that \( Y - N \) is a homeomorph of \([1, \Omega] \times \gamma N\).

**Definition 1.1.** A P-point of \( \beta N - N \) is said to be P-point of order 1 for \( \beta N - N \). Suppose that for \( n \in N \), we have defined a P-point of order \( n \). Then we define a P-point of order \( n + 1 \) to be a P-point of the derived set of a countable set of P-points each being of order \( n \) in \( \beta N - N \).

We will now proceed to get a scattered compactification for \( N \cup \{ p \} \) where \( p \) is a P-point of order 1 for \( \beta N - N \), by constructing a suitable quotient space of \( \beta N \) which is scattered and Hausdorff and which contains \( N \cup \{ p \} \) as a dense subspace. The following two lemmas are easy to prove and their proofs are omitted.

**Lemma 1.2.** Let \( p \) be a P-point of order 1 for \( \beta N - N \). Then using continuum hypothesis \( \beta N - N - \{ p \} \) can be written as the union of a collection \( \{ F_\alpha \}_{\alpha \in [1, \Omega]} \) of clopen sets in \( \beta N - N \) such that \( F_\alpha \subset F_\beta \) for all \( \alpha, \beta \in [1, \Omega] \) such that \( \alpha < \beta \).

**Lemma 1.3.** Let \( \pi \) be a partition of \( \beta N - N \) such that the quotient space \( (\beta N - N) / \pi \) is Hausdorff in its quotient topology. Let \( \tilde{\pi} \) be the partition of \( \beta N \) where each member of \( N \) is a member of \( \tilde{\pi} \) and each member of \( \pi \) is also a member of \( \tilde{\pi} \). Then \( Y = \beta N / \tilde{\pi} \).
is compact and Hausdorff and the image of $N$ in $Y$ is an open discrete dense subspace of $Y$.

Further, if $(\beta N - N)/\pi$ is scattered in quotient topology, $Y$ is also scattered in quotient topology.

**Lemma 1.4.** Let $p \in \beta N - N$. Let $\pi$ be a partition of $\beta N - N$ such that $(p) \in \pi$ and $(\beta N - N)/\pi$ is Hausdorff. Let $\bar{\pi}$ be the partition of $\beta N$ as described in Lemma 1.3. Let $\bar{q}: \beta N \to \beta N/\bar{\pi} = Y$ be the canonical map. Then $\bar{q}$ is a homeomorphism when restricted to $N \cup \{p\}$.

**Proof.** Clearly $\bar{q}|(N \cup \{p\}): N \cup \{p\} \to N \cup \{p\}$ is continuous, one-to-one and onto. Also $\bar{q}: \beta N \to \beta N/\bar{\pi}$ is continuous, $\beta N$ is compact and by Lemma 1.3, $Y$ is $T_2$. Therefore $\bar{q}$ is a closed map and hence upper semi-continuous. Let $O \subset N \cup \{p\}$ be open relative to $N \cup \{p\}$. Then $O = (N \cup \{p\}) \cap U$ where $U$ is open in $\beta N$. Let $W$ be the union of all partition classes with respect to $\bar{\pi}$ within $U$. Then, by the upper semicontinuity of $\bar{q}$, $W$ is open in $\beta N$. Since $W$ is also saturated under $\bar{\pi}$, $\bar{q}(W)$ is open in $\beta N/\bar{\pi}$. Also $W \cap (N \cup \{p\}) = O$ and hence $\bar{q}(W) \cap \bar{q}(N \cup \{p\}) = \bar{q}(O)$. Therefore, $\bar{q}(O)$ is open relative to $\bar{q}(N \cup \{p\})$. Thus, $\bar{q}|(N \cup \{p\})$ is an open map. Therefore, $\bar{q}|(N \cup \{p\})$ is a homeomorphism.

**Lemma 1.5.** Let $p$ be a $P$-point of $\beta N - N$. Then there exists a partition $\pi$ for $\beta N - N$ such that (i) $(p) \in \pi$ and (ii) the induced quotient space $X = (\beta N - N)/\pi$ is homeomorphic to $[1, \Omega]$.

**Proof.** By Lemma 1.2, $\beta N - N - \{p\}$ can be written as $\bigcup_{a \in [1, \Omega]} F_a$ such that $F_a$ is clopen in $\beta N - N$ for each $a$ and $F_a \subset F_{\beta_a}$, $\beta_a \in [1, \Omega]$ such that $\alpha < \beta$. Put $H_a = F_1$ and for each $a$ such that $1 < \alpha < \Omega$, put $H_a = F_a - \bigcup_{1 \leq \alpha < a} F_\gamma$, and put $H_\Omega = \{p\}$. Then the collection $(H_a)_{a \in [1, \Omega]}$ forms a partition $\pi$ of $\beta N - N$ by closed sets in $\beta N - N$. Let $q: \beta N - N \to (\beta N - N)/\pi$ be the induced quotient map. Let $q(H_a) = b_a$ for all $a \in [1, \Omega]$. Let $\tau_1$ be the usual order topology induced on $\{b_a \mid 1 \leq \alpha \leq \Omega\}$ by the bijection $b_a \to \alpha$ from $\{b_a \mid 1 \leq \alpha \leq \Omega\}$ onto $[1, \Omega]$ and let $\tau_2$ be the quotient topology on $\{b_a \mid 1 \leq \alpha \leq \Omega\}$ induced on it by the partition $\pi$ of $\beta N - N$. Then the topologies $\tau_1$ and $\tau_2$ on $\{b_a \mid 1 \leq \alpha \leq \Omega\}$ are both compact and Hausdorff and comparable and hence they are homeomorphic.

**Theorem 1.6.** Let $p$ be a $P$-point of order 1 for $\beta N - N$. Then $N \cup \{p\}$ has a scattered compactification.

**Proof.** Let $\pi$ be the partition of $\beta N - N$ obtained as in Lemma
1.4. Then \( \{p\} \in \pi \) and the quotient space \((\beta N - N)/\pi = X\) is homeomorphic to \([1, \omega]\). Hence \( X \) is a compact, scattered and Hausdorff space. Let \( \tilde{\pi} \) be the partition of \( \beta N \) as in Lemma 1.3. Then, by Lemma 4, \( \beta N/\tilde{\pi} \) contains a homeomorphic copy of \( N \cup \{p\} \). Since \( N \) is dense in \( \beta N \), \( N \cup \{p\} \) is dense in \( \beta N/\tilde{\pi} \). Thus, \( \beta N/\tilde{\pi} \) is a scattered, Hausdorff compactification for \( N \cup \{p\} \).

**Remark 1.6a.** The above scattered Hausdorff compactification of \( N \cup \{p\} \) is a space \( X \) such that the remainder \( X - N \) is homeomorphic to \([1, \omega]\). This compact Hausdorff space \( X \) is called \( \gamma N \) by S. P. Franklin and M. Rajagopalan in [1].

2. Scattered Hausdorff compactification for \( N \cup \{p\} \) where \( p \) is \( P \)-point of order 2 in \( \beta N - N \):

**Notations.** Let \( p \in \beta N - N \). Let \( p \) be a \( P \)-point of order 2 in \( \beta N - N \). Then there exists a countable set \( \{p_1, p_2, \ldots, p_n, \ldots\} \) of distinct \( P \)-points in \( \beta N - N \) such that \( P \) is a \( P \)-point of the set

\[
B = \text{cl}_{\beta N - N} \{p_1, p_2, p_3, \ldots, p_n, \ldots\} - \{p_1, p_2, \ldots, p_n, \ldots\}.
\]

**Lemma 2.7.** There exists a countable collection \( \{O_n\}_{n \in N} \) of clopen sets in \( \beta N - N \) such that (i) \( O_n \cap O_m = \emptyset \) for \( n, m \in N \) such that \( n \neq m \) and (ii) \( p_n \in O_n \forall n = 1, 2, 3, \ldots \)

**Proof.** Using the zero dimensionality of \( \beta N - N \) and the fact that \( p_1 \) is a \( P \)-point for \( \beta N - N \), we can get a clopen set \( O_1 \) in \( \beta N - N \) containing \( p_1 \) and disjoint with \( \{p_2, p_3, \ldots, p_n, \ldots\} \cup \{p\} \). Since, \( p_2 \) is a \( P \)-point of \( \beta N - N \), we get a clopen set \( F_1 \) in \( \beta N - N \) containing \( p_2 \) and disjoint with \( p_1, p_3, p_4, \ldots, p_n, \ldots, p \). Put \( O_2 = F_1 - O_1 \). Proceeding like this, by induction, for each \( n \in N \), we can get a clopen set \( O_n \) in \( \beta N - N \) satisfying the conditions (i) and (ii) of the Lemma 2.7.

**Lemma 2.8.** Let \( O \) be any \( \sigma \)-compact subset of \( \beta N - N \). Then \( \text{cl}_{\beta N - N}^{(\sigma)} = \beta O \).

**Proof.** This follows from the fact that \( O \) is a dense subset of the compact set \( \text{cl}_{\beta N - N} (O) \) and any continuous function \( f: O \to [0, 1] \) admits a continuous extension to \( \beta N \).

**Corollary 2.9.** Let the collection \( \{O_n\}_{n \in N} \) be as in Lemma 2.7. Let \( \text{cl}_{\beta N - N} (\bigcup_{n=1}^{\infty} O_n) = M \). Then \( \bigcup_{n=1}^{\infty} O_n \) is a \( \sigma \)-compact subset of
Corollary 2.10. Let \( \{p_1, p_2, \ldots, p_n, \ldots\} \) be a countable collection of P-points of \( \beta N - N \). Let \( B = \text{cl}_{\beta N - N} \{p_1, p_2, \ldots, p_n, \ldots\} - \{p_1, p_2, \ldots, p_n, \ldots\} \). Then \( B \cup \{p_1, p_2, \ldots, p_n, \ldots\} = \beta(\{p_1, \ldots, p_n, \ldots\}) \).

Note 2.11. Let \( X \) be any Tychonoff space. Let \( A \subset X \) be clopen in \( X \). Then \( \text{cl}_{\beta X} A \) is clopen in \( \beta X \).

Proof. The function \( f: X \to [0, 1] \) given by
\[
\begin{align*}
f(x) &= 0, \text{ for all } x \in A \\
&= 1, \text{ for all } x \in X - A
\end{align*}
\]
is continuous on \( X \). Therefore, \( f \) admits a continuous extension \( \tilde{f}: \beta X \to [0, 1] \). Then, it is clear that \( \tilde{f}(x) = 0 \) for all \( x \in \text{cl}_{\beta X} A \) and \( \tilde{f}(x) = 1 \) for all \( x \in \beta X - \text{cl}_{\beta X} A \). Hence, the result follows.

Lemma 2.12. Let the collection \( \{O_n\}_{n \in \mathbb{N}} \) be as in Lemma 2.7. Let \( B \) be as in Corollary 2.10. Let \( \text{cl}_{\beta N - N}(\bigcup_{n=1}^{\infty} O_n) = M \). Let \( M - \bigcup_{n=1}^{\infty} O_n = K \). Then, there exists an increasing collection \( \{A_\alpha\}_{\alpha \in [1, \Omega)} \) of clopen sets relative to \( K \) such that \( \bigcup_{\alpha \in [1, \Omega)} A_\alpha = K - B \).

Proof. For each \( n \in \mathbb{N} \), \( p_n \) is a P-point of \( \beta N - N \) and \( p_n \in O_n \). Hence, \( p_n \) is a P-point of \( O_\alpha \) for all \( n = 1, 2, 3, \ldots \). Therefore, as in Lemma 1.2, using continuum hypothesis, for each \( n \in \mathbb{N} \), \( O_n - \{p_n\} \) can be expressed as the union of an increasing collection \( \{A_\alpha\}_{\alpha \in [1, \Omega)} \) of clopen sets relative to \( O_n \) (and hence relative to \( \beta N - N \) also). For each \( n \in \mathbb{N} \), put \( A_\alpha = [\text{cl}_{\beta N - N}(\bigcup_{n=1}^{\infty} A_{\alpha n})] \cap K \). Then, by Corollary 2.9 and Note 2.11 above, \( A_\alpha \) is clopen relative to \( K \) for all \( \alpha \in [1, \Omega) \). Since \( A_{\alpha n} \subset A_{\beta n} \) for \( \alpha < \beta \), \( \alpha, \beta \in [1, \Omega) \), it follows that \( A_\alpha \subset A_\beta \) for all \( \alpha, \beta \in [1, \Omega) \) such that \( \alpha < \beta \).

Now it remains to show that \( \bigcup_{\alpha \in [1, \Omega)} A_\alpha = K - B \). Clearly \( A_\alpha \cap B = \emptyset \) for all \( \alpha \in [1, \Omega) \) and hence \( \bigcup_{\alpha \in [1, \Omega)} A_\alpha \subset K - B \). To get the other inclusion, let \( x_0 \in K - B \). Now, \( K - B \) is open relative to \( K \) and \( K \) is zero-dimensional. Therefore, there exists a clopen set \( V \) relative to \( K \) such that \( x_0 \in V \subset K - B \). Since \( V \subset K \) is clopen in \( K \) and \( \beta N - N \) is zero dimensional, there exists a clopen set \( W \) in \( \beta N - N \) such that \( V = W \cap K \). Put \( W \cap O_n = W_n \) for all \( n = 1, 2, 3, \ldots \). We note that \( p_n \) can belong to \( W_n \) for at most a finite number of \( n \)'s. Therefore, \( \exists k_0 \in \mathbb{N} \) such that \( p_n \notin W_n \forall n > k_0 \). Hence, for each \( n > k_0 \), there exists a countable ordinal \( \alpha_n \) such that \( A_{\alpha_n} \supset W_n \). Let the supremum of \( \alpha_n \) for \( n > k_0 \), be \( \gamma \). Then \( A_\gamma \supset W_n \forall n > k_0 \). Therefore,
\[ \bigcup_{n=0}^{\infty} A_{\gamma n} \supset \bigcup_{n=0}^{\infty} W_n. \]

Hence,

\[ \bigcup_{n=1}^{\infty} A_{\gamma n} \cap K = A_{\gamma} = \bigcup_{n=0}^{\infty} A_{\lambda n} \cap K = \bigcup_{n=0}^{\infty} W_n \cap K = \bigcup_{n=1}^{\infty} (W \cap O_n) \cap K = W \cap M \cap K = V. \]

Also \( x_0 \in V \). Therefore, \( \bigcup_{\alpha \in [1, \Omega)} A_{\alpha} = K - B \).

**Lemma 2.13.** Let \( B \) be as defined in Corollary 2.10 and let \( K \) be as in Lemma 2.12. Then, there exists a collection \( \{X_\alpha\}_{\alpha \in [1, \Omega)} \) of clopen sets relative to \( K \) such that \( X_\alpha \subset X_\beta \forall \alpha, \beta \in [1, \Omega) \) such that \( \alpha < \beta \) and \( \bigcup_{\alpha \in [1, \Omega)} X_\alpha \cap B = B - \{p\} \).

**Proof.** Now, \( p \) is a P-point of \( B \) and hence, using continuum hypothesis, \( B - \{p\} \) can be written as the union of an ascending collection \( \{B_\alpha\}_{\alpha \in [1, \Omega)} \) of clopen sets relative to \( B \). Since, by Corollary 2.10, \( B \cup \{p_1, p_2, \ldots, p_n, \ldots\} = \beta([p_1, \ldots, p_n, \ldots]) \), each \( B_\alpha \) gives a subset \( N_\alpha = \{p_{\alpha_1}, \ldots, p_{\alpha_k}, \ldots\} \) of \( \{p_1, p_2, \ldots, p_n, \ldots\} \) such that

\[ \text{cl}_{\beta N - N}(N_\alpha) \cap B_\alpha = B. \]

Since \( B_\alpha \subset B_\beta \) for \( \alpha < \beta \), we have \( N_\alpha \) is almost contained in \( N_\beta \) for \( \alpha < \beta \). Put \( \text{cl}_{\beta N - N}(\bigcup_{k=1}^{n} O_{n_k}) \cap K = X_\alpha \forall \alpha \in [1, \Omega) \). Then \( X_\alpha \) is clopen in \( K \) \( \forall \alpha \in [1, \Omega) \), \( X_\alpha \subset X_\beta \) for \( \alpha < \beta \), \( X_\alpha \cap B = B_\alpha \forall \alpha \in [1, \Omega) \) and also \( \bigcup_{\alpha \in [1, \Omega)} X_\alpha \cap B = \bigcup_{\alpha \in [1, \Omega)} B_\alpha = B - \{p\} \).

**Lemma 2.14.** Let the collection \( \{O_\alpha\}_{\alpha \in N} \), \( M \) and \( K \) be as in Lemma 2.12. Let \( \beta N - N - M = T \). Let \( \{C_\alpha\} \alpha \in [1, \Omega) \) be an ascending collection of clopen sets relative to \( K \). Then, there exists an ascending collection \( \{I_\alpha\}_{\alpha \in [1, \Omega)} \) of subsets of \( T \cup K \) such that each \( I_\alpha \) is clopen in \( T_\alpha \cup K \), \( I_\alpha \cap K = C_\alpha \forall \alpha \in [1, \Omega) \) and \( \bigcup_{\alpha \in [1, \Omega)} I_\alpha - \bigcup_{\alpha \in [1, \Omega)} C_\alpha = T \).

**Proof.** Using the fact that \( \beta N - N \) is zero-dimensional and is of weight \( c \) and also using the fact that the clopen sets of \( \beta N - N \)}
satisfy the Dubois-Reymond separability condition, we can write $T$ as the union of an ascending collection $\{G_\alpha\}_{\alpha \in [1, \Omega)}$ of clopen sets in $\beta N - N$ such that $G_\alpha \cap M = \emptyset \forall \alpha \in [1, \Omega)$.

Now, $C_i$ is clopen in $K$. Since $\beta N - N$ is zero-dimensional, \exists a clopen set $J_i$ in $\beta N - N$ such that $J_i \cap K = C_i$. Put $[J_i \cap (T \cup K)] \cup G_i = I_i$. Then $I_i$ is clopen in $T \cup K$ and $I_i \cap K = C_i$. Suppose that we have constructed clopen sets $I_1, I_2, \ldots, I_n$ in $T \cup K$ for $n \in N$ such that $I_1 \subset I_2 \subset \cdots \subset I_n$ and $I_i \cap K = C_j$ for $j = 1, 2, \ldots, n$. Then we construct $I_{n+1}$ as follows: Since $C_{n+1}$ is clopen in $K$ and $\beta N - N$ is zero-dimensional, there exists a clopen set $J_{n+1}$ in $\beta N - N$ such that $J_{n+1} \cap K = C_{n+1}$. Put $I_{n+1} = [I_{n+1} \cap (T \cup K)] \cup I_n \cup G_{n+1}$. Then $I_{n+1}$ is clopen in $T \cup K$, $I_{n+1} \supset I_n$ and $I_{n+1} \cap K = C_{n+1}$. Having constructed $I_i \cap C_i \subset C_i$ we now proceed to construct $I_\omega$. First, we claim that $\beta N - N - (\bigcup_{n=1}^{\omega} I_n) \cap (K - C_\omega) = \emptyset$. For, let $x_0 \in K - C_\omega$, which is clopen in $K$. Since $\beta N - N$ is zero-dimensional, there exists a clopen set $H_\omega$ in $\beta N - N$ such that $H_\omega \cap K = K - C_\omega$. Let $H_\omega \cap I_n = H_\omega \cap I_n = 1, 2, 3, \ldots$. Then $H_\omega$ is clopen in $\beta N - N$. We will now prove that $H_\omega$ is also open in $\beta N - N$. Since, $I_n$ is clopen in $T \cup K$ and $\beta N - N$ is zero dimensional, there exists a clopen set $\Gamma_n$ in $\beta N - N$ such that $\Gamma_n \cap (T \cup K) = I_n$. Then $\Gamma_n \cap [(T \cup K) \cap K] = I_n \cap K = C_n$. Now

$$H_{n, \omega} = (H_\omega \cap I_n) = H_\omega \cap [\Gamma_n \cap (T \cup K)] = (H_\omega \cap \Gamma_n \cap T) \cup (H_\omega \cap \Gamma_n \cap K) = (H_\omega \cap \Gamma_n \cap T) \cup [(K - C_\omega) \cap \Gamma_n] = (H_\omega \cap \Gamma_n \cap T) \cup [C_n \cap (K - C_\omega)] = H_\omega \cap \Gamma_n \cap T$$

which is open in $\beta N - N$.

Therefore, $H_{n, \omega}$ is clopen in $\beta N - N$. Also $\beta N - N - O_1, \beta N - N - (O_1 \cup O_2), \ldots$ form a decreasing countable collection of clopen sets in $\beta N - N$ such that $(\beta N - N - \bigcup_{n=1}^{\omega} I_n) \cap \bigcap_{n=1}^{\omega} H_{n, \omega} \forall m, n = 1, 2, 3, \ldots$. Therefore, by Dubois-Reymond separability condition, there exists a clopen set $H$ in $\beta N - N$ such that $H \subset T$ and $H \supset \bigcup_{n=1}^{\omega} H_{n, \omega}$. Therefore, $(\beta N - N - H) \cap H_\omega$ is a clopen set in $\beta N - N$ and $x_0 \in (\beta N - N - H) \cap H_\omega$. Also $[(\beta N - N - H) \cap H_\omega] \cap (\bigcup_{n=1}^{\omega} I_n) = \emptyset$. Therefore $x_0 \notin \text{cl}_{\beta N - N} (\bigcup_{n=1}^{\omega} I_n)$. Hence, $(K - C_\omega) \cap \bigcap_{n=1}^{\omega} I_n = \emptyset$. Now $C_\omega \cup \text{cl}_{\beta N - N} (\bigcup_{n=1}^{\omega} I_n)$ and $K - C_\omega$ are disjoint closed sets in $\beta N - N$ which is normal. Therefore, there exist disjoint open sets $D_1, D_2$ in $\beta N - N$ such that

$$D_1 \supset C_\omega \cup \text{cl}_{\beta N - N} \left( \bigcup_{n=1}^{\omega} I_n \right) \quad \text{and} \quad D_2 \supset K - C_\omega.$$
Now $\beta N - N$ is zero dimensional, $C_\omega \cup \text{cl}_{\beta N - N} \left( \bigcup_{n=1}^{\infty} I_n \right)$ is a compact subset of $\beta N - N$ and $D_1$ is an open set in $\beta N - N$ containing $C_\omega \cup \text{cl}_{\beta N - N} \left( \bigcup_{n=1}^{\infty} I_n \right)$. Hence, there exists a clopen set $J_\omega$ in $\beta N - N$ such that $D_1 \supset J_\omega \supset C_\omega \cup \text{cl}_{\beta N - N} \left( \bigcup_{n=1}^{\infty} I_n \right)$. Now, $J_\omega \cap D_2 = \emptyset$ and hence $(K - C_\omega) \cap J_\omega = \emptyset$. Therefore, $J_\omega \cap K = C_\omega$. Take $J_\omega = [J_\omega \cap (T \cup K)] \cup H_\omega$. Then $J_\omega$ is clopen in $T \cup K$, $J_\omega \supset \bigcup_{n=1}^{\infty} I_n$ and $J_\omega \cap K = C_\omega$. Continuing this process, we get an increasing collection $\{I_\alpha\}_{\alpha \in [1, \omega)}$ of clopen sets in $T \cup K$ such that $I_\alpha \cap K = C_\alpha \forall \alpha \in [1, \omega)$. It can also be seen that $\bigcup_\alpha I_\alpha - \bigcup_\alpha C_\alpha = T$.

**Corollary 2.15.** Let the collection $\{A_\alpha\}_{\alpha \in [1, \omega)}$ be as in Lemma 2.12. Then, there exists a collection $\{S_\alpha\}_{\alpha \in [1, \omega)}$ of clopen sets in $T \cup K$ such that $S_\alpha \subset S_\beta \forall \alpha, \beta \in [1, \omega)$ such that $\alpha < \beta$, $S_\alpha \cap K = A_\alpha \forall \alpha \in [1, \omega)$ and $\bigcup_\alpha S_\alpha - \bigcup_\alpha A_\alpha = T$.

**Corollary 2.16.** Let the collection $\{x_\alpha\}_{\alpha \in [1, \omega)}$ be as in Lemma 2.13. Then, there exists an increasing collection $\{L_\alpha\}_{\alpha \in [1, \omega)}$ of clopen sets in $T \cup K$ such that $L_\alpha \cap K = X_\alpha \forall \alpha \in [1, \omega)$ and $\bigcup_\alpha L_\alpha - \bigcup_\alpha X_\alpha = T$.

**Definition 2.17.** Let $\sigma_1$ and $\sigma_2$ be two partitions of a nonempty set $X$. Then we define $\sigma_1 \cap \sigma_2$ to be the partition of $X$ given by the collection $\{A \cap B | A \in \sigma_1, B \in \sigma_2, A \cap B \neq \emptyset\}$ of nonempty subsets of $X$.

**Lemma 2.18.** Let $X$ be a compact Hausdorff space. Let $\sigma_1, \sigma_2$ be two Hausdorff partitions for $X$. Then $\sigma_1 \cap \sigma_2$ is also a Hausdorff partition for $X$.

**Proof.** Let $X/\sigma_1 = Y_1$ and $X/\sigma_2 = Y_2$. Let $q_1: X \to Y_1$ and $q_2: X \to Y_2$ be the corresponding quotient maps. Define $(q_1, q_2): X \to Y_1 \times Y_2$ by $(q_1, q_2)(x) = (q_1(x), q_2(x)) \forall x \in X$. This is a continuous function form $X$ into $Y_1 \times Y_2$. Now $Y_1 \times Y_2$ is Hausdorff. Consider $(q_1, q_2)$ as a map from $X$ onto $(q_1, q_2)(X)$. Let the partition induced on $X$ by this map be $\sigma$. Then $\sigma = \sigma_1 \cap \sigma_2$. Let $q: X \to X/\sigma$ be the corresponding quotient map. Let $g: X/\sigma \to (q_1, q_2)(X)$ be the natural fill-up map making the following diagram commutative.
Now \( X/\sigma \) is compact, \( (q_1, q_2)(X) \) is Hausdorff and \( g \) is one-to-one, onto and continuous. Hence \( g \) is a homeomorphism. Since \( (q_1, q_2)(X) \) is Hausdorff, it follows that \( X/\sigma \) is Hausdorff. Therefore \( \sigma_1 \cap \sigma_2 \) is a Hausdorff partition for \( X \).

In the above proof, we also note that the quotient space induced by \( \sigma_1 \cap \sigma_2 \) is homeomorphic to the range of the function \( (q_1, q_2) \) in \( Y_1 \times Y_2 \).

**Lemma 2.19.** Let \( T \) and \( K \) be as in Lemma 2.14. Let \( B \) and \( p \) be as in Lemma 2.13. Then, there exists a Hausdorff partition for \( T \cup K \) with \( \{p\} \) as a separate partition class.

**Proof.** Let the collection \( \{S_\alpha\}_{\alpha \in [1, \Omega)} \) be as in Corollary 2.15 and let the collection \( \{L_\alpha\}_{\alpha \in [1, \Omega)} \) be as in Corollary 2.16. Put \( H_1 = S_1 \) and for each \( \alpha \in [2, \Omega) \), \( H_\alpha = S_\alpha - \bigcup_{\xi < \alpha} S_\xi \) and \( H_\Omega = K - \bigcup_\alpha A_\alpha = B \). Also, let \( M_1 = L_1 \); for each \( \alpha \in [2, \Omega) \), \( M_\alpha = L_\alpha - \bigcup_{\xi < \alpha} L_\xi \) and \( M_\Omega = K - \bigcup_{\alpha \in [1, \Omega)} X_\alpha \). Then, the collection \( \{H_\alpha\}_{\alpha \in [1, \Omega)} \) gives a partition \( \pi_1 \) for \( T \cup K \) such that the quotient space \( (T \cup K)/\pi_1 \) is homeomorphic to \([1, \Omega]\). Therefore, \( \pi_1 \) is a Hausdorff partition for \( T \cup K \). Similarly, the collection \( \{M_\alpha\}_{\alpha \in [1, \Omega]} \) gives a Hausdorff partition \( \pi_2 \) for \( T \cup K \). Let \( \pi_1 \cap \pi_2 = \pi_3 \). Then, by Lemma 2.18, \( \pi_3 \) is a Hausdorff partition for \( T \cup K \). Also

\[
H_\Omega \cap M_\Omega = B \cap \left(K - \bigcup_{\alpha} X_\alpha\right) = B - \bigcup_{\alpha} (B \cap X_\alpha)
= B - \bigcup_{\alpha} B_\alpha = \{p\}.
\]

**Lemma 2.20.** Let \( X \) be a topological space. Let \( A_1 \) and \( A_2 \) be closed in \( X \). Let \( A_1 \cup A_2 = X \). Let \( A \subset X \) be such that \( A \cap A_1 \) is open relative to \( A_1 \) and \( A \cap A_2 \) is open relative to \( A_2 \). Then \( A \) is open in \( X \).

**Proof.** This follows from the fact that

\[
A = (O_1 - A_2) \cup (O_2 - A_1) \cup (O_1 \cap O_2).
\]

**Lemma 2.21.** Let \( \pi_3 \) be the partition of \( T \cup K \) as obtained in the proof of Lemma 2.19. Let the collection of sets \( \{A_\alpha\}_{\alpha \in [1, \Omega]} \) be as obtained in the proof of Lemma 2.12. Let \( \{p_1, p_2, \ldots, p_n, \ldots\} \) be as in Corollary 2.10. For each \( k \in N \), let \( D_{\alpha k} = A_{\alpha k} - \bigcup_{\xi < \alpha} A_{\xi k} \). Then the collection of sets \( \{D_{\alpha k}\}_{\alpha \in [1, \Omega]} \) and \( \{p_\alpha\}_{\alpha \in N} \) together with the members of \( \pi_3 \) form a Hausdorff partition \( \pi_4 \) for \( \beta N - N \).
Proof. Clearly \( \pi_4 \) is a partition for \( \beta N - N \). We will now prove that \( (\beta N - N)/\pi_4 \) is Hausdorff. Given any two partition classes \( C_1 \) and \( C_2 \) of \( \beta N - N \) with respect to \( \pi_4 \), we must prove that there exists a clopen set \( Y \) in \( \beta N - N \) containing \( C_1 \), disjoint with \( C_2 \) and saturated under \( \pi_4 \). The cases where either \( C_1 \) or \( C_2 \) is a \( D_{\alpha k} \) or a \( p_\alpha \) are easy to handle and we consider the following cases:

Case 1. Let \( C_1 = H_\alpha \cap M_\beta \) and \( C_2 = H_\alpha \cap M_\gamma \) where \( \alpha, \beta, \gamma \in [1, \Omega] \) and \( \beta \neq \gamma \). Without loss of generality, we can assume that \( \beta < \gamma \). Now, by definition \( X_\beta = \text{cl}_{\beta N - N}(\bigcup_{n=1}^N O_{n \beta}) \cap K \) where \( \text{cl}_{\beta N - N}((p_{n \alpha}^\beta, \cdots, p_{n \beta}^\alpha, \cdots)) \cap B = B_\beta \) (see the proof of Lemma 2.13). Also \( L_\beta \cap K = X_\beta \) where \( L_\beta \) is clopen in \( T \cup K \) (see Corollary 2.16). Now, \( Y_1 = L_\beta \cup \text{cl}_{\beta N - N}(\bigcup_{n=1}^N O_{n \beta}) \) is closed in \( N \). Further, \( Y_1 \cap C_1 \) and \( Y_1 \cap C_2 \) are disjoint and \( Y_1 \) is saturated under \( \pi_4 \).

Case 2. Let \( C_1 = H_\alpha \cap M_\beta \) and \( C_2 = H_\gamma \cap M_\delta \) where \( \alpha, \beta, \gamma, \delta \in [1, \Omega] \) and \( \alpha \neq \gamma \). Without loss of generality, we can assume that \( \alpha < \gamma \). In this case, using Lemma 2.20, we can verify that the set \( Y_1 = \text{cl}_{\beta N - N}(\bigcup_{n=1}^N A_{n \alpha}) \cup S_\alpha \) is clopen in \( \beta N - N \). Further, \( Y_1 \cap C_1 \) and \( Y_1 \cap C_2 \) are disjoint and \( Y_1 \) is saturated under \( \pi_4 \). Therefore, \( \pi_4 \) is a Hausdorff partition for \( \beta N - N \).

Lemma 2.22. Let \( \pi_4 \) be the Hausdorff partition of \( \beta N - N \) as given in Lemma 2.21. Let \( \pi_5 \) be the partition of \( M \) given by \( \pi_5 = \pi_4 \mid \{X \cap M \mid X \in \pi_4\} \). Then \( \pi_5 \) is a Hausdorff partition for \( M \).

Proof. Let \( D_{\alpha k}, p_\alpha, B \) and \( O_\alpha \) be as in above lemmas. Let \( E_\alpha = A_\alpha - \bigcap_{\alpha \leq \alpha' < \alpha} A_{\alpha'} \) for all \( \alpha \in [2, \Omega] \). Then, it is easy to see that the partition \( \pi_5 \) of \( M \) given by the collection \( \{D_{\alpha k}\}_{\alpha \in [1, \Omega]} \) and \( \{E_\alpha\}_{\alpha \in [1, \Omega]} \) and \( B \) is a Hausdorff partition for \( M \). Let \( K_1 = X_\alpha \) and \( K_\alpha = X_\alpha - \bigcup_{\alpha' \leq \alpha < \alpha'} X_{\alpha'} \) for all \( \alpha \in [1, \Omega] \). Also, let \( K_0 = K - \bigcup_{\alpha \in [1, \Omega]} X_{\alpha} \). Then, the partition \( \pi_7 \) of \( M \) given by the collection \( \{O_\alpha\}_{\alpha \in N} \) and \( \{K_\alpha\}_{\alpha \in [1, \Omega]} \) is also a Hausdorff partition for \( M \). Further, \( \pi_5 = \pi_5 \cap \pi_7 \). Hence, by Lemma 2.18, \( \pi_5 \) is a Hausdorff partition for \( M \).

Lemma 2.23. Let \( M, \pi_4 \) and \( \pi_5 \) be as in previous lemmas. Then \( M/\pi_5 \) is homeomorphic to \( (\beta N - N)/\pi_4 \).

Proof. Let \( (\beta N - N)/\pi_4 = Y \) and let \( q_4 : \beta N - N \rightarrow Y \) be the quotient map induced by the partition \( \pi_4 \) of \( \beta N - N \). Then, by Lemma 2.21, \( Y \) is Hausdorff. Now, the map \( q_4 : M \rightarrow Y \) is a continuous function from \( M \) onto \( Y \) where \( M \) is compact and \( Y \) is Hausdorff. Hence, the topology of \( Y \) is the quotient topology of \( M \) induced on
it by the function $q_{JM}$. But $g_4$ induces the partition $\pi_\delta$ on $M$. Therefore, $M/\pi_\delta$ is homeomorphic to $Y = (\beta N - N)/\pi_\delta$.

**Lemma 2.24.** Let all notations be as in previous lemmas. Then $M/\pi_\delta$ is homeomorphic to $\gamma N \times [1, \Omega]$ where $\gamma N$ is the compactification of $N$ constructed by S. P. Franklin and M. Rajagopalan in [1]. (See also remark 1.6a).

**Proof.** Now $\pi_\delta = \pi_\delta \cap \pi_\gamma$ where $\pi_\delta$ and $\pi_\gamma$ are Hausdorff partitions of $M$ as given in the proof of Lemma 2.22. Let $q_\delta : M \to M/\pi_\delta$ and $q_\gamma : M \to M/\pi_\gamma$ be the corresponding quotient maps. Consider the function $(q_\delta, q_\gamma) : M \to M/\pi_\delta \times M/\pi_\gamma$ given by $(q_\delta(x), q_\gamma(x)) \forall x \in M$. Since $\pi_\delta \cap \pi_\gamma = \pi_\delta$, it follows from Lemma 2.18 that $M/\pi_\delta$ is homeomorphic to the range of the function $(q_\delta, q_\gamma)$ from $M$ into $M/\pi_\delta \times M/\pi_\gamma$. But it can be seen that $M/\pi_\delta$ is homeomorphic to $[1, \Omega] \times [1, \omega]$ with its usual product topology and $M/\pi_\gamma$ is homeomorphic to $\gamma N$ and that the range of the map $(q_\delta, q_\gamma)$ is homeomorphic to $[1, \Omega] \times \gamma N$. Hence, $M/\pi_\delta$ is homeomorphic to $[1, \Omega] \times \gamma N$.

**Theorem 2.25.** $N \cup \{p\}$ has a scattered Hausdorff compactification, when $p$ is a $P$-point of order 2 for $\beta N - N$.

**Proof.** Consider the partition $\pi_\gamma$ of $\beta N - N$ given in Lemma 2.21. Let $\pi_\gamma$ be the partition of $\beta N$ whose members are the members of $\pi_\gamma$ and the singletons in $N$. Since, $(\beta N - N)/\pi_\gamma$ is Hausdorff, by Lemma 1.3, it follows that $\beta N/\pi_\gamma$ is Hausdorff. Since $\beta N$ is compact, we have $\beta N/\pi_\gamma$ is compact. Since $(\beta N - N)/\pi_\gamma$ is homeomorphic to $[1, \Omega] \times \gamma N$ which is scattered, we have that $\beta N/\pi_\gamma$ is also scattered. Since $N$ is dense in $\beta N$ and $N \cup \{p\}$ maps homeomorphically onto itself under the quotient map from $\beta N$ onto $\beta N/\pi_\gamma$ (Lemma 1.4), it follows that $N \cup \{p\}$ is dense in $\beta N/\pi_\gamma$. Thus, $\beta N/\pi_\gamma$ is a scattered Hausdorff compactification for $N \cup \{p\}$. Hence the theorem.

**References**


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