RELATIONS BETWEEN PACKING AND COVERING NUMBERS OF A TREE

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Let \( P_k \) denote the size of the largest subset of nodes of a tree \( T \) with \( n \) nodes such that the distance between any two nodes in the subset is at least \( k + 1 \); let \( C_k \) denote the size of the smallest subset of nodes of \( T \) such that every node of \( T \) is at distance at most \( k \) from some node in the subset. We determine various relations involving \( P_k \) and \( C_k \); in particular, we show that \( P_k + kC_k \leq n \) if \( n \geq k + 1 \) and that \( P_2 = C_2 \).

1. Introduction. The distance between nodes \( x \) and \( y \) in a graph \( G \) is the number \( d(x, y) \) of edges in any shortest path in \( G \) that joins \( x \) and \( y \). (For definitions not given here see [1] or [5].) A subset \( \mathcal{P} \) of nodes of \( G \) is a \( k \)-packing if \( d(x, y) > k \) for all pairs of distinct nodes \( x \) and \( y \) of \( \mathcal{P} \); the \( k \)-packing number of \( G \) is the number \( P_k = P_k(G) \) of nodes in any largest \( k \)-packing in \( G \). A subset \( \mathcal{C} \) of nodes of \( G \) is a \( k \)-covering if for every node \( x \) in \( G \) there is at least one node \( y \) in \( \mathcal{C} \) such that \( d(x, y) \leq k \); the \( k \)-covering number of \( G \) is the number \( C_k = C_k(G) \) of nodes in any smallest \( k \)-covering of \( G \).

Our object here is to establish various relations between \( P_k(T) \) and \( C_k(T) \) when \( T \) is a tree with \( n \) nodes. We consider the case \( k = 1 \) in §2 and determine those values of \( \alpha \) and \( \beta \) for which there exists a tree \( T \) such that \( P_1(T) = \alpha \) and \( C_1(T) = \beta \). We derive upper bounds for \( P_k(T) \) and \( C_k(T) \) in §3. In §4 we show that \( P_k(T) + kC_k(T) \leq n \) for any tree \( T \) with \( n \) nodes when \( n \geq k + 1 \) and we show that this inequality is, in a sense, best possible. Finally, in §5 we show that \( P_2 = C_2 \).

The quantities \( P_1(G) \) and \( C_1(G) \) have been considered before under different names. For example, \( P_1(G) \) and \( C_1(G) \) are called the independence number and the domination number of \( G \) in [5; Chap. 13]; and they are called the coefficients of internal and external stability in [1; Chap. 4]. Some inequalities for \( P_1(G) \) and \( C_1(G) \) are given in [2; Chaps. 13 and 14] but some of these are unnecessarily weak when \( G \) is a tree.

2. Relations between \( P_1 \) and \( C_1 \). In what follows \( T \) will always denote an arbitrary tree with \( n \) nodes. For convenience, we shall frequently write \( P \) and \( C \) for \( P_1(T) \) and \( C_1(T) \).
THEOREM 1. If \( n \geq 2 \), then \( P + C \leq n \).

Proof. If \( \mathcal{P} \) denotes a 1-packing of \( P \) nodes in \( T \) then each node of \( \mathcal{P} \) must be joined to at least one node not in \( \mathcal{P} \) if \( n \geq 2 \). Thus the \( n - P \) nodes not in \( \mathcal{P} \) constitute a 1-covering of \( T \). Hence, \( C \leq n - P \), as required.

COROLLARY 1. If \( n \geq 2 \), then \( 1 \leq C \leq (1/2)n \leq P \leq n - 1 \).

Proof. It is obvious that \( C \geq 1 \) and \( P \leq n - 1 \) when \( n \geq 2 \). The remaining inequalities follow from Theorem 1 and the easily established fact that \( C \leq P \) (see [5; p. 211]); they may also be proved directly by observing that the sets of nodes of \( T \) whose distances from a given node \( x \) are odd or even, respectively, are both 1-packings and 1-coverings. We remark that the inequalities \( C(G) \leq (1/2)n \leq P(G) \) hold for any nontrivial connected bipartite graph \( G \) with \( n \) nodes.

THEOREM 2. If \( n \geq 1 \), then \( P + 2C \geq n + 1 \).

Proof. Let \( \mathcal{C} \) denote a 1-covering of \( C \) nodes of \( T \) and let \( R \) denote the subgraph determined by the \( n - C \) nodes not in \( \mathcal{C} \). If \( R \) has \( j \) components and \( e \) edges then \( e = n - C - j \) (see [5; p. 68]) and it is easy to see that \( P \geq j \). Since each node of \( R \) is joined to at least one node of \( \mathcal{C} \) and since \( T \) has \( n - 1 \) edges altogether it follows that

\[
e \leq (n - 1) - (n - C) = C - 1.
\]

Hence,

\[P \geq j = n - C - e \geq (n - C) - (C - 1) = n - 2C + 1,
\]
as required. (It will follow from Theorem 3 that the inequalities \( 1/2(n + 1 - P) \leq C \leq n - P \), implied by Theorems 1 and 2 are, in a sense, best possible.)

The next result is obtained by combining the inequalities \( P \geq (1/2)n \) and \( P + 2C \geq n + 1 \).

COROLLARY 2. If \( n \geq 1 \) and \( 0 \leq \lambda \leq 2 \), then

\[P + \lambda C \geq \frac{1}{2} \left( 1 + \frac{1}{2} \lambda \right) n + \frac{1}{2} \lambda ;
\]
in particular,

\[P + C \geq \left\{ \frac{3}{4} n + \frac{1}{2} \right\},
\]
where \( \lfloor x \rfloor \) denotes the least integer not less than \( x \).

It is not difficult to construct trees for which equality holds in the last inequality. We remark that it follows from results in [3] and [4] that the average value of \( P + C \) over the \( n^{n-1} \) trees with \( n \) labelled nodes is approximately \( .927n \) for large values of \( n \).

**Theorem 3.** If \( \alpha \) and \( \beta \) are positive integers such that

\[
\begin{align*}
\alpha & \geq \frac{1}{2} n, \\
\alpha + \beta & \leq n, \\
\alpha + 2\beta & \geq n + 1,
\end{align*}
\]

then there exists a tree \( T \) with \( n \) nodes such that \( P(T) = \alpha \) and \( C(T) = \beta \).

**Proof.** Let \( \nu = n - \alpha - \beta \). It follows from (1) that \( \beta + \nu \leq (1/2)n \) and this implies that \( n + 1 - 2\beta - 2\nu \geq 1 \); furthermore, it follows from (3) that \( \nu \leq \beta - 1 \) or \( \beta - 1 - \nu \geq 0 \). Let \( T \) denote the tree constructed as follows: \( n - 1 \) nodes are split into \( \nu \) sets of four nodes, \( \beta - 1 - \nu \) sets of two nodes, and \( n + 1 - 2\nu - 2\beta \) sets consisting of a single node; a path is formed on the nodes in each set and the node at one end of each of these paths is joined to an \( n \)th node. (The tree arising when \( n = 13, \alpha = 7, \) and \( \beta = 4 \) is illustrated in Figure 1.) It is not difficult to verify that this construction

\[
\begin{align*}
P(T) = 2\nu + (\beta - 1 - \nu) + (n + 1 - 2\nu - 2\beta) &= n - \beta - \nu = \alpha \\
C(T) = 1 + \nu + (\beta - 1 - \nu) &= \beta,
\end{align*}
\]
as required.

3. Upper bounds for $P_k$ and $C_k$. In what follows $k$ and $n$ will denote arbitrary positive integers.

**Theorem 4.** If $n \geq \lceil 1/2(k + 3) \rceil$ then

\begin{align*}
(4) & \quad P_k \leq \lceil 2n/(k + 2) \rceil \\
& \text{if $k$ is even, and }
(5) & \quad P_k \leq \lceil (2n - 2)/(k + 1) \rceil \\
& \text{if $k$ is odd.}
\end{align*}

**Proof.** If $x$ is any node in any $k$-packing $\mathcal{P}$ with $P_k$ nodes of $T$, let $N(x) = \{u: u \in T \text{ and } d(x, u) \leq j\}$ where $j = \lfloor 1/2k \rfloor$. Since the tree $T$ is connected and has at least $\lceil 1/2(k + 3) \rceil \geq j + 1$ nodes, it follows that $|N(x)| \geq j + 1$ for all $x \in \mathcal{P}$. Furthermore, the sets \{N(x): x \in \mathcal{P}\} are disjoint; for if $u \in N(x) \cap N(y)$ where $x \neq y$, then $d(x, y) = d(x, u) + d(u, y) \leq 2j + 1 = k$ and this would contradict the definition of $\mathcal{P}$. Hence,

$$n \geq \sum_{x \in \mathcal{P}} |N(x)| \geq P_k \cdot (j + 1)$$

and this implies inequality (4).

If $k = 2j + 1$ we may further assert that no edge joins a node $u$ of any set $N(x)$ to a node $v$ of any other set $N(y)$ where $x \neq y$; for if there were such an edge, then $d(x, y) = d(x, u) + d(u, v) + d(v, y) \leq 2j + 1 = k$ and this would again contradict the definition of $\mathcal{P}$. If $P_k = 1$ inequality (5) certainly holds. If $P_k \geq 2$ there must exist at least one node of $T$ that is not in any set $N(x)$, where $x \in \mathcal{P}$, for $T$ would not be connected otherwise. Hence,

$$n \geq 1 + \sum_{x \in \mathcal{P}} |N(x)| \geq 1 + P_k \cdot (j + 1)$$

when $k = 2j + 1$, and this implies inequality (5).

If $\mathcal{P}$ is any maximal $k$-packing of $P_k$ nodes in a tree $T$, then $\mathcal{P}$ is also a $k$-covering of $T$; for if there were a node $x$ in $T$ such that $d(x, y) > k$ for each node $y$ in $\mathcal{P}$ then $\mathcal{P} \cup \{x\}$ would be a larger $k$-packing in $T$ which is impossible. This implies that $C_k \leq P_k$ for any tree $T$ (this result is given in [5; p. 211] when $k = 1$, as was mentioned earlier). Hence, Theorem 4 provides an upper bound for $C_k$ also; a better bound is given in the following result.

**Theorem 5.** If $n \geq k + 1$, then
Proof. Suppose one of the longest paths in $T$ joins nodes $x$ and $y$. Let

$$D_i = \{ u: u \in T \text{ and } d(x, u) \equiv i \text{ (mod } k+1) \}$$

for $0 \leq i \leq k$. We may assume $D_i \neq \emptyset$ for each $i$, for otherwise the node $x$ itself would constitute a $k$-covering and inequality (6) would certainly hold. We now show that each set $D_i$ is a $k$-covering of $T$.

Let $z$ denote any node of $T$ and suppose $d(x, z) = l$. If $l \geq i$ then $i + m(k+1) \leq l < i + (m + 1)(k + 1)$ for some nonnegative integer $m$. Let $u$ denote the unique node on the path joining $x$ and $z$ such that $d(x, u) = i + m(k + 1)$; then $u \in D_i$ and $d(u, z) \leq k$ as required. If $l < i$ let $v$ denote the unique node on the path joining $x$ and $y$ such that $d(x, v) = i$; then $v \in D_i$ and

$$d(z, v) = d(z, y) - d(v, y) \leq d(x, y) - d(v, y) = d(x, v) = i \leq k,$$

as required.

The $k$-coverings $\{D_i: 0 \leq i \leq k\}$ are disjoint and together they exhaust the nodes of $T$; hence, at least one of them has at most $[n/(k + 1)]$ nodes. This suffices to complete the proof of the theorem.

It is not difficult to construct trees for which equality holds in (4), (5), and (6) for all admissible values of $k$ and $n$.

4. A relation between $P_k$ and $C_k$.

Theorem 6. If $n \geq k + 1$, then

$$P_k + kC_k \leq n.$$  

Proof. If $k = 1$ this is the same as Theorem 1, so we shall assume henceforth that $k \geq 2$.

Let $\mathcal{P}$ denote a $k$-packing of $P_k$ nodes in $T$. If $x \in \mathcal{P}$ let $E(x) = \{ u: u \in T \text{ and } d(u, x) = 1 \}$; these sets are nonempty and disjoint when $k \geq 2$ and no edge joins two nodes of the same set $E(x)$. Select one node $u_x$ from each set $E(x)$ and let $R$ denote the graph obtained from $T$ as follows: remove each node $x$ of $\mathcal{P}$ and all edges incident with $x$, and insert new edges joining each node $u_x$ to each of the other nodes of $E(x)$. It is not difficult to see that $R$ is a tree with $n - P_k$ nodes.

If $r$ and $s$ are nodes in $E(x)$ and $E(y)$, respectively, where $x \neq$
y, then $d(r, s) \geq k - 1$; for, if $d(r, s) \leq k - 2$ then $d(x, y) \leq k$ and this would contradict the definition of $S$. This implies the following observation:

(*) If a path in $R$ of length at most $k - 1$ contains a new edge of the type $ru_s$ where $r \in E(x)$, then the path does not contain any nodes of any other set $E(y)$ where $y \neq x$.

Let $C$ denote any smallest $(k - 1)$-covering of $R$. We shall show that the nodes of $C$ constitute a $k$-covering of $T$. Let $z$ denote any node of $T$. If $z \in C$ then $z \in R$ and there exists a node $v \in C$ such that $d(v, z) \leq k - 1$ in $R$. If there are no new edges in the path $p(v, z)$ from $v$ to $z$ in $R$ then all the edges of $p(v, z)$ are in $T$ and $d(v, z) \leq k - 1$ in $T$ also. If there is just one new edge in the path $p(v, z)$ of the type $ru_s$ where $r \in E(x)$, then $d(v, z) \leq k$ in $T$ since the edge $ru_s$ can be replaced by the two edges $rx$ and $xsu_s$ in $T$. If there is more than one new edge in the path $p(v, z)$ then these new edges must all join pairs of nodes from the same set $E(x)$, in view of observation (*). But all new edges of this type are incident with the node $u_s$. Hence, there can be only two such edges in $p(v, z)$, they must occur consecutively, and they must be of the form $ru_s$ and $u_sx$. But then $d(v, z) \leq k - 1$ in $T$ also since the edges $ru_s$ and $u_sx$ in $p(v, z)$ can be replaced by the edges $rx$ and $xs$ in $T$.

If $z \in C$ then there exist nodes $r \in E(z)$ and $v \in C$ such that $d(v, r) \leq k - 1$ in $R$ and the path $p(v, r)$ from $v$ to $r$ does not pass through any other nodes of $E(z)$. This path cannot contain any new edges by observation (*). Hence, $d(v, z) = d(v, r) + 1 \leq k$ in $T$, as required.

If $n = k + 1$ then $C_k = P_k = 1$ and inequality (7) certainly holds. If $n \geq k + 2$, it follows from Theorem 4 that $n - P_k \geq k$. Hence, when $n \geq k + 2$, we may apply Theorem 5 to the tree $R$ and conclude that $|C| \leq (n - P_k)/k$. Since $C_k \leq |C|$, this implies that $P_k + kC_k \leq n$, as required.

We now show that inequality (7) is best possible when $n = m(k + 1)$ for $m = 1, 2, \ldots$. Let $H$ denote the tree with $n$ nodes constructed as follows: the $n$ nodes are split into $m$ sets of $k + 1$ nodes each; a path of length $k$ is formed on the nodes in each set; and, finally, the nodes at one end of these paths are joined so as to form a path of length $m - 1$. (The tree $H$ arising when $n = 20$ and $k = 3$ is illustrated in Figure 2.) It is not difficult to verify that $P_k + kC_k = m + km = n$ for the tree $H$. We leave it as an exercise for the reader to show that there exists a tree with $n$ nodes for which $C_k =$
\[(n - P_k)/k\] for arbitrary values of \(n\) and \(k\) such that \(n \geq k + 1\).

No inequality of the type
\[(1 + \varepsilon)P_k + (k - \varepsilon)C_k \leq n ,\]
where \(\varepsilon\) is any positive constant, can be valid for all trees with sufficiently many nodes. To show this let \(J\) denote the tree with \(n = m(k + 1) + 1\) nodes formed by joining a new node to one of the nodes of \(H\) in the way illustrated in Figure 3 when \(n = 21\) and \(k = 3\). It is easy to see that

\[(1 + \varepsilon)P_k + (k - \varepsilon)C_k = (1 + \varepsilon)(m + 1) + (k - \varepsilon)m = n + \varepsilon\]

for the tree \(T\). It might be of some interest to determine best possible upper bounds in terms of \(n\) for \(lP_k + (k + 1 - l)C_k\) when \(l > 1\).

It might also be of some interest to determine best possible upper bounds in term of \(n\) for \(P_k + C_k\). It follows from Theorems 4 and 6 that \(P_k + C_k \leq 3n/(k + 2)\) when \(k\) is even, but this is probably not best possible in general.

There does not seem to be any natural nontrivial analogue of Theorem 2 when \(k \geq 2\), at least one that does not involve additional parameters or assumptions, since it is easy to construct trees for which \(P_k = C_k = 1\) when \(k \geq 2\).

5. The equality of \(P_{2k}\) and \(C_k\).
THEOREM 7. If \( k \geq 1 \), then \( P_{2k} = C_k \).

Proof. Let \( \mathcal{P} \) denote a \( 2k \)-packing consisting of \( P_{2k} \) nodes of the tree \( T \) and let \( \mathcal{C} \) denote a \( k \)-covering consisting of \( C_k \) nodes of \( T \). It is easy to see that for each node \( x \) in \( \mathcal{C} \) the set \( N(x) = \{ u : u \in T \text{ and } d(x, u) \leq k \} \) contains at most one node \( y \) in \( \mathcal{P} \). Since every node \( y \) in \( \mathcal{P} \) belongs to at least one set \( N(x) \) it follows that \( P_{2k} \leq C_k \). It remains to show that \( P_{2k} \geq C_k \).

Let \( \{ x_0, x_1, \ldots, x_m \} \) denote the nodes of any longest path in the tree \( T \). If \( m \leq 2k \), then \( P_{2k} = C_k = 1 \); so we may suppose that \( m \geq 2k + 1 \). Let \( T' \) denote the smallest subtree of \( T \) containing all nodes \( z \) of \( T \) such that \( d(x_k, z) > k \); that is, \( T' \) is the subtree determined by all nodes \( v \) of \( T \) such that either \( d(x_k, v) > k \) or there exists some node, say \( z_v \), such that \( d(x_k, z_v) > k \) and the unique path joining \( z_v \) and \( x_m \) in \( T \) contains \( v \). The subtree \( T' \) is nonempty since \( d(x_k, x_m) > k \).

Let \( \mathcal{P}' \) denote a largest \( 2k \)-packing consisting of \( P'_{2k} \) nodes of \( T' \) and let \( \mathcal{C}' \) denote a smallest \( k \)-covering consisting of \( C'_k \) nodes of \( T' \). It is easy to see that \( \mathcal{C} = \mathcal{C}' \cup \{ x_k \} \) is a \( k \)-covering of \( T \); consequently,

\[
C_k \leq C'_k + 1.
\]

Suppose there exists a node \( y \) in \( \mathcal{P}' \) such that \( d(x_k, y) \leq k \). Let \( B_y \) denote the subtree of \( T' \) determined by all nodes \( s \) of \( T' \) such that the unique path from \( s \) to \( x_m \) contains \( y \); in particular, the node \( z_s \), defined earlier, is in \( B_y \). We assert that \( y \) is the only node of \( \mathcal{P}' \) in \( B_y \). For, if there were a second such node, say \( w \), then \( d(w, y) \geq 2k + 1 \); this would imply that

\[
d(w, x_m) = d(w, y) + d(y, x_m) \geq 2k + 1 + d(x_k, x_m) - d(y, x_k)
\geq 2k + 1 + m - k - k = m + 1,
\]

contradicting the assumption that \( \{ x_0, x_1, \ldots, x_m \} \) was a longest path in \( T \).

The foregoing observations imply that we may replace each node \( y \) in \( \mathcal{P}' \) for which \( d(x_k, y) \leq k \) by a node \( z_y \) in \( T' \) for which \( d(x_k, z_y) > k \) and still have a \( 2k \)-packing. We may thus suppose, without loss of generality, that \( d(x_k, y) > k \) for every node \( y \) in \( \mathcal{P}' \); this implies that \( d(x_0, y) > 2k \) for every node \( y \) in \( \mathcal{P}' \). Thus the set \( \mathcal{P}' \cup \{ x_0 \} \) is a \( 2k \)-packing of \( T \) and, consequently,

\[
P_{2k} \geq P'_{2k} + 1.
\]

The tree \( T' \) has fewer nodes than \( T \) so we may assume, as our
induction hypothesis, that

\[(10) \quad P_{2k} \geq C_k.\]

It now follows, by inequalities (8), (9), and (10) that \( P_{2k} \geq C_k \), as required, and this completes the proof of the theorem.

Theorems 6 and 7 imply the following result.

**Corollary 3.** If \( n \geq k + 1 \), then

\[ P_k + kP_{2k} \leq n; \]

if \( n > 2k + 1 \), then

\[ C_k + 2kC_{2k} \leq n. \]

We remark that in general these packing and covering sets are not identical; in particular, for arbitrary \( k \) it is easy to construct a tree none of whose largest \( 2k \)-packings are smallest \( k \)-coverings. Furthermore, trees are not the only graphs \( G \) with the property that \( P_{2k}(G) = C_k(G) \) for all \( k \). For example, any graph with a node joined to all the remaining nodes has this property. It seems difficult to characterize such graphs in general.

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