

Pacific Journal of Mathematics

**A RATIO LIMIT THEOREM FOR A STRONGLY SUBADDITIVE
SET FUNCTION IN A LOCALLY COMPACT AMENABLE
GROUP**

JOHN CRONAN KIEFFER

A RATIO LIMIT THEOREM FOR A STRONGLY SUBADDITIVE SET FUNCTION IN A LOCALLY COMPACT AMENABLE GROUP

J. C. KIEFFER

It is the purpose of this paper to prove that the following property holds: Given a locally compact, amenable, unimodular group G , if S is a strongly subadditive, nonpositive, right invariant set function defined on the class \mathcal{H} of relatively compact Borel subsets of G , and if $\{A_\alpha\}$ is a net in \mathcal{H} satisfying an appropriate growth condition, then

$$\lim_\alpha \lambda(A_\alpha)^{-1} S(A_\alpha)$$

exists independently of $\{A_\alpha\}$, where λ is Haar measure on G .

Let G be a locally compact group. Let λ be right Haar outer measure defined on the subsets of G . Let \mathcal{H} be the class of relatively compact Borel subsets of G . If A is a subset of G and $K \in \mathcal{H}$, let $[A]_K = \{g \in A : Kg \subset A\} = \bigcap_{k \in K \cup \{1\}} k^{-1}A$, where 1 is the identity of G . In this paper, we call a locally compact, amenable, unimodular group a *leau* group.

DEFINITION 1. Following [1], we define a net $\{A_\alpha\}$ in \mathcal{H} to be a regular net in the locally compact group G if

(D. 1.1) $\lambda(A_\alpha) > 0$ for each α ;

(D. 1.2) $\lim_\alpha \lambda(KA_\alpha)^{-1} \lambda([A_\alpha]_K) = 1, K \in \mathcal{H}, K \neq \phi$.

(Even though $K A_\alpha$ and $[A_\alpha]_K$ may not be Borel measurable, (D. 1.2) makes sense because we required λ to be right Haar outer measure, which is defined for *all* subsets of G .)

LEMMA 1. *A locally compact group G possesses a regular net if and only if G is a leau group.*

Proof. A locally compact group G is amenable if and only if for any $\varepsilon > 0$, and for any nonempty compact subset K of G , there exists a compact subset U of G , of positive measure, such that $\lambda^*(U)^{-1} \lambda^*(KU) < 1 + \varepsilon$, where λ^* is left Haar measure. (See [2].) We call this necessary and sufficient condition for amenability of G condition (A).

Now suppose G possesses a regular net $\{A_\alpha\}$. Then (D. 1.2) implies that

$$(1) \quad \lim_{\alpha} \lambda(KA_{\alpha})^{-1} \lambda(A_{\alpha}) = 1, K \in \mathcal{K}, K \neq \phi.$$

Taking $K = \{g\}$, where g is any element of G , we see that $\Delta(g) = 1$. Thus G is unimodular. It then follows that (1) implies condition (A), and thus G is also amenable.

Conversely, suppose now G is *lcu*. Given $\varepsilon > 0$ and a nonempty compact subset K of G , we may find by condition (A) a compact set $U = U_{(K, \varepsilon)}$, of positive measure, such that $\lambda(U)^{-1} \lambda(K^2 U) < 1 + \varepsilon$. We direct the set $W = \{(K, \varepsilon) : K \text{ a nonempty compact set in } G, \varepsilon > 0\}$ as follows: $(K_1, \varepsilon_1) > (K_2, \varepsilon_2)$ if and only if $K_1 \supset K_2$ and $\varepsilon_1 < \varepsilon_2$. Then $\{V_{(K, \varepsilon)} : (K, \varepsilon) \in W\}$ is a regular net of compact subsets of G , where $V_{(K, \varepsilon)} = KU_{(K, \varepsilon)}$.

DEFINITION 2. Let G be a regular group. Throughout this paper, we consider a set function $S: \mathcal{K} \rightarrow R$, the set of real numbers, which satisfies the following properties:

$$(D. 2.1) \quad S(\phi) = 0.$$

$$(D. 2.2) \quad S \text{ is strongly subadditive; that is, } S(A \cap B) + S(A \cup B) \leq S(A) + S(B), A, B \in \mathcal{K}.$$

$$(D. 2.3) \quad S(A) \leq 0, A \in \mathcal{K}.$$

$$(D. 2.4) \quad S(Ag) = S(A), A \in \mathcal{K}, g \in G.$$

The main result we will prove in this note is the following theorem.

THEOREM 1. *Let G be a lcau group. Let $S: \mathcal{K} \rightarrow R$ satisfy Definition 2. Then there is an extended real number r^* such that $\lim_{\alpha} \lambda(A_{\alpha})^{-1} S(A_{\alpha}) = r^*$ for every regular net $\{A_{\alpha}\}$ in \mathcal{K} .*

A special case of this theorem, for vector groups, was proved in [7] in order to define entropy in statistical mechanics for classical continuous systems. The theorem can be used to define the entropy of a measurable partition relative to a discrete amenable group of measure-preserving transformations on a probability space, thereby enabling one to generalize the concept of the Kolmogorov-Sinai invariant [5].

One may construct a set function S satisfying Definition 2 as follows: Let (Ω, \mathcal{M}) be a measurable space. For each element g of the regular group G , let T^g be a measurable transformation from Ω to Ω . We suppose that $T^{g_1} \cdot T^{g_2} = T^{g_1 g_2}$, $g_1, g_2 \in G$. Let \mathcal{F} be a fixed sub-sigmafield of \mathcal{M} . If E is a nonempty subset of G , let \mathcal{F}_E be the smallest sub-sigmafield of \mathcal{M} containing $\bigcup_{g \in E} (T^g)^{-1} \mathcal{F}$. Define $\mathcal{F}_{\neq} = \{\phi, \Omega\}$. Let P, Q be probability measures on \mathcal{M} , such that P is stationary with respect to $\{T^g : g \in G\}$ and the fields $\{(T^g)^{-1} \mathcal{F} : g \in G\}$

are independent with respect to Q . For each $E \in \mathcal{H}$, let $S(E)$ be the negative of the entropy of P with respect to Q over \mathcal{F}_E , which we assume finite. The function $S: \mathcal{H} \rightarrow \mathbb{R}$ defined in this way can be shown to satisfy Definition 2 in a manner analogous to that employed in [7] for vector groups.

LEMMA 2. *If Theorem holds for all sigma-compact leau groups it holds for all leau groups.*

Proof. Let d be a complete metric on R^* , the set of extended real numbers, which induces the usual topology on R^* . Let $\{A_\alpha\}$ be a regular net for a non-sigma-compact leau group G . Suppose $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$ does not exist. Then for some $\varepsilon > 0$, we may find a sequence $\{F_n\}_0^\infty$ of elements of $\{A_\alpha\}$ and a sequence $\{E_n\}_0^\infty$ in \mathcal{H} such that

(a) F_0 is any A_α and E_0 is an open symmetric neighborhood of the identity.

(b) $d(\lambda(F_n)^{-1}S(F_n), \lambda(F_{n-1})^{-1}S(F_{n-1})) > \varepsilon, n \geq 1$.

(c) $\lambda(E_{n-1}F_n)^{-1}\lambda([F_n]_{E_{n-1}}) > 1 - n^{-1}, n \geq 1$.

(d) E_n is an open symmetric set containing the closure of $[E_{n-1} \cup F_n]^2, n \geq 1$.

Let $G' = \bigcup_n E_n$. It is easily seen that G' is an open, sigma-compact subgroup of G .

If we restrict λ to G' , we get right Haar measure on G' . Thus $\{F_n\}$ is a regular sequence for G' , and G' is a leau group. Assuming Theorem 1 holds for sigma-compact leau groups, $\lim_n \lambda(F_n)^{-1}S(F_n)$ would have to exist, a contradiction of b). Thus $\lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$ exists. Let $\{B_\beta\}$ be another regular net in G . Let $s_1 = \lim_\alpha \lambda(A_\alpha)^{-1}S(A_\alpha)$, $s_2 = \lim \lambda(B_\beta)^{-1}S(B_\beta)$. We show that $s_1 = s_2$. Define sequences $\{C_n\}_1^\infty, \{D_n\}_1^\infty, \{E_n\}_0^\infty$ in \mathcal{H} such that

(a) E_0 is an open symmetric neighborhood of the identity, $\{C_n\} \subset \{A_\alpha\}, \{D_n\} \subset \{B_\beta\}$.

(b) $d(\lambda(C_n)^{-1}S(C_n), s_1) < n^{-1}, d(\lambda(D_n)^{-1}S(D_n), s_2) < n^{-1}, n \geq 1$.

(c) $\lambda(E_{n-1}C_n)^{-1}\lambda([C_n]_{E_{n-1}}) \geq 1 - n^{-1}, \lambda(E_{n-1}D_n)^{-1}\lambda([D_n]_{E_{n-1}}) \geq 1 - n^{-1}, n \geq 1$.

(d) E_n is open, symmetric and contains the closure of $[E_{n-1} \cup C_n \cup D_n]^2, n \geq 1$.

It follows that $G' = \bigcup_n E_n$ is an open, sigma-compact, leau subgroup of G and that $\{C_n\}$ and $\{D_n\}$ are regular sequences for G' . Therefore, $\lim_n \lambda(C_n)^{-1}S(C_n) = \lim_n \lambda(D_n)^{-1}S(D_n)$, and so $s_1 = s_2$ by b).

DEFINITION 3. If G is a locally compact group, if $S: \mathcal{H} \rightarrow \mathbb{R}$ satisfies Definition 2, and if $A, B \in \mathcal{H}$ with $A \cap B = \phi$, define $S(A|B) = S(A \cup B) - S(B)$.

LEMMA 3. *Let G be a locally compact group, and let $S: \mathcal{K} \rightarrow R$ satisfy Definition 2. Then S obeys the following laws:*

(L. 3.1) $S(A) \leq S(B)$ if $A \supset B, A, B \in \mathcal{K}$.

(L. 3.2) *If A_1, A_2, \dots, A_k are elements of \mathcal{K} which partition A , then $S(A) = \sum_{i=1}^k S(A_i | \bigcup_{j=1}^{i-1} A_j)$, where an empty union is the null set.*

(L. 3.3) $S(E | D_1) \leq S(E | D_2), D_1 \supset D_2, E \cap D_1 = \phi, E, D_1, D_2 \in \mathcal{K}$.

(L. 3.4) $S(E | D) \leq S(E) \leq 0, E, D \in \mathcal{K}, E \cap D = \phi$.

Proof. (L. 3.2) follows easily from Definition 2. The strong subadditivity of S is equivalent to saying $S(A \setminus B | B) \leq S(A \setminus B | A \cap B), A, B \in \mathcal{K}$. Letting $A = E \cup D_2$ and $B = D_1$, where E, D_1, D_2 satisfy $D_1 \cap E = \phi$ and $D_1 \supset D_2$, we have $A \cap B = D_2$ and $A \setminus B = E$, whence (L. 3.3) follows. In (L. 3.3) if we take $D_2 = \phi$, (L. 3.4) follows because $S(E | \phi) = S(E)$. If $A \supset B$, where $A, B \in \mathcal{K}$, then $S(A) = S(B) + S(A \setminus B | B) \leq S(B)$, and thus (L. 3.1) follows.

DEFINITION 4. We define a locally compact group G to be a P -group if there exists for some positive integer n a triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ such that:

(D. 4.1) K is a nonempty relatively compact Borel set in G .

(D. 4.2) $\{G_i\}_1^n$ and $\{H_i\}_1^n$ are sequences of closed subgroups of G satisfying $G_1 \subset H_1 \subset G_2 \subset H_2 \subset \dots \subset G_n \subset H_n$.

(D. 4.3) The index of G_i in H_i is countable, $i = 1, 2, \dots, n$.

(D. 4.4) If E_i is any set of coset representatives of the right cosets $\{G_i h: h \in H_i\}$ of G_i in $H_i, i = 1, 2, \dots, n$, then each $g \in G$ has a unique factorization in the form $g = ke_1 e_2 \dots e_n, k \in K, e_i \in E_i, i = 1, 2, \dots, n$. Also, $K(\prod_{j=1}^{i-1} E_j)G_i = K(\prod_{j=1}^{i-1} E_j), i = 1, 2, \dots, n$, where an empty product is the identity in G .

In order to prove Theorem 1 for sigma-compact *lcac* groups, we need to show that such groups are P -groups. This we now do, by means of several lemmas. To see how the following lemma may be proved, see [2], page 379.

LEMMA 4. *Let G' be a closed normal subgroup of a connected Lie group G . Let $\phi: G \rightarrow G/G'$ be the canonical homomorphism. Then there exists a map $\tau: G/G' \rightarrow G$ such that*

(L. 4.1) τ is a cross-section; that is, $\phi \cdot \tau$ is the identity map on G/G' .

(L. 4.2) *If U is a relatively compact subset of G/G' , then $\tau(U)$ is a relatively compact subset of G .*

(L. 4.3) *If U is a Borel set in G/G' and V is a Borel set in G' , then $\tau(U)V$ is a Borel set in G .*

LEMMA 5. *Let G be a connected Lie group and G' a closed normal subgroup of G such that G/G' is either a vector group or compact. Then if G' is a P -group, so is G .*

Proof. Let $\tau: G/G' \rightarrow G$ be the cross-section map provided by Lemma 4. Since G/G' is a vector group or compact, it is easy to see that there exists a closed countable subgroup G'' of G/G' and a relatively compact Borel set K' in G/G' such that $\{K'g: g \in G''\}$ partitions G/G' . If G' is a P -group with respect to the triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$, then G is a P -group with respect to the triple $(\tau(K')K, \{G_i\}_1^{n+1}, \{H_i\}_1^{n+1})$, where $G_{n+1} = G'$ and $H_{n+1} = \phi^{-1}(G'')$.

LEMMA 6. *If G is a sigma-compact locally compact group and G' is an open subgroup of G which is a P -group, then G is a P -group.*

Proof. Let G' be a P -group with respect to the triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$. Then G is a P -group with respect to the triple $(K, \{G_i\}_1^{n+1}, \{H_i\}_1^{n+1})$, where $G_{n+1} = G'$, $H_{n+1} = G$.

LEMMA 7. *If G is a locally compact group and G' is a compact normal subgroup of G such that G/G' is a P -group, then G is a P -group.*

Proof. Suppose G/G' is a P -group with respect to the triple $(K, \{G_i\}_1^n, \{H_i\}_1^n)$. Let $\phi: G \rightarrow G/G'$ be the canonical homomorphism. Then G is a P -group with respect to the triple $(\phi^{-1}(K), \{\phi^{-1}(G_i)\}_1^n, \{\phi^{-1}(H_i)\}_1^n)$.

THEOREM 2. *Every sigma-compact locally compact amenable group is a P -group.*

Proof. Every connected amenable Lie group G possesses a series of closed subgroups $G_0 \subset G_1 \subset G_2 \subset \dots \subset G_n = G$, where G_0 is the identity, G_i is normal in G_{i+1} , and G_{i+1}/G_i is either a vector group or compact, $i = 0, 1, \dots, n - 1$. (See [3], Theorem 3.3.2, and [4], Lemma 3.3.) Now G_0 is clearly a P -group, so by using Lemma 5 repeatedly we conclude every connected amenable Lie group is a P -group. Applying Lemma 6, every sigma-compact amenable Lie group is a P -group. For every locally compact group G there exists an open subgroup G' of G and a compact normal subgroup K of G' such that G'/K is a Lie group. (See [6], page 153.) Assuming G in addition is sigma-compact and amenable, so is G'/K . Thus G'/K is a P -group and then so is G' by Lemma 7. Then G is a P -group by Lemma 6.

We fix G to be a sigma-compact *locu* group for the rest of the paper. We need to show Theorem 1 holds for G . This we accomplish by means of some lemmas and Theorem 3.

Let $(K, \{G_i\}_1^n, \{H_i\}_1^n)$ be a triple with respect to which G is a P -group. Let E_i be a set of coset representatives of the right cosets of G_i in H_i such that $1 \in E_i, i = 1, 2, \dots, n$, where 1 is the identity of G . For each i , let \bar{H}_i be the collection of right cosets of G_i in H_i . (Since G_i is not necessarily normal in H_i, \bar{H}_i need not be a group.) For each i , let $\phi_i: H_i \rightarrow \bar{H}_i$ be the map such that $\phi_i(h) = G_i h, h \in H_i$; let $\tau_i: \bar{H}_i \rightarrow E_i$ be the unique map such that $\phi_i \cdot \tau_i$ is the identity map on \bar{H}_i . By a total order $<$ on a set W , we mean a transitive relation such that for $x, y \in W$ exactly one of the following hold: $x < y, x = y$, or $y < x$. For each i , let $<^i$ be a total order on E_i ; if $h \in H_i$, let $<_h^i$ be the total order on E_i such that if $e, e' \in E_i$ then $e <_h^i e'$ if and only if $\tau_i \cdot \phi_i(eh) <^i \tau_i \cdot \phi_i(e'h)$. If $h \in H_i$, let $P_h^i(e) = \{e' \in E_i: e' <_h^i e\}$. Let $E = E_1 E_2 \dots E_n$. Let H be the locally compact amenable group $H = H_1 \times H_2 \times \dots \times H_n$. If $h = (h_1, h_2, \dots, h_n) \in H$, let $<_h$ be the lexicographical order on E defined as follows: if $e = e_1 e_2 \dots e_n$ and $e' = e'_1 e'_2 \dots e'_n$ are elements of E , where $e_i, e'_i \in E_i$, then $e <_h e'$ if and only if there exists an integer $k, n \geq k \geq 1$, such that $e_k <_{h_k}^k e'_k$ and for $n \geq j > k, e_j = e'_j$. If $h \in H, e \in E$, let $P_h(e) = \{e' \in E: e' <_h e\}$. If $A \in \mathcal{N}, e \in E$, let $\phi_A^e: H \rightarrow R$ be the function such that $\phi_A^e(h) = S(Ke | KP_h(e) \cap Ae) = S(K | KP_h(e)e^{-1} \cap A), h \in H$.

LEMMA 8. *If $A \in \mathcal{N}$ and $e \in E$, then $\phi_A^e \in L^\infty(H)$, the space of bounded Borel-measurable real-valued functions with domain H .*

Proof. Fix $A \in \mathcal{N}, e \in E$. By (L. 3.4), $\phi_A^e \leq 0$. To achieve a lower bound, let $E' = \{e' \in E: Ke' \cap Ae \neq \phi\}$. Since $KE' \subset KK^{-1}Ae, E'$ is finite. Let $F = \{e\} \cup E'$. By (L. 3.2), $S(KF) = \sum_{f \in F} S(Kf | KP_h(f) \cap KF)$. By (L. 3.3) and (L. 3.4), $S(KF) \leq S(Ke | KP_h(e) \cap KF) \leq S(Ke | KF_h(e) \cap Ae) = \phi_A^e(h)$, where the fact that $KF \supset Ae$ was used. Thus ϕ_A^e is a bounded function. We now show that it is a Borel measurable function. It is easily seen that ϕ_A^e is a simple function with possible values $S(Ke | KF' \cap Ae), F' \subset F$. If $F' \subset F$, then $\phi_A^e = S(Ke | KF' \cap Ae)$ on the set $\{h \in H: P_h(e) \cap F' = F'\}$, which is equal to the intersection of the sets $\bigcap_{f \in F'} \{h: f \in P_h(e)\}$ and $\bigcap_{f \in F \setminus F'} \{h: f \notin P_h(e)\}$. Thus ϕ_A^e is Borel measurable if for each $f \in F, \{h \in H: f \in P_h(e)\}$ is a Borel set. If $f = e$, this set is empty. Thus, fix $f \in F, f \neq e$. Let $f = f_1 f_2 \dots f_n$, and $e = e_1 e_2 \dots e_n$, where $e_i, f_i \in E_i$ for each i . Let $j = \max \{i: f_i \neq e_i\}$. Then $\{h \in H: f \in P_h(e)\} = \{h \in H: f_j \in P_{h_j}^j(e_j)\}$, where $h_j \in H_j$ is the j^{th} component of $h \in H$. This is a Borel set in H if $\{h \in H_j: f_j \in P_h^j(e_j)\}$ is a Borel set in H_j . Now this latter set is the union of the sets $\{h \in H_j: G_j f_j h = G_j g_1, G_j e_j h = G_j g_2\}$ where (g_1, g_2) ranges over all ordered

pairs such that $g_1, g_2 \in E_j$ and $g_1 \prec^j g_2$. Since the union is a countable union of closed subsets of H_j , Borel measurability follows.

LEMMA 9. *Let μ be a left invariant mean on $L^\infty(H)$. Then $\mu(\phi_A^e) = \mu(\phi_A^1)$, $A \in \mathcal{K}$, $e \in E$.*

Proof. Fix $A \in \mathcal{K}$, $e \in E$. We observe that

$$\begin{aligned} KP_h(e)e^{-1} &= \left[\bigcup_{i=1}^n K \left(\prod_{j=1}^{i-1} E_j \right) P_{h_i}^i(e_i) e_{i+1} \cdots e_n \right] e^{-1} \\ &= \bigcup_{i=1}^n \left[K \left(\prod_{j=1}^{i-1} E_j \right) G_i P_{h_i}^i(e_i) e_i^{-1} \cdots e_2^{-1} e_1^{-1} \right], \end{aligned}$$

by (D.4.4), where $h = (h_1, h_2, \dots, h_n) \in H$ and $e = e_1 e_2 \cdots e_n$. It is routine to show that $G_i P_{h_i}^i(e_i) = G_i P_1^i(\tau_i \cdot \phi_i(e_i h_i)) h_i^{-1}$. Also, since $e_j \in G_i$ for $j < i$, we have $\phi_i(e_i h_i) = \phi_i(e_1 e_2 \cdots e_i h_i)$. Thus, $KP_h(e)e^{-1} = \bigcup_{i=1}^n [K(\prod_{j=1}^{i-1} E_j) P_1^i(\tau_i \cdot \phi_i(e_1 \cdots e_i h_i))(e_1 \cdots e_i h_i)^{-1}] = KP_{mh}(1)$, where $m = (m_1, m_2, \dots, m_n) \in H$ satisfies $m_i = \prod_{j=1}^i e_j$, $i = 1, 2, \dots, n$. Thus $\phi_A^e(h) = \phi_A^1(mh)$, $h \in H$, from which the lemma follows.

THEOREM 3. *Let $\{A_\alpha\}$ be a regular net in the sigma-compact leau group G . Then $\lim_\alpha \lambda(A_\alpha)^{-1} S(A_\alpha) = \inf_{B \in \mathcal{K}} \lambda(K)^{-1} \mu(\phi_B^1)$.*

Proof. Fix the regular net $\{A_\alpha\}$. Now $KE'_\alpha \subset A_\alpha \subset KE_\alpha$, where $E_\alpha = \{e \in E: Ke \cap A_\alpha \neq \phi\}$, $E'_\alpha = \{e \in E: Ke \subset A_\alpha\}$. Thus by (L.3.1), $S(KE_\alpha) \leq S(A_\alpha) \leq S(KE'_\alpha)$. We show that $\limsup_\alpha \lambda(A_\alpha)^{-1} S(KE'_\alpha) \leq L$ and $\liminf_\alpha \lambda(A_\alpha)^{-1} S(KE_\alpha) \geq L$, where $L = \inf_{B \in \mathcal{K}} \lambda(K)^{-1} \mu(\phi_B^1)$. Now $S(KE_\alpha) = \sum_{e \in E_\alpha} S(Ke | KP_h(e) \cap KE_\alpha) \geq \sum_{e \in E_\alpha} \phi_{B_\alpha}^e$, where $B_\alpha = \bigcup_{e \in E_\alpha} KE_\alpha e^{-1}$. Applying μ to the inequality and using Lemma 9, $S(KE_\alpha) \geq |E_\alpha| \mu(\phi_{B_\alpha}^1) \geq |E_\alpha| \lambda(K) L = \lambda(KE_\alpha) L$, where $|E_\alpha|$ denotes the cardinality of E_α . Since $KE_\alpha \subset KK^{-1}A_\alpha$ we have $\lim_\alpha \lambda(A_\alpha)^{-1} \lambda(KE_\alpha) = 1$, by the regularity of $\{A_\alpha\}$. Thus $\liminf_\alpha \lambda(A_\alpha)^{-1} S(KE_\alpha) \geq L$. Fix $B \in \mathcal{K}$. We suppose that $B \supset K$. Now $S(KE'_\alpha) = \sum_{e \in E'_\alpha} S(Ke | KP_h(e) \cap KE'_\alpha) \leq \sum_{e \in E'_\alpha} \phi_B^e$ where $F_\alpha = \{e \in E'_\alpha: KE'_\alpha e^{-1} \supset B\}$. Applying μ , $S(KE'_\alpha) \leq \lambda(KF_\alpha) \lambda(K)^{-1} \mu(\phi_B^1)$. We could conclude that $\limsup_\alpha \lambda(A_\alpha)^{-1} S(KE'_\alpha) \leq L$, provided $\lim_\alpha \lambda(A_\alpha)^{-1} \lambda(KF_\alpha) = 1$. This limit is one by the regularity of $\{A_\alpha\}$, since $[A_\alpha]_{KK^{-1}BK^{-1}} \subset KF_\alpha$. To see this, let $x \in [A_\alpha]_{KK^{-1}BK^{-1}}$. By definition, $KK^{-1}BK^{-1}x \subset A_\alpha$. Now $x \in Ke$ for some $e \in E$. We have $Ke \subset KK^{-1}x \subset KK^{-1}BK^{-1}x \subset A_\alpha$. Thus $e \in E'_\alpha$. It will follow that $x \in KF_\alpha$ if $Be \subset KE'_\alpha$. To see this, let $y \in Be$. Then $y \in Ke'$ for some $e' \in E$. Now $Ke' \subset KK^{-1}y \subset KK^{-1}Be \subset KK^{-1}BK^{-1}x \subset A_\alpha$. Thus $e' \in E'_\alpha$ and $y \in KE'_\alpha$.

REFERENCES

1. M. F. Driscoll, J. N. McDonald, and N. A. Weiss, *LLN for weakly stationary processes on locally compact abelian groups*, Ann. of Prob., **2** (1974), 1168-1171.
2. W. R. Emerson and F. P. Greenleaf, *Covering properties and Folner conditions for locally compact groups*, Math. Zeitschr., **102** (1967), 370-384.
3. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand, 1969.
4. K. Iwasawa, *On some types of topological groups*, Ann. of Math., **50** (1949), 507-558.
5. J. C. Kieffer, *An entropy equidistribution property for a measurable partition under the action of an amenable group*, Bull. Amer. Math. Soc., **81** (1975), 464-466.
6. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, 1955.
7. D. W. Robinson and D. Ruelle, *Mean entropy of states in classical statistical mechanics*, Commun. Math. Phys., **5** (1967), 288-300.

Received May 15, 1975.

UNIVERSITY OF MISSOURI-ROLLA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)
University of California
Los Angeles, California 90024

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. A. BEAUMONT
University of Washington
Seattle, Washington 98105

D. GILBARG AND J. MILGRAM
Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Jiří Adámek, V. Koubek and Věra Trnková, <i>Sums of Boolean spaces represent every group</i>	1
Richard Neal Ball, <i>Full convex l-subgroups and the existence of a^*-closures of lattice ordered groups</i>	7
Joseph Becker, <i>Normal hypersurfaces</i>	17
Gerald A. Beer, <i>Starshaped sets and the Hausdorff metric</i>	21
Dennis Dale Berkey and Alan Cecil Lazer, <i>Linear differential systems with measurable coefficients</i>	29
Harald Boehme, <i>Glättungen von Abbildungen 3-dimensionaler Mannigfaltigkeiten</i>	45
Stephen LaVern Campbell, <i>Linear operators for which T^*T and $T + T^*$ commute</i>	53
H. P. Dikshit and Arun Kumar, <i>Absolute summability of Fourier series with factors</i>	59
Andrew George Earnest and John Sollion Hsia, <i>Spinor norms of local integral rotations. II</i>	71
Erik Maurice Ellentuck, <i>Semigroups, Horn sentences and isolic structures</i>	87
Ingrid Fotino, <i>Generalized convolution ring of arithmetic functions</i>	103
Michael Randy Gabel, <i>Lower bounds on the stable range of polynomial rings</i>	117
Fergus John Gaines, <i>Kato-Taussky-Wielandt commutator relations and characteristic curves</i>	121
Theodore William Gamelin, <i>The polynomial hulls of certain subsets of C^2</i>	129
R. J. Gazik and Darrell Conley Kent, <i>Coarse uniform convergence spaces</i>	143
Paul R. Goodey, <i>A note on starshaped sets</i>	151
Eloise A. Hamann, <i>On power-invariance</i>	153
M. Jayachandran and M. Rajagopalan, <i>Scattered compactification for $N \cup \{P\}$</i>	161
V. Karunakaran, <i>Certain classes of regular univalent functions</i>	173
John Cronan Kieffer, <i>A ratio limit theorem for a strongly subadditive set function in a locally compact amenable group</i>	183
Siu Kwong Lo and Harald G. Niederreiter, <i>Banach-Buck measure, density, and uniform distribution in rings of algebraic integers</i>	191
Harold W. Martin, <i>Contractibility of topological spaces onto metric spaces</i>	209
Harold W. Martin, <i>Local connectedness in developable spaces</i>	219
A. Meir and John W. Moon, <i>Relations between packing and covering numbers of a tree</i>	225
Hiroshi Mori, <i>Notes on stable currents</i>	235
Donald J. Newman and I. J. Schoenberg, <i>Splines and the logarithmic function</i>	241
M. Ann Piech, <i>Locality of the number of particles operator</i>	259
Fred Richman, <i>The constructive theory of KT-modules</i>	263
Gerard Sierksma, <i>Carathéodory and Helly-numbers of convex-product-structures</i>	275
Raymond Earl Smithson, <i>Subcontinuity for multifunctions</i>	283
Gary Roy Spoar, <i>Differentiability conditions and bounds on singular points</i>	289
Rosario Strano, <i>Azumaya algebras over Hensel rings</i>	295