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**A RATIO LIMIT THEOREM FOR A STRONGLY SUBADDITIVE  
SET FUNCTION IN A LOCALLY COMPACT AMENABLE  
GROUP**

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## A RATIO LIMIT THEOREM FOR A STRONGLY SUBADDITIVE SET FUNCTION IN A LOCALLY COMPACT AMENABLE GROUP

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**It is the purpose of this paper to prove that the following property holds: Given a locally compact, amenable, unimodular group  $G$ , if  $S$  is a strongly subadditive, nonpositive, right invariant set function defined on the class  $\mathcal{K}$  of relatively compact Borel subsets of  $G$ , and if  $\{A_\alpha\}$  is a net in  $\mathcal{K}$  satisfying an appropriate growth condition, then**

$$\lim_\alpha \lambda(A_\alpha)^{-1} S(A_\alpha)$$

**exists independently of  $\{A_\alpha\}$ , where  $\lambda$  is Haar measure on  $G$ .**

Let  $G$  be a locally compact group. Let  $\lambda$  be right Haar outer measure defined on the subsets of  $G$ . Let  $\mathcal{K}$  be the class of relatively compact Borel subsets of  $G$ . If  $A$  is a subset of  $G$  and  $K \in \mathcal{K}$ , let  $[A]_K = \{g \in A: Kg \subset A\} = \bigcap_{k \in K \cup \{1\}} k^{-1}A$ , where  $1$  is the identity of  $G$ . In this paper, we call a locally compact, amenable, unimodular group a *leau* group.

DEFINITION 1. Following [1], we define a net  $\{A_\alpha\}$  in  $\mathcal{K}$  to be a regular net in the locally compact group  $G$  if

$$(D.1.1) \quad \lambda(A_\alpha) > 0 \text{ for each } \alpha;$$

$$(D.1.2) \quad \lim_\alpha \lambda(KA_\alpha)^{-1} \lambda([A_\alpha]_K) = 1, K \in \mathcal{K}, K \neq \phi.$$

(Even though  $K A_\alpha$  and  $[A_\alpha]_K$  may not be Borel measurable, (D.1.2) makes sense because we required  $\lambda$  to be right Haar outer measure, which is defined for *all* subsets of  $G$ .)

LEMMA 1. *A locally compact group  $G$  possesses a regular net if and only if  $G$  is a leau group.*

*Proof.* A locally compact group  $G$  is amenable if and only if for any  $\varepsilon > 0$ , and for any nonempty compact subset  $K$  of  $G$ , there exists a compact subset  $U$  of  $G$ , of positive measure, such that  $\lambda^*(U)^{-1} \lambda^*(KU) < 1 + \varepsilon$ , where  $\lambda^*$  is left Haar measure. (See [2].) We call this necessary and sufficient condition for amenability of  $G$  condition (A).

Now suppose  $G$  possesses a regular net  $\{A_\alpha\}$ . Then (D.1.2) implies that

$$(1) \quad \lim_{\alpha} \lambda(KA_{\alpha})^{-1} \lambda(A_{\alpha}) = 1, K \in \mathcal{H}, K \neq \phi.$$

Taking  $K = \{g\}$ , where  $g$  is any element of  $G$ , we see that  $\Delta(g) = 1$ . Thus  $G$  is unimodular. It then follows that (1) implies condition (A), and thus  $G$  is also amenable.

Conversely, suppose now  $G$  is *lcav*. Given  $\varepsilon > 0$  and a nonempty compact subset  $K$  of  $G$ , we may find by condition (A) a compact set  $U = U_{(K, \varepsilon)}$ , of positive measure, such that  $\lambda(U)^{-1} \lambda(K^2 U) < 1 + \varepsilon$ . We direct the set  $W = \{(K, \varepsilon) : K \text{ a nonempty compact set in } G, \varepsilon > 0\}$  as follows:  $(K_1, \varepsilon_1) > (K_2, \varepsilon_2)$  if and only if  $K_1 \supset K_2$  and  $\varepsilon_1 < \varepsilon_2$ . Then  $\{V_{(K, \varepsilon)} : (K, \varepsilon) \in W\}$  is a regular net of compact subsets of  $G$ , where  $V_{(K, \varepsilon)} = KU_{(K, \varepsilon)}$ .

**DEFINITION 2.** Let  $G$  be a regular group. Throughout this paper, we consider a set function  $S: \mathcal{H} \rightarrow \mathbb{R}$ , the set of real numbers, which satisfies the following properties:

$$(D. 2.1) \quad S(\phi) = 0.$$

$$(D. 2.2) \quad S \text{ is strongly subadditive; that is, } S(A \cap B) + S(A \cup B) \leq S(A) + S(B), A, B \in \mathcal{H}.$$

$$(D. 2.3) \quad S(A) \leq 0, A \in \mathcal{H}.$$

$$(D. 2.4) \quad S(Ag) = S(A), A \in \mathcal{H}, g \in G.$$

The main result we will prove in this note is the following theorem.

**THEOREM 1.** *Let  $G$  be a lcav group. Let  $S: \mathcal{H} \rightarrow \mathbb{R}$  satisfy Definition 2. Then there is an extended real number  $r^*$  such that  $\lim_{\alpha} \lambda(A_{\alpha})^{-1} S(A_{\alpha}) = r^*$  for every regular net  $\{A_{\alpha}\}$  in  $\mathcal{H}$ .*

A special case of this theorem, for vector groups, was proved in [7] in order to define entropy in statistical mechanics for classical continuous systems. The theorem can be used to define the entropy of a measurable partition relative to a discrete amenable group of measure-preserving transformations on a probability space, thereby enabling one to generalize the concept of the Kolmogorov-Sinai invariant [5].

One may construct a set function  $S$  satisfying Definition 2 as follows: Let  $(\Omega, \mathcal{M})$  be a measurable space. For each element  $g$  of the regular group  $G$ , let  $T^g$  be a measurable transformation from  $\Omega$  to  $\Omega$ . We suppose that  $T^{g_1} \cdot T^{g_2} = T^{g_1 g_2}$ ,  $g_1, g_2 \in G$ . Let  $\mathcal{F}$  be a fixed sub-sigmafield of  $\mathcal{M}$ . If  $E$  is a nonempty subset of  $G$ , let  $\mathcal{F}_E$  be the smallest sub-sigmafield of  $\mathcal{M}$  containing  $\bigcup_{g \in E} (T^g)^{-1} \mathcal{F}$ . Define  $\mathcal{F}_{\phi} = \{\phi, \Omega\}$ . Let  $P, Q$  be probability measures on  $\mathcal{M}$ , such that  $P$  is stationary with respect to  $\{T^g : g \in G\}$  and the fields  $\{(T^g)^{-1} \mathcal{F} : g \in G\}$

are independent with respect to  $Q$ . For each  $E \in \mathcal{H}$ , let  $S(E)$  be the negative of the entropy of  $P$  with respect to  $Q$  over  $\mathcal{F}_E$ , which we assume finite. The function  $S: \mathcal{H} \rightarrow R$  defined in this way can be shown to satisfy Definition 2 in a manner analogous to that employed in [7] for vector groups.

**LEMMA 2.** *If Theorem holds for all sigma-compact lcgu groups it holds for all lcgu groups.*

*Proof.* Let  $d$  be a complete metric on  $R^*$ , the set of extended real numbers, which induces the usual topology on  $R^*$ . Let  $\{A_\alpha\}$  be a regular net for a non-sigma-compact lcgu group  $G$ . Suppose  $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$  does not exist. Then for some  $\varepsilon > 0$ , we may find a sequence  $\{F_n\}_0^\infty$  of elements of  $\{A_\alpha\}$  and a sequence  $\{E_n\}_0^\infty$  in  $\mathcal{H}$  such that

- (a)  $F_0$  is any  $A_\alpha$  and  $E_0$  is an open symmetric neighborhood of the identity.
- (b)  $d(\lambda(F_n)^{-1}S(F_n), \lambda(F_{n-1})^{-1}S(F_{n-1})) > \varepsilon, n \geq 1$ .
- (c)  $\lambda(E_{n-1}F_n)^{-1}\lambda([F_n]_{E_{n-1}}) > 1 - n^{-1}, n \geq 1$ .
- (d)  $E_n$  is an open symmetric set containing the closure of  $[E_{n-1} \cup F_n]^2, n \geq 1$ .

Let  $G' = \bigcup_n E_n$ . It is easily seen that  $G'$  is an open, sigma-compact subgroup of  $G$ .

If we restrict  $\lambda$  to  $G'$ , we get right Haar measure on  $G'$ . Thus  $\{F_n\}$  is a regular sequence for  $G'$ , and  $G'$  is a lcgu group. Assuming Theorem 1 holds for sigma-compact lcgu groups,  $\lim_n \lambda(F_n)^{-1}S(F_n)$  would have to exist, a contradiction of b). Thus  $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$  exists. Let  $\{B_\beta\}$  be another regular net in  $G$ . Let  $s_1 = \lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$ ,  $s_2 = \lim \lambda(B_\beta)^{-1}S(B_\beta)$ . We show that  $s_1 = s_2$ . Define sequences  $\{C_n\}_1^\infty, \{D_n\}_1^\infty, \{E_n\}_0^\infty$  in  $\mathcal{H}$  such that

- (a)  $E_0$  is an open symmetric neighborhood of the identity,  $\{C_n\} \subset \{A_\alpha\}, \{D_n\} \subset \{B_\beta\}$ .
- (b)  $d(\lambda(C_n)^{-1}S(C_n), s_1) < n^{-1}, d(\lambda(D_n)^{-1}S(D_n), s_2) < n^{-1}, n \geq 1$ .
- (c)  $\lambda(E_{n-1}C_n)^{-1}\lambda([C_n]_{E_{n-1}}) \geq 1 - n^{-1}, \lambda(E_{n-1}D_n)^{-1}\lambda([D_n]_{E_{n-1}}) \geq 1 - n^{-1}, n \geq 1$ .
- (d)  $E_n$  is open, symmetric and contains the closure of  $[E_{n-1} \cup C_n \cup D_n]^2, n \geq 1$ .

It follows that  $G' = \bigcup_n E_n$  is an open, sigma-compact, lcgu subgroup of  $G$  and that  $\{C_n\}$  and  $\{D_n\}$  are regular sequences for  $G'$ . Therefore,  $\lim_n \lambda(C_n)^{-1}S(C_n) = \lim_n \lambda(D_n)^{-1}S(D_n)$ , and so  $s_1 = s_2$  by b).

**DEFINITION 3.** If  $G$  is a locally compact group, if  $S: \mathcal{H} \rightarrow R$  satisfies Definition 2, and if  $A, B \in \mathcal{H}$  with  $A \cap B = \phi$ , define  $S(A|B) = S(A \cup B) - S(B)$ .

LEMMA 3. Let  $G$  be a locally compact group, and let  $S: \mathcal{K} \rightarrow \mathcal{R}$  satisfy Definition 2. Then  $S$  obeys the following laws:

(L. 3.1)  $S(A) \leq S(B)$  if  $A \supset B$ ,  $A, B \in \mathcal{K}$ .

(L. 3.2) If  $A_1, A_2, \dots, A_k$  are elements of  $\mathcal{K}$  which partition  $A$ , then  $S(A) = \sum_{i=1}^k S(A_i | \bigcup_{j=1}^{i-1} A_j)$ , where an empty union is the null set.

(L. 3.3)  $S(E|D_1) \leq S(E|D_2)$ ,  $D_1 \supset D_2$ ,  $E \cap D_1 = \phi$ ,  $E, D_1, D_2 \in \mathcal{K}$ .

(L. 3.4)  $S(E|D) \leq S(E) \leq 0$ ,  $E, D \in \mathcal{K}$ ,  $E \cap D = \phi$ .

*Proof.* (L. 3.2) follows easily from Definition 2. The strong subadditivity of  $S$  is equivalent to saying  $S(A \setminus B|B) \leq S(A \setminus B|A \cap B)$ ,  $A, B \in \mathcal{K}$ . Letting  $A = E \cup D_2$  and  $B = D_1$ , where  $E, D_1, D_2$  satisfy  $D_1 \cap E = \phi$  and  $D_1 \supset D_2$ , we have  $A \cap B = D_2$  and  $A \setminus B = E$ , whence (L. 3.3) follows. In (L. 3.3) if we take  $D_2 = \phi$ , (L. 3.4) follows because  $S(E|\phi) = S(E)$ . If  $A \supset B$ , where  $A, B \in \mathcal{K}$ , then  $S(A) = S(B) + S(A \setminus B|B) \leq S(B)$ , and thus (L. 3.1) follows.

DEFINITION 4. We define a locally compact group  $G$  to be a  $P$ -group if there exists for some positive integer  $n$  a triple  $(K, \{G_i\}_1^n, \{H_i\}_1^n)$  such that:

(D. 4.1)  $K$  is a nonempty relatively compact Borel set in  $G$ .

(D. 4.2)  $\{G_i\}_1^n$  and  $\{H_i\}_1^n$  are sequences of closed subgroups of  $G$  satisfying  $G_1 \subset H_1 \subset G_2 \subset H_2 \subset \dots \subset G_n \subset H_n$ .

(D. 4.3) The index of  $G_i$  in  $H_i$  is countable,  $i = 1, 2, \dots, n$ .

(D. 4.4) If  $E_i$  is any set of coset representatives of the right cosets  $\{G_i h: h \in H_i\}$  of  $G_i$  in  $H_i$ ,  $i = 1, 2, \dots, n$ , then each  $g \in G$  has a unique factorization in the form  $g = ke_1 e_2 \dots e_n$ ,  $k \in K$ ,  $e_i \in E_i$ ,  $i = 1, 2, \dots, n$ . Also,  $K(\prod_{j=1}^{i-1} E_j)G_i = K(\prod_{j=1}^{i-1} E_j)$ ,  $i = 1, 2, \dots, n$ , where an empty product is the identity in  $G$ .

In order to prove Theorem 1 for sigma-compact *lcac* groups, we need to show that such groups are  $P$ -groups. This we now do, by means of several lemmas. To see how the following lemma may be proved, see [2], page 379.

LEMMA 4. Let  $G'$  be a closed normal subgroup of a connected Lie group  $G$ . Let  $\phi: G \rightarrow G/G'$  be the canonical homomorphism. Then there exists a map  $\tau: G/G' \rightarrow G$  such that

(L. 4.1)  $\tau$  is a cross-section; that is,  $\phi \cdot \tau$  is the identity map on  $G/G'$ .

(L. 4.2) If  $U$  is a relatively compact subset of  $G/G'$ , then  $\tau(U)$  is a relatively compact subset of  $G$ .

(L. 4.3) If  $U$  is a Borel set in  $G/G'$  and  $V$  is a Borel set in  $G'$ , then  $\tau(U)V$  is a Borel set in  $G$ .

LEMMA 5. *Let  $G$  be a connected Lie group and  $G'$  a closed normal subgroup of  $G$  such that  $G/G'$  is either a vector group or compact. Then if  $G'$  is a  $P$ -group, so is  $G$ .*

*Proof.* Let  $\tau: G/G' \rightarrow G$  be the cross-section map provided by Lemma 4. Since  $G/G'$  is a vector group or compact, it is easy to see that there exists a closed countable subgroup  $G''$  of  $G/G'$  and a relatively compact Borel set  $K'$  in  $G/G'$  such that  $\{K'g: g \in G''\}$  partitions  $G/G'$ . If  $G'$  is a  $P$ -group with respect to the triple  $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ , then  $G$  is a  $P$ -group with respect to the triple  $(\tau(K')K, \{G_i\}_1^{n+1}, \{H_i\}_1^{n+1})$ , where  $G_{n+1} = G'$  and  $H_{n+1} = \phi^{-1}(G'')$ .

LEMMA 6. *If  $G$  is a sigma-compact locally compact group and  $G'$  is an open subgroup of  $G$  which is a  $P$ -group, then  $G$  is a  $P$ -group.*

*Proof.* Let  $G'$  be a  $P$ -group with respect to the triple  $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ . Then  $G$  is a  $P$ -group with respect to the triple  $(K, \{G_i\}_1^{n+1}, \{H_i\}_1^{n+1})$ , where  $G_{n+1} = G'$ ,  $H_{n+1} = G$ .

LEMMA 7. *If  $G$  is a locally compact group and  $G'$  is a compact normal subgroup of  $G$  such that  $G/G'$  is a  $P$ -group, then  $G$  is a  $P$ -group.*

*Proof.* Suppose  $G/G'$  is a  $P$ -group with respect to the triple  $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ . Let  $\phi: G \rightarrow G/G'$  be the canonical homomorphism. Then  $G$  is a  $P$ -group with respect to the triple  $(\phi^{-1}(K), \{\phi^{-1}(G_i)\}_1^n, \{\phi^{-1}(H_i)\}_1^n)$ .

THEOREM 2. *Every sigma-compact locally compact amenable group is a  $P$ -group.*

*Proof.* Every connected amenable Lie group  $G$  possesses a series of closed subgroups  $G_0 \subset G_1 \subset G_2 \subset \dots \subset G_n = G$ , where  $G_0$  is the identity,  $G_i$  is normal in  $G_{i+1}$ , and  $G_{i+1}/G_i$  is either a vector group or compact,  $i = 0, 1, \dots, n - 1$ . (See [3], Theorem 3.3.2, and [4], Lemma 3.3.) Now  $G_0$  is clearly a  $P$ -group, so by using Lemma 5 repeatedly we conclude every connected amenable Lie group is a  $P$ -group. Applying Lemma 6, every sigma-compact amenable Lie group is a  $P$ -group. For every locally compact group  $G$  there exists an open subgroup  $G'$  of  $G$  and a compact normal subgroup  $K$  of  $G'$  such that  $G'/K$  is a Lie group. (See [6], page 153.) Assuming  $G$  in addition is sigma-compact and amenable, so is  $G'/K$ . Thus  $G'/K$  is a  $P$ -group and then so is  $G'$  by Lemma 7. Then  $G$  is a  $P$ -group by Lemma 6.

We fix  $G$  to be a sigma-compact *lcu* group for the rest of the paper. We need to show Theorem 1 holds for  $G$ . This we accomplish by means of some lemmas and Theorem 3.

Let  $(K, \{G_i\}_1^n, \{H_i\}_1^n)$  be a triple with respect to which  $G$  is a  $P$ -group. Let  $E_i$  be a set of coset representatives of the right cosets of  $G_i$  in  $H_i$  such that  $1 \in E_i$ ,  $i = 1, 2, \dots, n$ , where 1 is the identity of  $G$ . For each  $i$ , let  $\bar{H}_i$  be the collection of right cosets of  $G_i$  in  $H_i$ . (Since  $G_i$  is not necessarily normal in  $H_i$ ,  $\bar{H}_i$  need not be a group.) For each  $i$ , let  $\phi_i: H_i \rightarrow \bar{H}_i$  be the map such that  $\phi_i(h) = G_i h$ ,  $h \in H_i$ ; let  $\tau_i: \bar{H}_i \rightarrow E_i$  be the unique map such that  $\phi_i \cdot \tau_i$  is the identity map on  $\bar{H}_i$ . By a total order  $<$  on a set  $W$ , we mean a transitive relation such that for  $x, y \in W$  exactly one of the following hold:  $x < y$ ,  $x = y$ , or  $y < x$ . For each  $i$ , let  $<^i$  be a total order on  $E_i$ ; if  $h \in H_i$ , let  $<^i_h$  be the total order on  $E_i$  such that if  $e, e' \in E_i$  then  $e <^i_h e'$  if and only if  $\tau_i \cdot \phi_i(eh) <^i \tau_i \cdot \phi_i(e'h)$ . If  $h \in H_i$ , let  $P_h^i(e) = \{e' \in E_i: e' <^i_h e\}$ . Let  $E = E_1 E_2 \dots E_n$ . Let  $H$  be the locally compact amenable group  $H = H_1 \times H_2 \times \dots \times H_n$ . If  $h = (h_1, h_2, \dots, h_n) \in H$ , let  $<_h$  be the lexicographical order on  $E$  defined as follows: if  $e = e_1 e_2 \dots e_n$  and  $e' = e'_1 e'_2 \dots e'_n$  are elements of  $E$ , where  $e_i, e'_i \in E_i$ , then  $e <_h e'$  if and only if there exists an integer  $k$ ,  $n \geq k \geq 1$ , such that  $e_k <^k_h e'_k$  and for  $n \geq j > k$ ,  $e_j = e'_j$ . If  $h \in H$ ,  $e \in E$ , let  $P_h(e) = \{e' \in E: e' <_h e\}$ . If  $A \in \mathcal{K}$ ,  $e \in E$ , let  $\phi_A^e: H \rightarrow R$  be the function such that  $\phi_A^e(h) = S(Ke | KP_h(e) \cap Ae) = S(K | KP_h(e)e^{-1} \cap A)$ ,  $h \in H$ .

LEMMA 8. *If  $A \in \mathcal{K}$  and  $e \in E$ , then  $\phi_A^e \in L^\infty(H)$ , the space of bounded Borel-measurable real-valued functions with domain  $H$ .*

*Proof.* Fix  $A \in \mathcal{K}$ ,  $e \in E$ . By (L. 3.4),  $\phi_A^e \leq 0$ . To achieve a lower bound, let  $E' = \{e' \in E: Ke' \cap Ae \neq \emptyset\}$ . Since  $KE' \subset KK^{-1}Ae$ ,  $E'$  is finite. Let  $F = \{e\} \cup E'$ . By (L. 3.2),  $S(KF) = \sum_{f \in F} S(Kf | KP_h(f) \cap KF)$ . By (L. 3.3) and (L. 3.4),  $S(KF) \leq S(Ke | KP_h(e) \cap KF) \leq S(Ke | KF_h(e) \cap Ae) = \phi_A^e(h)$ , where the fact that  $KF \supset Ae$  was used. Thus  $\phi_A^e$  is a bounded function. We now show that it is a Borel measurable function. It is easily seen that  $\phi_A^e$  is a simple function with possible values  $S(Ke | KF' \cap Ae)$ ,  $F' \subset F$ . If  $F' \subset F$ , then  $\phi_A^e = S(Ke | KF' \cap Ae)$  on the set  $\{h \in H: P_h(e) \cap F = F'\}$ , which is equal to the intersection of the sets  $\bigcap_{f \in F'} \{h: f \in P_h(e)\}$  and  $\bigcap_{f \in F \setminus F'} \{h: f \notin P_h(e)\}$ . Thus  $\phi_A^e$  is Borel measurable if for each  $f \in F$ ,  $\{h \in H: f \in P_h(e)\}$  is a Borel set. If  $f = e$ , this set is empty. Thus, fix  $f \in F$ ,  $f \neq e$ . Let  $f = f_1 f_2 \dots f_n$ , and  $e = e_1 e_2 \dots e_n$ , where  $e_i, f_i \in E_i$  for each  $i$ . Let  $j = \max\{i: f_i \neq e_i\}$ . Then  $\{h \in H: f \in P_h(e)\} = \{h \in H: f_j \in P_{h_j}^j(e_j)\}$ , where  $h_j \in H_j$  is the  $j^{\text{th}}$  component of  $h \in H$ . This is a Borel set in  $H$  if  $\{h \in H_j: f_j \in P_h^j(e_j)\}$  is a Borel set in  $H_j$ . Now this latter set is the union of the sets  $\{h \in H_j: G_j f_j h = G_j g_1, G_j e_j h = G_j g_2\}$  where  $(g_1, g_2)$  ranges over all ordered

pairs such that  $g_1, g_2 \in E_j$  and  $g_1 \triangleleft^j g_2$ . Since the union is a countable union of closed subsets of  $H_j$ , Borel measurability follows.

LEMMA 9. *Let  $\mu$  be a left invariant mean on  $L^\infty(H)$ . Then  $\mu(\phi_A^e) = \mu(\phi_A^1)$ ,  $A \in \mathcal{K}$ ,  $e \in E$ .*

*Proof.* Fix  $A \in \mathcal{K}$ ,  $e \in E$ . We observe that

$$\begin{aligned} KP_h(e)e^{-1} &= \left[ \bigcup_{i=1}^n K \left( \prod_{j=1}^{i-1} E_j \right) P_{h_i}^i(e_i) e_{i+1} \cdots e_n \right] e^{-1} \\ &= \bigcup_{i=1}^n \left[ K \left( \prod_{j=1}^{i-1} E_j \right) G_i P_{h_i}^i(e_i) e_i^{-1} \cdots e_2^{-1} e_1^{-1} \right], \end{aligned}$$

by (D. 4.4), where  $h = (h_1, h_2, \dots, h_n) \in H$  and  $e = e_1 e_2 \cdots e_n$ . It is routine to show that  $G_i P_{h_i}^i(e_i) = G_i P_1^i(\tau_i \cdot \phi_i(e_i h_i)) h_i^{-1}$ . Also, since  $e_j \in G_i$  for  $j < i$ , we have  $\phi_i(e_i h_i) = \phi_i(e_1 e_2 \cdots e_i h_i)$ . Thus,  $KP_h(e)e^{-1} = \bigcup_{i=1}^n [K(\prod_{j=1}^{i-1} E_j) P_1^i(\tau_i \cdot \phi_i(e_1 \cdots e_i h_i))(e_1 \cdots e_i h_i)^{-1}] = KP_{mh}(1)$ , where  $m = (m_1, m_2, \dots, m_n) \in H$  satisfies  $m_i = \prod_{j=1}^i e_j$ ,  $i = 1, 2, \dots, n$ . Thus  $\phi_A^e(h) = \phi_A^1(mh)$ ,  $h \in H$ , from which the lemma follows.

THEOREM 3. *Let  $\{A_\alpha\}$  be a regular net in the sigma-compact leau group  $G$ . Then  $\lim_\alpha \lambda(A_\alpha)^{-1} S(A_\alpha) = \inf_{B \in \mathcal{X}} \lambda(K)^{-1} \mu(\phi_B^1)$ .*

*Proof.* Fix the regular net  $\{A_\alpha\}$ . Now  $KE'_\alpha \subset A_\alpha \subset KE_\alpha$ , where  $E_\alpha = \{e \in E: Ke \cap A_\alpha \neq \emptyset\}$ ,  $E'_\alpha = \{e \in E: Ke \subset A_\alpha\}$ . Thus by (L. 3.1),  $S(KE_\alpha) \leq S(A_\alpha) \leq S(KE'_\alpha)$ . We show that  $\limsup_\alpha \lambda(A_\alpha)^{-1} S(KE'_\alpha) \leq L$  and  $\liminf_\alpha \lambda(A_\alpha)^{-1} S(KE_\alpha) \geq L$ , where  $L = \inf_{B \in \mathcal{X}} \lambda(K)^{-1} \mu(\phi_B^1)$ . Now  $S(KE_\alpha) = \sum_{e \in E_\alpha} S(Ke | KP_h(e) \cap KE_\alpha) \geq \sum_{e \in E_\alpha} \phi_{B_\alpha}^e$ , where  $B_\alpha = \bigcup_{e \in E_\alpha} KE_\alpha e^{-1}$ . Applying  $\mu$  to the inequality and using Lemma 9,  $S(KE_\alpha) \geq |E_\alpha| \mu(\phi_{B_\alpha}^1) \geq |E_\alpha| \lambda(K) L = \lambda(KE_\alpha) L$ , where  $|E_\alpha|$  denotes the cardinality of  $E_\alpha$ . Since  $KE_\alpha \subset KK^{-1}A_\alpha$  we have  $\lim_\alpha \lambda(A_\alpha)^{-1} \lambda(KE_\alpha) = 1$ , by the regularity of  $\{A_\alpha\}$ . Thus  $\liminf_\alpha \lambda(A_\alpha)^{-1} S(KE_\alpha) \geq L$ . Fix  $B \in \mathcal{X}$ . We suppose that  $B \supset K$ . Now  $S(KE'_\alpha) = \sum_{e \in E'_\alpha} S(Ke | KP_h(e) \cap KE'_\alpha) \leq \sum_{e \in F_\alpha} \phi_B^e$  where  $F_\alpha = \{e \in E'_\alpha: KE'_\alpha e^{-1} \supset B\}$ . Applying  $\mu$ ,  $S(KE'_\alpha) \leq \lambda(KF_\alpha) \lambda(K)^{-1} \mu(\phi_B^1)$ . We could conclude that  $\limsup_\alpha \lambda(A_\alpha)^{-1} S(KE'_\alpha) \leq L$ , provided  $\lim_\alpha \lambda(A_\alpha)^{-1} \lambda(KF_\alpha) = 1$ . This limit is one by the regularity of  $\{A_\alpha\}$ , since  $[A_\alpha]_{KK^{-1}BK^{-1}} \subset KF_\alpha$ . To see this, let  $x \in [A_\alpha]_{KK^{-1}BK^{-1}}$ . By definition,  $KK^{-1}BK^{-1}x \subset A_\alpha$ . Now  $x \in Ke$  for some  $e \in E$ . We have  $Ke \subset KK^{-1}x \subset KK^{-1}BK^{-1}x \subset A_\alpha$ . Thus  $e \in E'_\alpha$ . It will follow that  $x \in KF_\alpha$  if  $Be \subset KE'_\alpha$ . To see this, let  $y \in Be$ . Then  $y \in Ke'$  for some  $e' \in E$ . Now  $Ke' \subset KK^{-1}y \subset KK^{-1}Be \subset KK^{-1}BK^{-1}x \subset A_\alpha$ . Thus  $e' \in E'_\alpha$  and  $y \in KE'_\alpha$ .



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