A RATIO LIMIT THEOREM FOR A STRONGLY SUBADDITIVE SET FUNCTION IN A LOCALLY COMPACT AMENABLE GROUP

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It is the purpose of this paper to prove that the following property holds: Given a locally compact, amenable, unimodular group $G$, if $S$ is a strongly subadditive, nonpositive, right invariant set function defined on the class $\mathcal{K}$ of relatively compact Borel subsets of $G$, and if $\{A_a\}$ is a net in $\mathcal{K}$ satisfying an appropriate growth condition, then

$$\lim_a \lambda(A_a)^{-1} S(A_a)$$

exists independently of $\{A_a\}$, where $\lambda$ is Haar measure on $G$.

Let $G$ be a locally compact group. Let $\lambda$ be right Haar outer measure defined on the subsets of $G$. Let $\mathcal{K}$ be the class of relatively compact Borel subsets of $G$. If $A$ is a subset of $G$ and $K \in \mathcal{K}$, let $[A]_K = \{g \in A : Kg \subset A\} = \bigcap_{k \in K, k \neq 1} k^{-1} A$, where 1 is the identity of $G$. In this paper, we call a locally compact, amenable, unimodular group a leau group.

DEFINITION 1. Following [1], we define a net $\{A_a\}$ in $\mathcal{K}$ to be a regular net in the locally compact group $G$ if

(D. 1.1) $\lambda(A_a) > 0$ for each $a$;
(D. 1.2) $\lim_a \lambda(KA_a)^{-1}\lambda([A_a]_K) = 1$, $K \in \mathcal{K}$, $K \neq \phi$.

(Even though $KA_a$ and $[A_a]_K$ may not be Borel measurable, (D. 1.2) makes sense because we required $\lambda$ to be right Haar outer measure, which is defined for all subsets of $G$.)

LEMMA 1. A locally compact group $G$ possesses a regular net if and only if $G$ is a leau group.

Proof. A locally compact group $G$ is amenable if and only if for any $\varepsilon > 0$, and for any nonempty compact subset $K$ of $G$, there exists a compact subset $U$ of $G$, of positive measure, such that $\lambda^*(U)^{-1}\lambda^*(KU) < 1 + \varepsilon$, where $\lambda^*$ is left Haar measure. (See [2].) We call this necessary and sufficient condition for amenability of $G$ condition (A).

Now suppose $G$ possesses a regular net $\{A_a\}$. Then (D. 1.2) implies that
Taking $K = \{g\}$, where $g$ is any element of $G$, we see that $A(g) = 1$. Thus $G$ is unimodular. It then follows that (1) implies condition (A), and thus $G$ is also amenable.

Conversely, suppose now $G$ is lcnu. Given $\epsilon > 0$ and a nonempty compact subset $K$ of $G$, we may find by condition (A) a compact set $U = U_{(K, \epsilon)}$, of positive measure, such that $\lambda(U)^{-1}\lambda(K^2U) < 1 + \epsilon$. We direct the set $W = \{(K, \epsilon): K$ a nonempty compact set in $G, \epsilon > 0\}$ as follows: $(K_1, \epsilon_1) > (K_2, \epsilon_2)$ if and only if $K_1 \supset K_2$ and $\epsilon_1 < \epsilon_2$. Then $\{V_{(K, \epsilon)}: (K, \epsilon) \in W\}$ is a regular net of compact subsets of $G$, where $V_{(K, \epsilon)} = KU_{(K, \epsilon)}$.

**DEFINITION 2.** Let $G$ be a regular group. Throughout this paper, we consider a set function $S: \mathcal{H} \to \mathbb{R}$, the set of real numbers, which satisfies the following properties:

1. $S(\emptyset) = 0$.
2. $S$ is strongly subadditive; that is, $S(A \cap B) + S(A \cup B) \leq S(A) + S(B)$, $A, B \in \mathcal{H}$.
3. $S(A) \leq 0$, $A \in \mathcal{H}$.
4. $S(A_g) = S(A)$, $A \in \mathcal{H}$, $g \in G$.

The main result we will prove in this note is the following theorem.

**THEOREM 1.** Let $G$ be a lcnu group. Let $S: \mathcal{H} \to \mathbb{R}$ satisfy Definition 2. Then there is an extended real number $r^*$ such that $\lim_a \lambda(A_a)^{-1}S(A_a) = r^*$ for every regular net $\{A_a\}$ in $\mathcal{H}$.

A special case of this theorem, for vector groups, was proved in [7] in order to define entropy in statistical mechanics for classical continuous systems. The theorem can be used to define the entropy of a measurable partition relative to a discrete amenable group of measure-preserving transformations on a probability space, thereby enabling one to generalize the concept of the Kolmogorov-Sinai invariant [5].

One may construct a set function $S$ satisfying Definition 2 as follows: Let $(\Omega, \mathcal{M})$ be a measurable space. For each element $g$ of the regular group $G$, let $T_g$ be a measurable transformation from $\Omega$ to $\Omega$. We suppose that $T_{g_1} \cdot T_{g_2} = T_{g_1g_2}$, $g_1, g_2 \in G$. Let $\mathcal{F}$ be a fixed sub-sigmafield of $\mathcal{M}$. If $E$ is a nonempty subset of $G$, let $\mathcal{F}_E$ be the smallest sub-sigmafield of $\mathcal{M}$ containing $\bigcup_{g \in E} (T_g)^{-1}\mathcal{F}$. Define $\mathcal{F}_\emptyset = \{\emptyset, \Omega\}$. Let $P, Q$ be probability measures on $\mathcal{M}$, such that $P$ is stationary with respect to $\{T_g: g \in G\}$ and the fields $\{(T_g)^{-1}\mathcal{F}: g \in G\}$
are independent with respect to Q. For each $E \in \mathcal{X}$, let $S(E)$ be the negative of the entropy of $P$ with respect to $Q$ over $\mathcal{F}_E$, which we assume finite. The function $S: \mathcal{X} \to \mathbb{R}$ defined in this way can be shown to satisfy Definition 2 in a manner analogous to that employed in [7] for vector groups.

**Lemma 2.** If Theorem holds for all sigma-compact lcau groups it holds for all lcau groups.

**Proof.** Let $d$ be a complete metric on $\mathbb{R}^*$, the set of extended real numbers, which induces the usual topology on $\mathbb{R}^*$. Let $(A_n)$ be a regular net for a non-sigma-compact lcau group $G$. Suppose $\lim_n \lambda(A_n)^{-1}S(A_n)$ does not exist. Then for some $\varepsilon > 0$, we may find a sequence $(F_n)$ of elements of $(A_n)$ and a sequence $(E_n)$ in $\mathcal{X}$ such that

(a) $F_0$ is any $A_n$ and $E_0$ is an open symmetric neighborhood of the identity.

(b) $d(\lambda(F_n)^{-1}S(F_n), \lambda(F_{n-1})^{-1}S(F_{n-1})) > \varepsilon$, $n \geq 1$.

(c) $\lambda(E_{n-1}F_n)^{-1}\lambda([F_n]_{E_{n-1}}) > 1 - n^{-1}$, $n \geq 1$.

(d) $E_n$ is an open symmetric set containing the closure of $[E_{n-1} \cup F_n]^*$, $n \geq 1$.

Let $G' = \bigcup_n E_n$. It is easily seen that $G'$ is an open, sigma-compact subgroup of $G$.

If we restrict $\lambda$ to $G'$, we get right Haar measure on $G'$. Thus $(F_n)$ is a regular sequence for $G'$, and $G'$ is a lcau group. Assuming Theorem 1 holds for sigma-compact lcau groups, $\lim_n \lambda(F_n)^{-1}S(F_n)$ would have to exist, a contradiction of b). Thus $\lim_n \lambda(A_n)^{-1}S(A_n)$ exists. Let $(B_n)$ be another regular net in $G$. Let $s_1 = \lim_n \lambda(A_n)^{-1}S(A_n)$, $s_2 = \lim_n \lambda(B_n)^{-1}S(B_n)$. We show that $s_1 = s_2$. Define sequences $(C_n)^*$, $(D_n)^*$, $(E_n)^*$ in $\mathcal{X}$ such that

(a) $E_0$ is an open symmetric neighborhood of the identity, $(C_n) \subset [A_n]$, $(D_n) \subset [B_n]$.

(b) $d(\lambda(C_n)^{-1}S(C_n), s_1) < n^{-1}$, $d(\lambda(D_n)^{-1}S(D_n), s_2) < n^{-1}$, $n \geq 1$.

(c) $\lambda(E_{n-1}C_n)^{-1}\lambda([C_n]_{E_{n-1}}) \geq 1 - n^{-1}$, $\lambda(E_{n-1}D_n)^{-1}\lambda([D_n]_{E_{n-1}}) \geq 1 - n^{-1}$, $n \geq 1$.

(d) $E_n$ is open, symmetric and contains the closure of $[E_{n-1} \cup C_n]$, $C_n \cup D_n]^*$, $n \geq 1$.

It follows that $G' = \bigcup_n E_n$ is an open, sigma-compact, lcau subgroup of $G$ and that $(C_n)$ and $(D_n)$ are regular sequences for $G'$. Therefore, $\lim_n \lambda(C_n)^{-1}S(C_n) = \lim_n \lambda(D_n)^{-1}S(D_n)$, and so $s_1 = s_2$ by b).

**Definition 3.** If $G$ is a locally compact group, if $S: \mathcal{X} \to \mathbb{R}$ satisfies Definition 2, and if $A, B \in \mathcal{X}$ with $A \cap B = \emptyset$, define $S(A \mid B) = S(A \cup B) - S(B)$. 
Lemma 3. Let G be a locally compact group, and let S : \mathcal{A} \to \mathbb{R}

satisfy Definition 2. Then S obeys the following laws:

(L. 3.1) \( S(A) \leq S(B) \) if \( A \supset B, A, B \in \mathcal{A} \).

(L. 3.2) If \( A_1, A_2, \ldots, A_k \) are elements of \( \mathcal{A} \) which partition \( A \), then \( S(A) = \sum_{i=1}^{k} S(A_i \mid \bigcup_{i=1}^{k} A_i) \), where an empty union is the null set.

(L. 3.3) \( S(\emptyset) \leq S(E \setminus D) \leq S(\emptyset) \leq 0 \), \( E, D \in \mathcal{A} \), \( E \cap D = \emptyset \).

(L. 3.4) \( S(\emptyset) \leq S(E) \leq 0 \), \( E, D \in \mathcal{A} \), \( E \cap D = \emptyset \).

Proof. (L. 3.2) follows easily from Definition 2. The strong subadditivity of \( S \) is equivalent to saying \( S(A \setminus B \setminus B) \leq S(A \setminus B \setminus A \cap B) \), \( A, B \in \mathcal{A} \). Letting \( A = E \cup D \) and \( B = D_1 \), where \( E, D_1, D_2 \) satisfy \( D_1 \cap E = \emptyset \) and \( D_1 \supset D_2 \), we have \( A \cap B = D_1 \) and \( A \setminus B = E \), whence (L. 3.3) follows. In (L. 3.3) if we take \( D_2 = \emptyset \), (L. 3.4) follows because \( S(E \setminus \emptyset) = S(E) \). If \( A \supset B \), where \( A, B \in \mathcal{A} \), then \( S(A) = S(B) + S(A \setminus B \setminus B) \leq S(B) \), and thus (L. 3.1) follows.

Definition 4. We define a locally compact group \( G \) to be a \( P \)-group if there exists for some positive integer \( n \) a triple \((K, \{G_i\}^n, \{H_i\}^n)\) such that:

(D. 4.1) \( K \) is a nonempty relatively compact Borel set in \( G \).

(D. 4.2) \( \{G_i\}^n \) and \( \{H_i\}^n \) are sequences of closed subgroups of \( G \) satisfying \( G_1 \subset H_1 \subset G_2 \subset H_2 \subset \cdots \subset G_n \subset H_n \).

(D. 4.3) The index of \( G_i \) in \( H_i \) is countable, \( i = 1, 2, \ldots, n \).

(D. 4.4) If \( E_i \) is any set of coset representatives of the right cosets \( \{G_i h : h \in H_i\} \) of \( G_i \) in \( H_i \), \( i = 1, 2, \ldots, n \), then each \( g \in G \) has a unique factorization in the form \( g = k e_i e_{i-1} \cdots e_1 k_1 e_1 \cdots e_{n-1} k_{n-1} e_{n-1} \cdots k e_n \), \( i = 1, 2, \ldots, n \). Also, \( K = \prod_{i=1}^{n} E_i G_i = K(\prod_{i=1}^{n} E_i), i = 1, 2, \ldots, n \), where an empty product is the identity in \( G \).

In order to prove Theorem 1 for sigma-compact leau groups, we need to show that such groups are \( P \)-groups. This we now do, by means of several lemmas. To see how the following lemma may be proved, see [2], page 379.

Lemma 4. Let \( G' \) be a closed normal subgroup of a connected Lie group \( G \). Let \( \phi : G \to G/G' \) be the canonical homomorphism. Then there exists a map \( \tau : G/G' \to G \) such that

(L. 4.1) \( \tau \) is a cross-section; that is, \( \phi \cdot \tau \) is the identity map on \( G/G' \).

(L. 4.2) If \( U \) is a relatively compact subset of \( G/G' \), then \( \tau(U) \) is a relatively compact subset of \( G \).

(L. 4.3) If \( U \) is a Borel set in \( G/G' \) and \( V \) is a Borel set in \( G' \), then \( \tau(U)V \) is a Borel set in \( G \).
**Lemma 5.** Let $G$ be a connected Lie group and $G'$ a closed normal subgroup of $G$ such that $G/G'$ is either a vector group or compact. Then if $G'$ is a $P$-group, so is $G$.

*Proof.* Let $\tau: G/G' \to G$ be the cross-section map provided by Lemma 4. Since $G/G'$ is a vector group or compact, it is easy to see that there exists a closed countable subgroup $G''$ of $G/G'$ and a relatively compact Borel set $K'$ in $G/G'$ such that $\{K'g: g \in G''\}$ partitions $G/G'$. If $G'$ is a $P$-group with respect to the triple $(K, \{G_i\}^\infty_i, \{H_i\}^\infty_i)$, then $G$ is a $P$-group with respect to the triple $(\tau(K)K, \{G_i\}^{n+1}_i, \{H_i\}^{n+1}_i)$, where $G_{n+1} = G'$ and $H_{n+1} = \phi^{-1}(G'')$.

**Lemma 6.** If $G$ is a sigma-compact locally compact group and $G'$ is an open subgroup of $G$ which is a $P$-group, then $G$ is a $P$-group.

*Proof.* Let $G'$ be a $P$-group with respect to the triple $(K, \{G_i\}^\infty_i, \{H_i\}^\infty_i)$. Then $G$ is a $P$-group with respect to the triple $(K, \{G_i\}^{n+1}_i, \{H_i\}^{n+1}_i)$, where $G_{n+1} = G'$, $H_{n+1} = G$.

**Lemma 7.** If $G$ is a locally compact group and $G'$ is a compact normal subgroup of $G$ such that $G/G'$ is a $P$-group, then $G$ is a $P$-group.

*Proof.* Suppose $G/G'$ is a $P$-group with respect to the triple $(K, \{G_i\}^\infty_i, \{H_i\}^\infty_i)$. Let $\phi: G \to G/G'$ be the canonical homomorphism. Then $G$ is a $P$-group with respect to the triple $(\phi^{-1}(K), \{\phi^{-1}(G_i)\}^\infty_i, \{\phi^{-1}(H_i)\}^\infty_i)$.

**Theorem 2.** Every sigma-compact locally compact amenable group is a $P$-group.

*Proof.* Every connected amenable Lie group $G$ possesses a series of closed subgroups $G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G$, where $G_0$ is the identity, $G_i$ is normal in $G_{i+1}$, and $G_{i+1}/G_i$ is either a vector group or compact, $i = 0, 1, \cdots, n-1$. (See [3], Theorem 3.3.2, and [4], Lemma 3.3.) Now $G_0$ is clearly a $P$-group, so by using Lemma 5 repeatedly we conclude every connected amenable Lie group is a $P$-group. Applying Lemma 6, every sigma-compact amenable Lie group is a $P$-group. For every locally compact group $G$ there exists an open subgroup $G'$ of $G$ and a compact normal subgroup $K$ of $G'$ such that $G'/K$ is a Lie group. (See [6], page 153.) Assuming $G$ in addition is sigma-compact and amenable, so is $G'/K$. Thus $G'/K$ is a $P$-group and then so is $G'$ by Lemma 7. Then $G$ is a $P$-group by Lemma 6.
We fix \( G \) to be a sigma-compact lcau group for the rest of the paper. We need to show Theorem 1 holds for \( G \). This we accomplish by means of some lemmas and Theorem 3.

Let \( (K, \{G_i\}_i; \{H_i\}_i) \) be a triple with respect to which \( G \) is a \( P \)-group. Let \( E_i \) be a set of coset representatives of the right cosets of \( G_i \) in \( H_i \) such that \( 1 \in E_i \), \( i = 1, 2, \ldots, n \), where 1 is the identity of \( G \). For each \( i \), let \( \tilde{H}_i \) be the collection of right cosets of \( G_i \) in \( H_i \). (Since \( G_i \) is not necessarily normal in \( H_i \), \( \tilde{H}_i \) need not be a group.) For each \( i \), let \( \phi_i: H_i \rightarrow \tilde{H}_i \) be the map such that \( \phi_i(h) = G_i h, h \in H_i \); let \( \tau_i: \tilde{H}_i \rightarrow E_i \) be the unique map such that \( \phi_i \cdot \tau_i \) is the identity map on \( \tilde{H}_i \). By a total order \( < \) on a set \( W \), we mean a transitive relation such that for \( x, y \in W \) exactly one of the following holds: \( x < y \), \( x = y \), or \( y < x \). For each \( i \), let \( <' \) be a total order on \( E_i \); if \( h \in H_i \), let \( <' \) be the total order on \( E_i \) such that if \( e, e' \in E_i \), then \( e <' e' \) if and only if only if \( \tau_i \cdot \phi_i(eh) <' \tau_i \cdot \phi_i(e'h) \). If \( h \in H_i \), let \( P_i(e) = \{e' \in E_i; e <' e'\} \). Let \( E = E_1 E_2 \cdots E_n \). Let \( H \) be the locally compact amenable group \( H = H_1 \times H_2 \times \cdots \times H_n \). If \( h = (h_1, h_2, \ldots, h_n) \in H \), let \( < \) be the lexicographical order on \( E \) defined as follows: if \( e = e_1 e_2 \cdots e_n \) and \( e' = e_1' e_2' \cdots e_n' \) are elements of \( E \), where \( e_i, e_i' \in E_i \), then \( e < e' \) if and only if there exists an integer \( k \), \( n \geq k \geq 1 \), such that \( e_k < e_k' \) and for \( n \geq j > k \), \( e_j = e_j' \). If \( h \in H \), \( e \in E \), let \( P(e) = \{e' \in E; e < e'\} \). If \( A \in \mathcal{X} \), \( e \in E \), let \( \phi_A: H \rightarrow R \) be the function such that \( \phi_A(h) = S(Ke|KF_k(e) \cap Ae) = S(K|KF_k(e)e^{-1} \cap A), h \in H \).

**Lemma 8.** If \( A \in \mathcal{X} \) and \( e \in E \), then \( \phi_A \in L^0(H) \), the space of bounded Borel-measurable real-valued functions with domain \( H \).

**Proof.** Fix \( A \in \mathcal{X} \), \( e \in E \). By (L. 3.4), \( \phi_A \leq 0 \). To achieve a lower bound, let \( E' = \{e \in E; Ke\cap Ae \neq \phi\} \). Since \( KE' \subset KK' \cap Ae \), \( E' \) is finite. Let \( F = \{e\} \cup E' \). By (L. 3.2), \( S(KF) = \sum_{f \in F} S(Kf|KP_k(f) \cap KF) \). By (L. 3.3) and (L. 3.4), \( S(KF) \leq S(Ke|KF_k(e) \cap KF) \leq S(Ke|KF_k(e) \cap Ae) = \phi_A(h) \), where the fact that \( KF \supset Ae \) was used. Thus \( \phi_A \) is a bounded function. We now show that it is a Borel measurable function. It is easily seen that \( \phi_A \) is a simple function with possible values \( S(Ke|KF' \cap Ae), F' \subset F \). If \( F' \subset F \), then \( \phi_A = S(Ke|KF' \cap Ae) \) on the set \{\( h \in H; P_k(e) \cap F = F' \)\}, which is equal to the intersection of the sets \( \bigcap_{f \in F'} \{h; f \in P_k(e)\} \) and \( \bigcap_{f \in F'} \{h; f \in P_k(e)\} \). Thus \( \phi_A \) is Borel measurable if for each \( f \in F \), \( \{h \in H; f \in P_k(e)\} \) is a Borel set. If \( f = e \), this set is empty. Thus, fix \( f \in F, f \neq e \). Let \( f = f_1 f_2 \cdots f_n, e = e_1 e_2 \cdots e_n \), where \( e_i, f_i \in E_i \) for each \( i \). Let \( j = \max \{i; f_i \neq e_i\} \). Then \( \{h \in H; f \in P_k(e)\} = \{h \in H; f_j \in P_k(e_j)\} \), where \( h_j \in H_j \) is the \( j \)th component of \( h \in H \). This is a Borel set in \( H \) if \( \{h \in H_j; f_j \in P_k(e_j)\} \) is a Borel set in \( H_j \). Now this latter set is the union of the sets \( \{h \in H_j; g \in G_j, \ G_je_1h = G_je_2\} \) where \( (g_1, g_2) \) ranges over all ordered
pairs such that $g_1, g_2 \in E_j$ and $g_1 \prec g_2$. Since the union is a countable union of closed subsets of $H_j$, Borel measurability follows.

**Lemma 9.** Let $\mu$ be a left invariant mean on $L^*(H)$. Then $\mu(\phi_n) = \mu(\phi_i)$, $A \in \mathcal{X}$, $e \in E$.

**Proof.** Fix $A \in \mathcal{X}$, $e \in E$. We observe that

$$K \prod_{i=1}^{n} \left( \prod_{j=1}^{n} E_i \right) = \left[ \bigcup_{i=1}^{n} K \prod_{i=1}^{n} \left( \prod_{j=1}^{n} E_i \right) \right].$$

by (D. 4.4), where $h = (h_1, h_2, \ldots, h_n) \in H$ and $e = e_1 e_2 \cdots e_n$. It is routine to show that $G_t \prod_{i=1}^{n} \left( \prod_{j=1}^{n} E_i \right) = G_t \prod_{i=1}^{n} \left( \prod_{j=1}^{n} E_i \right)$. Also, since $e_3 \prec e_2 \prec e_1$ for $j < i$, we have $\phi_i(e_1 e_2 e_3) = \phi_i(e_0 e_1 e_2 e_3)$. Thus, $K \prod_{i=1}^{n} \left( \prod_{j=1}^{n} E_i \right) = \left[ \bigcup_{i=1}^{n} K \prod_{i=1}^{n} \left( \prod_{j=1}^{n} E_i \right) \right].$

**Theorem 3.** Let $\{A_n\}$ be a regular net in the sigmacompact lcav group $G$. Then $\lim_{n} \lambda(A_n)^{-1} S(A_n) = \inf_{\beta \in \mathcal{X}} \lambda(K)^{-1} \mu(\phi_\beta)$.

**Proof.** Fix the regular net $\{A_n\}$. Now $KE_a \subset A_n \subset KE_a$, where $E_a = \{e \in E: Ke \cap A_n \neq \emptyset\}, E_a' = \{e \in E: Ke \subset A_n\}$. Thus by (L. 3.1), $S(KE_a) \leq S(A_n) \leq S(KE_a')$. We show that $\limsup_n \lambda(A_n) \lambda(K) \lambda(KE_a) \leq L$, where $L = \inf_{\beta \in \mathcal{X}} \lambda(K)^{-1} \mu(\phi_\beta)$. Now $S(KE_a) = \sum_{e \in E_a} S(Ke) \lambda(KP_h(e) \cap KE_a) \geq \sum_{e \in E_a} \phi_h(e)$, where $B_a = \bigcup_{e \in E_a} KE_a \lambda^{-1} e^{-1}$. Applying $\mu$ to the inequality and using Lemma 9, $S(KE_a) \geq |E_a| \mu(\phi_\beta) \geq |E_a| \lambda(K) L = \lambda(KE_a) L$, where $|E_a|$ denotes the cardinality of $E_a$. Since $KE_a \subset KK^{-1} A_n$, we have $\liminf_n \lambda(A_n)^{-1} \lambda(KE_a) = 1$, by the regularity of $\{A_n\}$. Thus $\liminf_n \lambda(A_n)^{-1} S(KE_a) = L$. Fix $B \in \mathcal{X}$. We suppose that $B \supset K$. Now $S(KE_a) = \sum_{e \in E_a} S(Ke) \lambda(KP_h(e) \cap KE_a) \leq \sum_{e \in E_a} \phi_h(e) \lambda(KF_a) \lambda(K)^{-1} \mu(\phi_\beta)$. Applying $\mu$, $S(KE_a) \leq \lambda(KF_a) \lambda(K)^{-1} \mu(\phi_\beta)$. We could conclude that $\limsup_n \lambda(A_n) \lambda(K) \lambda(KE_a) \leq L$, provided $\liminf_n \lambda(A_n)^{-1} \lambda(KF_a) = 1$. This limit is one by the regularity of $\{A_n\}$, since $[A_n]_{KK^{-1} e \in \mathcal{X}} \subset [A_n]_{KK^{-1} e \in \mathcal{X}}$. By definition, $KK^{-1} BK^{-1} x \subset A_n$. Now $x \in Ke$ for some $e \in E$. We have $Ke \subset KK^{-1} x \subset KK^{-1} BK^{-1} x \subset A_n$. Thus $e \in E_a'$. It will follow that $x \in KF_a$ if $Be \subset KE_a'$. To see this, let $x \in [A_n]_{KK^{-1} e \in \mathcal{X}}$. By definition, $KK^{-1} BK^{-1} x \subset A_n$. Now $x \in Ke$ for some $e \in E$. We have $Ke \subset KK^{-1} y \subset KK^{-1} BK^{-1} x \subset A_n$. Thus $e' \in E_a'$ and $y \in KE_a'$. 


REFERENCES


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