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SCATTERED COMPACTIFICATION FOR $N \cup \{p\}$

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In this paper, it is shown that the scattered space $N \cup \{p\}$ admits a scattered Hausdorff compactification for a large class of points p in $\beta N - N$. This gives a partial solution to the following problem raised by Z. Semadeni in 1959: "Is there a scattered Hausdorff compactification for the space $N \cup \{p\}$ where p is any point of $\beta N - N$?" (See "Sur les ensembles clairsemés," Rozprawy Matematyczne, 19 (1959).) The proofs are purely topological and the compactifications are easy to visualize.

In 1970, C. Ryll-Nardzewski and R. Telgarsky [5], using deep results from Boolean Algebras, have proved that $N \cup \{p\}$ has a scattered compactification if p is a P-point of $\beta N - N$. In the first section of this paper, it is shown that the space γN constructed by S. P. Franklin and M. Rajagopalan [1] serves as a scattered compactification for $N \cup \{p\}$ when p is a P-point of $\beta N - N$. In the second section, a scattered Hausdorff compactification for $N \cup \{p\}$ is provided, when p is a P-point of order 2 for $\beta N - N$ (definition follows). In this case, it is also shown that the compactification of $N \cup \{p\}$ is a space Y such that $Y - N$ is a homeomorph of $[1, \Omega] \times \gamma N$.

DEFINITION 1.1. A P-point of $\beta N - N$ is said to be P-point of order 1 for $\beta N - N$. Suppose that for $n \in N$, we have defined a P-point of order n. Then we define a P-point of order $n + 1$ to be a P-point of the derived set of a countable set of P-points each being of order n in $\beta N - N$.

We will now proceed to get a scattered compactification for $N \cup \{p\}$ where p is a P-point of order 1 for $\beta N - N$, by constructing a suitable quotient space of βN which is scattered and Hausdorff and which contains $N \cup \{p\}$ as a dense subspace. The following two lemmas are easy to prove and their proofs are omitted.

LEMMA 1.2. *Let p be a P-point of order 1 for $\beta N - N$. Then using continuum hypothesis $\beta N - N - \{p\}$ can be written as the union of a collection $\{F_\alpha\}_{\alpha \in [1, \Omega)}$ of clopen sets in $\beta N - N$ such that $F_\alpha \subset F_\beta$ for all $\alpha, \beta \in [1, \Omega)$ such that $\alpha < \beta$.*

LEMMA 1.3. *Let π be a partition of $\beta N - N$ such that the quotient space $(\beta N - N)/\pi$ is Hausdorff in its quotient topology. Let $\tilde{\pi}$ be the partition of βN where each member of N is a member of $\tilde{\pi}$ and each member of π is also a member of $\tilde{\pi}$. Then $Y = \beta N/\tilde{\pi}$*

is compact and Hausdorff and the image of N in Y is an open discrete dense subspace of Y.

Further, if $(\beta N - N)/\pi$ is scattered in quotient topology, Y is also scattered in quotient topology.

LEMMA 1.4. *Let $p \in \beta N - N$. Let π be a partition of $\beta N - N$ such that $\{p\} \in \pi$ and $(\beta N - N)/\pi$ is Hausdorff. Let $\tilde{\pi}$ be the partition of βN as described in Lemma 1.3. Let $\tilde{q}\colon \beta N \to \beta N/\tilde{\pi} = Y$ be the canonical map. Then $\tilde{q}$ is a homeomorphism when restricted to $N \cup \{p\}$.*

Proof. Clearly $\tilde{q}|(N \cup \{p\})\colon N \cup \{p\} \to N \cup \{p\}$ is continuous, one-to-one and onto. Also $\tilde{q}\colon \beta N \to \beta N/\tilde{\pi}$ is continuous, βN is compact and by Lemma 1.3, Y is T_2. Therefore $\tilde{q}$ is a closed map and hence upper semi-continuous. Let $O \subset N \cup \{p\}$ be open relative to $N \cup \{p\}$. Then $O = (N \cup \{p\}) \cap U$ where U is open in βN. Let W be the union of all partition classes with respect to $\tilde{\pi}$ within U. Then, by the upper semicontinuity of $\tilde{q}$, W is open in βN. Since W is also saturated under $\tilde{\pi}$, $\tilde{q}(W)$ is open in $\beta N/\tilde{\pi}$. Also $W \cap (N \cup \{p\}) = O$ and hence $\tilde{q}(W) \cap \tilde{q}(N \cup \{p\} = \tilde{q}(O)$. The refore, $\tilde{q}(O)$ is open relative to $\tilde{q}(N \cup \{p\})$. Thus, $\tilde{q}|(N \cup \{p\})$ is an open map. Therefore, $\tilde{q}|(N \cup \{p\})$ is a homeomorphism.

LEMMA 1.5. *Let p be a P-point of $\beta N - N$. Then there exists a partition π for $\beta N - N$ such that* (i) *$\{p\} \in \pi$ and* (ii) *the induced quotient space $X = (\beta N - N)/\pi$ is homeomorphic to* $[1, \Omega]$.

Proof. By Lemma 1.2, $\beta N - N - \{p\}$ can be written as $\bigcup_{\alpha \in [1,\Omega)} F_\alpha$ such that F_α is clopen in $\beta N - N$ for each α and $F_\alpha \subset F_\beta \forall \alpha, \beta \in [1, \Omega)$ such that $\alpha < \beta$. Put $H_1 = F_1$ and for each α such that $1 < \alpha < \Omega$, put $H_\alpha = F_\alpha - \bigcup_{1 \leq \gamma < \alpha} F_\gamma$, and put $H_\Omega = \{p\}$. Then the collection $\{H_\alpha\}_{\alpha \in [1,\Omega]}$ forms a partition π of $\beta N - N$ by closed sets in $\beta N - N$. Let q: $\beta N - N \to (\beta N - N)/\pi$ be the induced quotient map. Let $q(H_\alpha) = b_\alpha$ for all $\alpha \in [1, \Omega]$. Let τ_1 be the usual order topology induced on $\{b_\alpha | 1 \leqq \alpha \leqq \Omega\}$ by the bijection $b_\alpha \to \alpha$ from $\{b_\alpha | 1 \leqq \alpha \leqq \Omega\}$ onto $[1, \Omega]$ and let τ_2 be the quotient topology on $\{b_\alpha | 1 \leqq \alpha \leqq \Omega\}$ induced on it by the partition π of $\beta N - N$. Then the topologies τ_1 and τ_2 on $\{b_\alpha | 1 \leqq \alpha \leqq \Omega\}$ are both compact and Hausdorff and comparable and hence they are homeomorphic.

THEOREM 1.6. *Let p be a P-point of order* 1 *for $\beta N - N$. Then $N \cup \{p\}$ has a scattered compactification.*

Proof. Let π be the partition of $\beta N - N$ obtained as in Lemma

1.4. Then $\{p\} \in \pi$ and the quotient space $(\beta N - N)/\pi = X$ is homeomorphic to $[1, \Omega]$. Hence X is a compact, scattered and Hausdorff space. Let $\tilde{\pi}$ be the partition of βN as in Lemma 1.3. Then, by Lemma 4, $\beta N/\tilde{\pi}$ contains a homeomorphic copy of $N \cup \{p\}$. Since N is dense in βN, $N \cup \{p\}$ is dense in $\beta N/\tilde{\pi}$. Thus, $\beta N/\tilde{\pi}$ is a scattered, Hausdorff compactification for $N \cup \{p\}$.

REMARK 1.6a. The above scattered Hausdorff compactification of $N \cup \{p\}$ is a space X such that the remainder $X - N$ is homeomorphic to $[1, \Omega]$. This compact Hausdorff space X is called γN by by S. P. Franklin and M. Rajagopalan in [1].

2. Scattered Hausdorff compactification for $N \cup \{p\}$ where p is P-point of order 2 in $\beta N - N$:

NOTATIONS. Let $p \in \beta N - N$. Let p be a P-point of order 2 in $\beta N - N$. Then there exists a countable set $\{p_1, p_2, \cdots p_n, \cdots\}$ of distinct P-points in $\beta N - N$ such that P is a P-point of the set

$$B = \mathrm{cl}_{\beta N - N} \{p_1, p_2, p_3, \cdots, \cdots, p_n, \cdots\} - \{p_1, p_2, \cdots p_n, \cdots\} .$$

LEMMA 2.7. *There exists a countable collection $\{O_n\}_{n \in N}$ of clopen sets in $\beta N - N$ such that* (i) $O_n \cap O_m = \emptyset$ *for $n, m \in N$ such that $n \neq m$ and* (ii) $p_n \in O_n \forall n = 1, 2, 3, \cdots$

Proof. Using the zero dimensionality of $\beta N - N$ and the fact that p_1, is a P-point for $\beta N - N$, we can get a clopen set O_1 in $\beta N - N$ containing p_1 and disjoint with $\{p_2, p_3, \cdots, p_n, \cdots\} \cup \{p\}$. Since, p_2 is a P-point of $\beta N - N$, we get a clopen set F_2 in $\beta N - N$ containing p_2 and disjoint with $p_1, p_3, p_4, \cdots, p_n, \cdots, p$. Put $O_2 = F_2 - O_1$. Proceeding like this, by induction, for each $n \in N$, we can get a clopen set O_n in $\beta N - N$ satisfying the conditions (i) and (ii) of the Lemma 2.7.

LEMMA 2.8. *Let O be any σ-compact subset of $\beta N - N$. Then* $\mathrm{cl}_{\beta N - N}{}^{(O)} = \beta O$.

Proof. This follows from the fact that O is a dense subset of the compact set $\mathrm{cl}_{\beta N - N}(O)$ and any continuous function $f: O \to [0, 1]$ admits a continuous extension to βN.

COROLLARY 2.9. *Let the collection $\{O_n\}_{n \in N}$ be as in Lemma* 2.7. *Let* $\mathrm{cl}_{\beta N - N}(\bigcup_{n=1} O_n) = M$. *Then $\bigcup_{n=1} O_n$ is a σ-compact subset of*

$\beta N - N$ and $M = \beta(\bigcup_{n=1} O_n)$.

COROLLARY 2.10. *Let $\{p_1, p_2, \cdots, p_n, \cdots\}$ be a countable collection of P-points of $\beta N - N$. Let $B = \mathrm{cl}_{\beta N-N}\{p_1, p_2, \cdots, p_n, \cdots\} - \{p_1, p_2, \cdots, p_n, \cdots\}$. Then $B \cup \{p_1, p_2, \cdots, p_n, \cdots\} = \beta(\{p_1, \cdots, p_n, \cdots\})$.*

NOTE 2.11. Let X be any Tychonoff space. Let $A \subset X$ be clopen in X. Then $\mathrm{cl}_{\beta X} A$ is clopen in βX.

Proof. The function $f: X \to [0, 1]$ given by

$$\begin{aligned} f(x) &= 0, \text{ for all } x \in A \\ &= 1, \text{ for all } x \in X - A \end{aligned}$$

is continuous on X. Therefore, f admits a continuous extension $\tilde{f}$: $\beta X \to [0, 1]$. Then, it is clear that $\tilde{f}(x) = 0$ for all $x \in \mathrm{cl}_{\beta X} A$ and $\tilde{f}(x) = 1$ for all $x \in \beta X - \mathrm{cl}_{\beta X} A$. Hence, the result follows.

LEMMA 2.12. *Let the collection $\{O_n\}_{n \in N}$ be as in Lemma 2.7. Let B be as in Corollary 2.10. Let $\mathrm{cl}_{\beta N-N}(\bigcup_{n=1} O_n) = M$. Let $M - \bigcup_{n=1} O_n = K$. Then, there exists an increasing collection $\{A_\alpha\}_{\alpha \in [1,\Omega)}$ of clopen sets relative to K such that $\bigcup_{\alpha \in [1,\Omega)} A_\alpha = K - B$.*

Proof. For each $n \in N$, p_n is a P-point of $\beta N - N$ and $p_n \in O_n$. Hence, p_n is a P-point of O_n for all $n = 1, 2, 3, \cdots$. Therefore, as in Lemma 1.2, using continuum hypothesis, for each $n \in N$, $O_n - \{p_n\}$ can be expressed as the union of an increasing collection $\{A_{\alpha n}\}_{\alpha \in [1,\Omega)}$ of clopen sets relative to O_n (and hence relative to $\beta N - N$ also). For each $n \in N$, put $A_\alpha = [\mathrm{cl}_{\beta N-N}(\bigcup_{n=1}^{\infty} A_{\alpha n})] \cap K$. Then, by Corollary 2.9 and Note 2.11 above, A_α is clopen relative to K for all $\alpha \in [1, \Omega)$. Since $A_{\alpha n} \subset A_{\beta n}$ for $\alpha < \beta$, $\alpha, \beta \in [1, \Omega)$, it follows that $A_\alpha \subset A_\beta$ for all $\alpha, \beta \in [1, \Omega)$ such that $\alpha < \beta$.

Now it remains to show that $\bigcup_{\alpha \in [1,\Omega)} A_\alpha = K - B$. Clearly $A_\alpha \cap B = \phi$ for all $\alpha \in [1, \Omega)$ and hence $\bigcup_\alpha A_\alpha \subset K - B$. To get the other inclusion, let $x_0 \in K - B$. Now, $K - B$ is open relative to K and K is zero-dimensional. Therefore, there exists a clopen set V relative to K such that $x_0 \in V \subset K - B$. Since $V \subset K$ is clopen in K and $\beta N - N$ is zero dimensional, there exists a clopen set W in $\beta N - N$ such that $V = W \cap K$. Put $W \cap O_n = W_n$ for all $n = 1, 2, 3, \cdots$ We note that p_n can belong to W_n for at most a finite number of n's. Therefore, $\exists k_0 \in N$ such that $p_n \notin W_n \forall n > k_0$. Hence, for each $n > k_0$, there exists a countable ordinal α_n such that $A_{\alpha_n n} \supset W_n$. Let the supremum of α_n for $n > k_0$, be γ. Then $A_{\gamma n} \supset W_n \forall n > k_0$. Therefore,

$$\bigcup_{n=k_0+1}^{\infty} A_{\gamma n} \supset \bigcup_{n=k_0+1}^{\infty} W_n .$$

Hence,

$$\begin{aligned}\overline{\bigcup_{n=1}^{\infty} A_{\gamma n}} \cap K = A_\gamma &= \overline{\bigcup_{n=k_0+1}^{\infty} A_{\lambda n}} \cap K \\ &= \overline{\bigcup_{n=k_0+1}^{\infty} W_n} \cap K \\ &= \overline{\bigcup_{n=1}^{\infty} W_n} \cap K \\ &= \overline{\bigcup_{n=1}^{\infty} (W \cap O_n)} \cap K \\ &= W \cap M \cap K \\ &= W \cap K \\ &= V .\end{aligned}$$

Also $x_0 \in V$. Therefore, $\bigcup_{\alpha \in [1,\Omega)} A_\alpha = K - B$.

LEMMA 2.13. *Let B be as defined in Corollary 2.10 and let K be as in Lemma 2.12. Then, there exists a collection $\{X_\alpha\}_{\alpha \in [1,\Omega)}$ of clopen sets relative to K such that $X_\alpha \subset X_\beta \forall \alpha, \beta \in [1, \Omega)$ such that $\alpha < \beta$ and $[\bigcup_{\alpha \in [1,\Omega)} X_\alpha] \cap B = B - \{p\}$.*

Proof. Now, p is a P-point of B and hence, using continuum hypothesis, $B - \{p\}$ can be written as the union of an ascending collection $\{B_\alpha\}_{\alpha \in [1,\Omega)}$ of clopen sets relative to B. Since, by Corollary 2.10, $B \cup \{p_1, p_2, \cdots, p_n, \cdots\} = \beta(\{p_1, \cdots, p_n, \cdots\})$, each B_α gives a subset $N_\alpha = \{p_{n_1^\alpha}, \cdots, p_{n_k^\alpha}, \cdots\}$ of $\{p_1, p_2, \cdots, p_n, \cdots\}$ such that

$$\mathrm{cl}_{\beta N - N}(N_\alpha) \cap B_\alpha = B .$$

Since $B_\alpha \subset B_\beta$ for $\alpha < \beta$, we have N_α is almost contained in N_β for $\alpha < \beta$. Put $[\mathrm{cl}_{\beta N - N}(\bigcup_{k=1} O_{n_k^\alpha})] \cap K = X_\alpha \forall \alpha \in [1, \Omega]$. Then X_α is clopen in K $\forall \alpha \in [1, \Omega)$, $X_\alpha \subset X_\beta$ for $\alpha < \beta$, $X_\alpha \cap B = B_\alpha \forall \alpha \in [1, \Omega)$ and also $(\bigcup_\alpha X_\alpha) \cap B = \bigcup_\alpha (X_\alpha \cap B) = \bigcup_\alpha B_\alpha = B - \{p\}$.

LEMMA 2.14. *Let the collection $\{O_n\}_{n \in N}$, M and K be as in Lemma 2.12. Let $\beta N - N - M = T$. Let $\{C_\alpha\}$ $\alpha \in [1, \Omega)$ be an ascending collection of clopen sets relative to K. Then, there exists an ascending collection $\{I_\alpha\}_{\alpha \in [1,\Omega)}$ of subsets of $T \cup K$ such that each I_α is clopen in $T_\alpha \cup K$, $I_\alpha \cap K = C_\alpha \forall \alpha \in [1, \Omega)$ and $\bigcup_\alpha I_\alpha - \bigcup_\alpha C_\alpha = T$.*

Proof. Using the fact that $\beta N - N$ is zero-dimensional and is of weight c and also using the fact that the clopen sets of $\beta N - N$

satisfy the Dubois-Reymond separability condition, we can write T as the union of an ascending collection $\{G_\alpha\}_{\alpha\in[1,\Omega)}$ of clopen sets in $\beta N - N$ such that $G_\alpha \cap M = \phi \forall \alpha \in [1, \Omega)$.

Now, C_1 is clopen in K. Since $\beta N - N$ is zero-dimensional, $\exists$ a clopen set J_1 in $\beta N - N$ such that $J_1 \cap K = C_1$. Put $[J_1 \cap (T \cup K)] \cup G_1 = I_1$. Then I_1 is clopen in $T \cup K$ and $I_1 \cap K = C_1$. Suppose that we have constructed clopen sets $I_1, I_2, \cdots, I_n$ in $T \cup K$ for $n \in N$ such that $I_1 \subset I_2 \subset \cdots \subset I_n$ and $I_j \cap K = C_j$ for $j = 1, 2, \cdots, n$. Then we construct I_{n+1} as follows: Since C_{n+1} is clopen in K and $\beta N - N$ is zero-dimensional, there exists a clopen set J_{n+1} in $\beta N - N$ such that $J_{n+1} \cap K = C_{n+1}$. Put $I_{n+1} = [J_{n+1} \cap (T \cup K)] \cup I_n \cup G_{n+1}$. Then I_{n+1} is clopen in $T \cup K$, $I_{n+1} \supset I_n$ and $I_{n+1} \cap K = C_{n+1}$. Having constructed $I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$ we now proceed to construct I_ω. First, we claim that $\mathrm{cl}_{\beta N - N} (\bigcup_{n=1}^\infty I_n) \cap (K - C_\omega) = \varnothing$. For, let $x_0 \in k - C_\omega$, which is clopen in K. Since $\beta N - N$ is zero-dimensional, there exists a clopen set H_ω in $\beta N - N$ such that $H_\omega \cap K = K - C_\omega$. Let $H_\omega \cap I_n = H_{n\omega} \forall_n = 1, 2, 3, \cdots$. Then $H_{n\omega}$ is closed in $\beta N - N$. We will now prove that $H_{n\omega}$ is also open in $\beta N - N$. Since, I_n is clopen in $T \cup K$ and $\beta N - N$ is zero dimensional, there exists a clopen set Γ_n in $\beta N - N$ such that $\Gamma_n \cap (T \cup K) = I_n$. Then $\Gamma_n \cap [(T \cup K) \cap K] = I_n \cap K = C_n$. Now

$$\begin{aligned} H_{n\omega} = (H_\omega \cap I_n) &= H_\omega \cap [\Gamma_n \cap (T \cup K)] \\ &= H_\omega \cap [(\Gamma_n \cap T) \cup (\Gamma_n \cap K)] \\ &= (H_\omega \cap \Gamma_n \cap T) \cup (H_\omega \cap \Gamma_n \cap K) \\ &= (H_\omega \cap \Gamma_n \cap T) \cup [(K - C_\omega) \cap \Gamma_n] \\ &= (H_\omega \cap \Gamma_n \cap T) \cup [K \cap (K - C_\omega) \cap \Gamma_n] \\ &= (H_\omega \cap \Gamma_n \cap T) \cup [(C_n \cap (K - C_\omega)] \\ &= H_\omega \cap \Gamma_n \cap T \text{ which is open in } \beta N - N. \end{aligned}$$

Therefore, $H_{n\omega}$ is clopen in $\beta N - N$. Also $\beta N - N - O_1$, $\beta N - N - (O_1 \cup O_2)$, $\cdots$ form a decreasing countable collection of clopen sets in $\beta N - N$ such that $(\beta N - N - \bigcup_{i=1}^n 0_i) \supset H_{m\omega} \forall m, n = 1, 2, 3, \cdots$ Therefore, by Dubois-Reymond separability condition, there exists a clopen set H in $\beta N - N$ such that $H \subset T$ and $H \supset \bigcup_{n=1}^\infty H_{n\omega}$. Therefore, $(\beta N - N - H) \cap H_\omega$ is a clopen set in $\beta N - N$ and $x_0 \in (\beta N - N - H) \cap H_\omega$. Also $[(\beta N - N - H) \cap H_\omega] \cap (\bigcup_{n=1}^\infty I_n) = \varnothing$. Therefore $x_0 \notin \mathrm{cl}_{\beta N - N} (\bigcup_{n=1} I_n)$. Hence, $(K - C_\omega) \cap \overline{(\bigcap_{n=1}^\infty I_n)} = \varnothing$. Now $C_\omega \cup \mathrm{cl}_{\beta N - N} (\bigcup_{n=1}^\infty I_n)$ and $K - C_\omega$ are disjoint closed sets in $\beta N - N$ which is normal. Threfore, there exist disjoint open sets D_1, D_2 in $\beta N - N$ such that

$$D_1 \supset C_\omega \cup \mathrm{cl}_{\beta N - N} \left(\bigcup_{n=1}^\infty I_n \right) \quad \text{and} \quad D_2 \supset K - C_\omega \,.$$

Now $\beta N - N$ is zero dimensional, $C_\omega \cup \mathrm{cl}_{\beta N - N}(\bigcup_{n=1}^\infty I_n)$ is a compact subset of $\beta N - N$ and D_1 is an open set in $\beta N - N$ containing $C_\omega \cup \mathrm{cl}_{\beta N - N}(\bigcup_{n=1}^\infty I_n)$. Hence, there exists a clopen set J_ω in $\beta N - N$ such that $D_1 \supset J_\omega \supset C_\omega \cup \mathrm{cl}_{\beta N = N}(\bigcup_{n-1}^\infty I_n)$. Now, $J_\omega \cap D_2 = \varnothing$ and hence $(K - C_\omega) \cap J_\omega = \varnothing$. Therefore, $J_\omega \cap K = C_\omega$. Take $I_\omega = [J_\omega \cap (T \cup K)] \cup H_\omega$. Then I_ω is clopen in $T \cup K$, $I_\omega \supset \bigcup_{n=1}^\infty I_n$ and $I_\omega \cap K = C_\omega$. Continuing this process, we get an increasing collection $\{I_\alpha\}_{\alpha \in [1,\Omega)}$ of clopen sets in $T \cup K$ such that $I_\alpha \cap K = C_\alpha \forall \alpha \in [1, \Omega)$. It can also be seen that $\bigcup_\alpha I_\alpha - \bigcup_\alpha C_\alpha = T$.

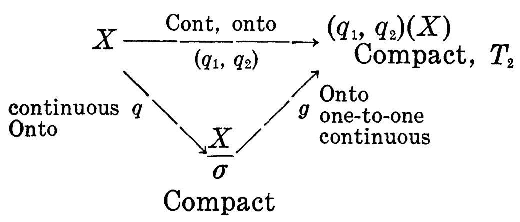
COROLLARY 2.15. *Let the collection* $\{A_\alpha\}_{\alpha \in [1,\Omega)}$ *be as in Lemma* 2.12. *Then, there exists a collection* $\{S_\alpha\}_{\alpha \in [1,\Omega)}$ *of clopen sets in* $T \cup K$ *such that* $S_\alpha \subset S_\beta \forall \alpha, \beta \in [1, \Omega)$ *such that* $\alpha < \beta$, $S_\alpha \cap K = A_\alpha \forall \alpha \in [1, \Omega)$ and $\bigcup_\alpha S_\alpha - \bigcup_\alpha A_\alpha = T$.

COROLLARY 2.16. *Let the collection* $\{x_\alpha\}_{\alpha \in [1,\Omega)}$ *be as in Lemma* 2.13. *Then, there exists an increasing collection* $\{L_\alpha\}_{\alpha \in [1,\Omega)}$ *of clopen sets in* $T \cup K$ *such that* $L_\alpha \bigcap_K = X_\alpha \forall \alpha \in [1, \Omega)$ *and* $\bigcup_\alpha L_\alpha - \bigcup_\alpha X_\alpha = T$.

DEFINITION 2.17. Let σ_1 and σ_2 be two partitions of a nonempty set X. Then we define $\sigma_1 \cap \sigma_2$ to be the partition of X given by the collection $\{A \cap B \mid A \in \sigma_1, B \in \sigma_2, A \cap B \neq \varnothing\}$ of nonempty subsets of X.

LEMMA 2.18. *Let X be a compact Hausdorff space. Let σ_1, σ_2 be two Hausdorff partitions for X. Then $\sigma_1 \cap \sigma_2$ is also a Hausdorff partition for X.*

Proof. Let $X/\sigma_1 = Y_1$ and $X/\sigma_2 = Y_2$. Let $q_1: X \to Y_1$ and $q_2: X \to Y_2$ be the corresponding quotient maps. Define $(q_1, q_2): X \to Y_1 \times Y_2$ by $(q_1, q_2)(x) = (q_1(x), q_2(x)) \forall x \in X$. This is a continuous function form X into $Y_1 \times Y_2$. Now $Y_1 \times Y_2$ is Hausdorff. Consider (q_1, q_2) as a map from X onto $(q_1, q_2)(X)$. Let the partition induced on X by this map be σ. Then $\sigma = \sigma_1 \cap \sigma_2$. Let $q: X \to X/\sigma$ be the corresponding quotient map. Let $g: X/\sigma \to (q_1, q_2)(X)$ be the natural fill-up map making the following diagram commutative.



Now X/σ is compact, $(q_1, q_2)(X)$ is Hausdorff and g is one-to-one, onto and continuous. Hence g is a homeomorphism. Since $(q_1, q_2)(X)$ is Hausdorff, it follows that X/σ is Hausdorff. Therefore $\sigma_1 \cap \sigma_2$ is a Hausdorff partition for X.

In the above proof, we also note that the quotient space induced by $\sigma_1 \cap \sigma_2$ is homeomorphic to the range of the function (q_1, q_2) in $Y_1 \times Y_2$.

LEMMA 2.19. *Let T and K be as in Lemma 2.14. Let B and p be as in Lemma 2.13. Then, there exists a Hausdorff partition for $T \cup K$ with $\{p\}$ as a separate partition class.*

Proof. Let the collection $\{S_\alpha\}_{\alpha \in [1,\Omega)}$ be as in Corollary 2.15 and let the collection $\{L_\alpha\}_{\alpha \in [1,\Omega)}$ be as in Corollary 2.16. Put $H_1 = S_1$ and for each $\alpha \in [2, \Omega)$, $H_\alpha = S_\alpha - \bigcup_{1 \leq \gamma < \alpha} S_\gamma$ and $H_\Omega = K - \bigcup_\alpha A_\alpha = B$. Also, let $M_1 = L_1$; for each $\alpha \in [2, \Omega)$, $M_\alpha = L_\alpha - \bigcup_{1 \leq \gamma < \alpha} L_\gamma$ and $M_\Omega = K - \bigcup_{\alpha \in [1,\Omega)} X_\alpha$. Then, the collection $\{H_\alpha\}_{\alpha \in [1,\Omega)}$ gives a partition π_1 for $T \cup K$ such that the quotient space $(T \cup K)/\pi_1$ is homeomorphic to $[1, \Omega]$. Therefore, π_1 is a Hausdorff partition for $T \cup K$. Similarly, the collection $\{M_\alpha\}_{\alpha \in [1,\Omega]}$ gives a Hausdorff partition π_2 for $T \cup K$. Let $\pi_1 \cap \pi_2 = \pi_3$. Then, by Lemma 2.18, π_3 is a Hausdorff partition for $T \cup K$. Also

$$\begin{aligned} H_\Omega \cap M_\Omega &= B \cap \Big(K - \bigcup_\alpha X_\alpha\Big) \\ &= B - \bigcup_\alpha (B \cap X_\alpha) \\ &= B - \bigcup_\alpha B_\alpha = \{p\} . \end{aligned}$$

LEMMA 2.20. *Let X be a topological space. Let A_1 and A_2 be closed in X. Let $A_1 \cup A_2 = X$. Let $A \subset X$ be such that $A \cap A_1$ is open relative to A_1 and $A \cap A_2$ is open relative to A_2 Then A is open in X.*

Proof. This follows from the fact that

$$A = (O_1 - A_2) \cup (O_2 - A_1) \cup (O_1 \cap O_2) .$$

LEMMA 2.21. *Let π_3 be the partition of $T \cup K$ as obtained in the proof of Lemma 2.19. Let the collection of sets $\{A_{\alpha_k}\}_{\alpha \in [1,\Omega), k \in N}$ be as obtained in the proof of Lemma 2.12.* Let *$\{p_1, p_2, \cdots, p_n, \cdots\}$ be as in Corollary 2.10. For each $k \in N$, let $D_{\alpha_k} = A_{\alpha_k} - \bigcup_{1 \leq \gamma < \alpha} A_{\gamma_k}$. Then the collection of sets $\{D_{\alpha_k}\}_{\alpha \in [1,\Omega), k \in N}$ and $\{p_n\}_{n \in N}$ together with the members of π_3 form a Hausdorff partition π_4 for $\beta N - N$.*

Proof. Clearly π_4 is a partition for $\beta N - N$. We will now prove that $(\beta N - N)/\pi_4$ is Hausdorff. Given any two partition classes C_1 and C_2 of $\beta N - N$ with respect to π_4, we must prove that there exists a clopen set Y_1 in $\beta N - N$ containing C_1, disjoint with C_2 and saturated under π_4. The cases where either C_1 or C_2 is a D_{α_k} or a p_n are easy to handle and we consider the following cases:

Case 1. Let $C_1 = H_\alpha \cap M_\beta$ and $C_2 = H_\alpha \cap M_\gamma$ where $\alpha, \beta, \gamma \in [1, \Omega]$ and $\beta \neq \gamma$. Without loss of generality, we can assume that $\beta < \gamma$. Now, by definition $X_\beta = \mathrm{cl}_{\beta N - N}(\bigcup_{k=1}^\infty O_{n_k^\beta}) \cap K$ where $\mathrm{cl}_{\beta N - N}(\{p_{n_1^\beta}, \cdots, p_{n_k^\beta}, \cdots\}) \cap B = B_\beta$ (see the proof of Lemma 2.13). Also $L_\beta \cap K = X_\beta$ where L_β is clopen in $T \cup K$ (see Corollary 2.16). Now, $Y_1 = L_\beta \cup \mathrm{cl}_{\beta N - N}(\bigcup_{k=1}^\infty O_{n\beta})$ is closed in $\beta N - N$ and using Lemma 2.20, we can see that it is also open in $\beta N - N$. Further $Y_1 \supset C_1$ and $Y_1 \cap C_2 = \varnothing$. Also, Y_1 is saturated under π_4.

Case 2. Let $C_1 = H_\alpha \cap M_\beta$ and $C_2 = H_\gamma \cap M_\delta$ where $\alpha, \beta, \gamma, \delta \in [1, \Omega]$ and $\alpha \neq \gamma$. Without loss of generality, we can assume that $\alpha < \gamma$. In this case, using Lemma 2.20, we can verify that the set $Y_1 = \mathrm{cl}_{\beta N - N}(\bigcup_{n=1}^\infty A_{\alpha_n}) \cup S_\alpha$ is clopen in $\beta N - N$. Further, $Y_1 \supset C_1$ and $Y_1 \cap C_2 = \varnothing$. Also Y_1 is saturated under π_4. Therefore, π_4 is a Hausdorff partition for $\beta N - N$.

LEMMA 2.22. *Let π_4 be the Hausdorff partition of $\beta N - N$ as given in Lemma 2.21. Let π_5 be the partition of M given by $\pi_5 = \pi_4 | M = \{X \cap M | X \in \pi_4\}$. Then π_5 is a Hausdorff partition for M.*

Proof. Let D_{α_k}, p_n, B and O_n be as in above lemmas. Let $E_1 = A_1$ and $E_\alpha = A_\alpha - \bigcap_{1 \leq \gamma < \alpha} A_\gamma$, $\forall \alpha \in [2, \Omega)$. Then, it is easy to see that the partition π_6 of M given by the collection $\{D_{\alpha_k}\} \alpha \in [1, \Omega] k \in N$, $[p_n]_{n \in N}$, $\{E_\alpha\}_{\alpha \in [1, \Omega)}$ and B is a Hausdorff partition for M. Let $K_1 = X_1$ and $K_\alpha = X_\alpha - \bigcup_{1 \leq \gamma < \alpha} X_\gamma \forall \alpha \in [1, \Omega)$. Also, let $K_\Omega = K - \bigcup_{\alpha \in [1, \Omega)} X_\alpha$. Then, the partition π_7 of M given by the collection $\{O_n\}_{n \in N}$ and $\{K_\alpha\}_{\alpha \in [1, \Omega]}$ is also a Hausdorff partition for M. Further $\pi_5 = \pi_6 \cap \pi_7$. Hence, by Lemma 2.18, π_5 is a Hausdorff partition for M.

LEMMA 2.23. *Let M, π_4 and π_5 be as in previous lemmas. Then M/π_5 is homeomorphic to $(\beta N - N)/\pi_4$.*

Proof. Let $(\beta N - N)/\pi_4 = Y$ and let $q_4 \colon \beta N - N \to Y$ be the quotient map induced by the partition π_4 of $\beta N - N$. Then, by Lemma 2.21, Y is Hausdorff. Now, the map $q_4/M \colon M \to Y$ is a continuous function from M onto Y where M is compact and Y is Hausdorff. Hence, the topology of Y is the quotient topology of M induced on it by the function q_4/M. But q_4 induces the partition π_5 on M. Therefore, M/π_5 is homeomorphic to $Y = (\beta N - N)/\pi_4$.

LEMMA 2.24. *Let all notations be as in previous lemmas. Then M/π_5 is homeomorphic to $\gamma N \times [1, \Omega]$ where γN is the compactification of N constructed by S. P. Frankline and M. Rajagopalan in* [1]. (See also remark 1.6a).

Proof. Now $\pi_5 = \pi_6 \cap \pi_7$ where π_6 and π_7 are Hausdorff partitions of M as given in the proof of Lemma 2.22. Let $q_6: M \to M/\pi_6$ and $q_7: M \to M/\pi_7$ be the corresponding quotient maps. Consider the function $(q_6, q_7): M \to M/\pi_6 \times M/\pi_7$ given by $(q_6, q_7)(x) = (q_6(x), q_7(x)) \forall x \in M$. Since $\pi_6 \cap \pi_7 = \pi_5$, it follows from Lemma 2.18 that M/π_5 is homeomorphic to the range of the function (q_6, q_7) from M into $M/\pi_6 \times M/\pi_7$. But it can be seen that M/π_6 is homeomorphic to $[1, \Omega] \times [1, \omega]$ with its usual product topology and M/π_7 is homeomorphic to γN and that the range of the map (q_6, q_7) is homeomorphic to $[1, \Omega] \times \gamma N$. Hence, M/π_5 is homeomorphic to $[1, \Omega] \times \gamma N$.

THEOREM 2.25. *$N \cup \{p\}$ has a scattered Hausdorff compactification, when p is a P-point of order 2 for $\beta N - N$.*

Proof. Consider the partition π_4 of $\beta N - N$ given in Lemma 2.21. Let $\tilde{\pi}_4$ be the partition of βN whose members are the members of π_4 and the singletons in N. Since, $(\beta N - N)/\pi_4$ is Hausdorff, by Lemma 1.3, it follows that $\beta N/\tilde{\pi}_4$ is Hausdorff. Since βN is compact, we have $\beta N/\tilde{\pi}_4$ is compact. Since $(\beta N - N)/\pi_4$ is homeomorphic to $[1, \Omega] \times \gamma N$ which is scattered, we have that $\beta N/\tilde{\pi}_4$ is also scaterted. Since N is dense in βN and $N \cup \{p\}$ maps homeomorphically onto itself under the quotient map from βN onto $\beta N/\tilde{\pi}$ (Lemma 1.4), it follows that $N \cup \{p\}$ is dense in $\beta N/\tilde{\pi}_4$ Thus, $\beta N/\tilde{\pi}_4$ is a scattered Hausdorff compactification for $N \cup \{p\}$. Hence the theorem.

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