NOTES ON STABLE CURRENTS

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With additional assumptions we answer a conjecture proposed by Lawson and Simons.

In a work [5], H. B. Lawson, Jr. and J. Simons proved that there exist no stable rectifiable currents on an $n$-dimensional unit sphere $S^n$ in the $(n + 1)$-dimensional Euclidean space $R^{n+1}$. And concerning to this fact, they conjectured as follows.

Conjecture. Let $M$ be a compact, simply-connected Riemannian manifold with the sectional curvature satisfying $1/4 < K_\sigma \leq 1$ for all tangent two planes $\sigma$. Then there exist no stable rectifiable currents on $M$.

We obtain the following results with respect to this conjecture.

Let $M$ be a compact, connected $n$-dimensional Riemannian manifold isometrically immersed in $(n + 1)$-dimensional Euclidean space $R^{n+1}$. Let $\delta$ be a constant with $0 < \delta \leq 1$, and suppose that at each point $x$ of $M$, with respect to a suitable unit normal, every principal curvature $\lambda_j$ of $M$ satisfies

$$\sqrt{\delta} \leq \lambda_j \leq 1$$

for $j = 1, \cdots, n$.

THEOREM. Let $M$ be a compact, connected Riemannian manifold satisfying the conditions expressed above. Associate to each $\mathcal{S} \in \mathcal{H}_p(M)$ a quadratic form $Q_\mathcal{S}$ on $\mathcal{T}$ as follows. For $V \in \mathcal{T}$, let $\phi_t$ be the flow generated by $V$ and set

$$Q_\mathcal{S}(V) = \frac{d^2}{dt^2} M(\phi_{t_0} \mathcal{S})_{t=0}.$$ 

Then for all $\mathcal{S} \in \mathcal{H}_p(M)$

$$\text{tr } Q_\mathcal{S} \leq p(p + 1 - n\delta - \delta)M(\mathcal{S}).$$

(For definitions of $\mathcal{T}$ and $\mathcal{H}_p(M)$, see below.)

COROLLARY 1. Under the assumptions of the Theorem, for all $p$ with $1 \leq p < n\delta + \delta - 1$, any rectifiable $p$-current $\mathcal{S} \in \mathcal{H}_p(M)$ is not stable. If $\delta$ satisfies $n/(n + 1) < \delta \leq 1$, then any rectifiable $p$-current $\mathcal{S} \in \mathcal{H}_p(M)$ is not stable for each $p$ with $1 \leq p \leq n - 1$. 

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COROLLARY 2. Under the assumptions of the Theorem, if \( \delta \) satisfies \( n/(n + 1) < \delta \leq 1 \), then

\[ H_p(M; \mathbb{Z}) = H_p(S^n; \mathbb{Z}) \]

for each \( p \) with \( 0 \leq p \leq n \). Therefore, in particular, if \( n = 2 \) or \( n \geq 5 \), then \( M \) is homeomorphic to \( S^n \). (This conclusion follows from weaker conditions.)

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1. In the following, we use the same notation as in [5]. Also see [5] for detailed definitions. Let \( M \) be a compact \( n \)-dimensional Riemannian manifold with Riemannian metric \( g \) and canonical connection \( \nabla \). For a point \( x \in M \), \( T_x(M) \) denotes the tangent space of \( M \) at \( x \). Let \( \mathcal{R}_p(M) \) be the set of all rectifiable \( p \)-currents on \( M \), where \( 0 \leq p \leq n \). For a current \( \mathcal{S} \in \mathcal{R}_p(M) \), \( \mathcal{T}_\mathcal{S} \) denotes an orientation of \( \mathcal{S} \), that is, for \( \mathcal{H}^p \)-almost all \( x \in \mathcal{S} \), \( \mathcal{T}_\mathcal{S} \) is a simple \( p \)-vector of unit length which represents \( T_x(\mathcal{S}) \), where \( \mathcal{H}^p \) is the Hausdorff \( p \)-measure on \( M \). Let \( V \) be a smooth vector field on \( M \). We define a linear mapping \( \mathcal{A}^r : T_x(M) \to T_x(M) \) by \( \mathcal{A}^r(X) = F \cdot V \) for \( X \in T_x(M) \). This mapping can be extended uniquely as a derivation to \( \Lambda^p T_x(M) \), that is, as a linear map \( \mathcal{A}^r : \Lambda^p T_x(M) \to \Lambda^p T_x(M) \) which for simple vectors is given by

\[ \mathcal{A}^r(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^p X_1 \wedge \cdots \wedge X_{i-1} \wedge \mathcal{A}^r X_i \wedge X_{i+1} \wedge \cdots \wedge X_p. \]

At \( x \in M \), we define also the linear map \( \mathcal{F}_r \cdot V : T_x(M) \to T_x(M) \) by \( \mathcal{A}_r \cdot V = \mathcal{F}_r \cdot \mathcal{F} - \mathcal{F}_r \cdot \mathcal{V} \) for \( X \in T_x(M) \), where \( \mathcal{X} \) is any extension of \( X \) to a local vector field. The definition is independent of any extension \( \mathcal{X} \), and the map carries over uniquely as a derivation to \( \Lambda^p T_x(M) \). Consider a current \( \mathcal{S} \in \mathcal{R}_p(M) \) and a vector field \( V \) on \( M \). Let \( \phi_t : M \to M \), \( t \in R \) be the 1-parameter group of diffeomorphisms generated by \( V \). Then for each \( t \in R \) we have the rectifiable current \( \phi_{it}(\mathcal{S}) \) which, as a linear functional on \( \Lambda^p(M) \), is given by

\[ (\phi_{it} \mathcal{S})(\omega) = \mathcal{S}(\phi_t^\ast \omega) \]

for \( \omega \in \Lambda^p(M) \), where \( \Lambda^p(M) \) is the space of all smooth \( p \)-forms on \( M \). Let \( M \) denote the usual norm of a linear functional on \( \Lambda^p(M) \) which has the usual Frechet topology. Then,

\[ M(\phi_{it} \mathcal{S}) = \int_M \sqrt{(\phi_t^\ast g)(\mathcal{S}, \mathcal{S})} d || \mathcal{S} || \]
where \( \|\mathcal{S}\| \) is a measure on \( M \) defined, by using the \( p \)-dimensional Hausdorff measure \( \mathcal{H}^p \) on \( M \), as follows: for a Borel set \( X \subset M \),
\[
\|\mathcal{S}\|(X) = \mathcal{H}^p(X \cap \mathcal{S}).
\]

**Definition.** A rectifiable \( p \)-current \( \mathcal{S} \in \mathcal{R}_p(M) \) is said to be stable if, for each vector field \( V \) the following two conditions hold:

\[
\begin{align*}
(s_1) & \quad \frac{d}{dt} M(\phi_{t\mathcal{S}})_{t=0} = 0 , \\
(s_2) & \quad \frac{d^2}{dt^2} M(\phi_{t\mathcal{S}})_{t=0} \geq 0 .
\end{align*}
\]

The following is obtained by Lawson and Simons in [5].

**Proposition 1.** Let \( M \) be a compact Riemannian manifold and \( V \) a vector field on \( M \) with associated flow \( \phi_t \). Then for any rectifiable \( p \)-current \( \mathcal{S} \in \mathcal{R}_p(M) \),

\[
\begin{align*}
(1) & \quad \frac{d}{dt} M(\phi_{t\mathcal{S}})_{t=0} = \int_M \langle \mathcal{S}, \mathcal{S} \rangle d\|\mathcal{S}\| , \\
(2) & \quad \frac{d^2}{dt^2} M(\phi_{t\mathcal{S}})_{t=0} = \int_M \{ -\langle \mathcal{S}V, \mathcal{S} \rangle \}^2 + \langle \mathcal{S}V, \mathcal{S} \rangle^2 + \langle \mathcal{S}V, \mathcal{S} \rangle \} d\|\mathcal{S}\|.
\end{align*}
\]

**Remark.** In the special case that \( V = \nabla f (= \) the gradient of \( f \) for some \( f \in C^\infty(M) \), the transformation \( \mathcal{S}V \) is symmetric and (2) simplifies to

\[
(2)' \quad \frac{d^2}{dt^2} M(\phi_{t\mathcal{S}})_{t=0} = \int_M \{ -\langle \mathcal{S}V, \mathcal{S} \rangle \}^2 + 2 |\mathcal{S}V| \}^2 + \langle F_{r,s} V, \mathcal{S} \rangle \} d\|\mathcal{S}\|.
\]

For future reference we shall write the integrand of (2)' at \( x \in M \) in terms of tangent vectors at \( x \). Let \( \{e_1, \ldots, e_p, e_{p+1}, \ldots, e_n\} \) be an orthonormal basis of \( T_x(M) \) and set \( \xi = e_1 \wedge \cdots \wedge e_p \). Then

\[
-\langle \mathcal{S}V, \xi \rangle^2 + 2 |\mathcal{S}V| \}^2 + \langle F_{r,s} V, \xi \rangle \}
= \left\{ \sum_{j=1}^p \langle \mathcal{S}V(e_j), e_j \rangle \right\}^2 + 2 \sum_{j=1}^p \sum_{a=p+1}^n \langle \mathcal{S}V(e_j), e_a \rangle^2
\]

\[
+ \sum_{i=1}^p \langle F_{r,s} V, e_j \rangle .
\]

where \( |\mathcal{S}V(\xi)| \) denotes the length of \( p \)-vector \( \mathcal{S}V(\xi) \).

2. Now we assume that \( M \) is isometrically immersed in \( (n + 1)-
dimensional Euclidean space $\mathbb{R}_{n+1}$ with canonical Riemannian metric $\langle \cdot , \cdot \rangle$ and canonical Riemannian connection $\nabla$. For all local formulas we may consider the isometric immersion $f$ of $M$ into $\mathbb{R}_{n+1}$ as an imbedding and thus identify $x \in M$ with $f(x) \in \mathbb{R}_{n+1}$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(\mathbb{R}_{n+1})$. The normal space $T_x \Sigma$ is the subspace of $T_x(\mathbb{R}_{n+1})$ consisting of all $\zeta \in T_x(\mathbb{R}_{n+1})$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric $\langle \cdot , \cdot \rangle$. For each point $x$ of $M$, choose a field $\zeta$ of unit normal vectors defined on a neighborhood $U$ of $x$. Then we have the basic formulas

$$
\nabla_x Y = \nabla_x Y + \langle A_\zeta X, Y \rangle \zeta
$$
$$
\nabla_x \zeta = -A_\zeta X
$$

where $X$ and $Y$ are smooth vector fields tangent to $M$, and $A_\zeta$ is a tensor field of type $(1, 1)$, called the second fundamental form associated with $\zeta$. The Gauss equation expresses the curvature tensor $R$ of $M$ as follows.

$$
R(X, Y)Z = \langle A_\zeta Y, Z \rangle A_\zeta X - \langle A_\zeta X, Z \rangle A_\zeta Y
$$

where $X$, $Y$ and $Z$ are smooth vector fields tangent to $M$.

Let $\delta$ be a constant with $0 < \delta \leq 1$, and suppose that at each point $x$ of $M$, with respect to a suitable field $\zeta$ of unit normals, every principal curvature $\lambda_j$ of $M$ satisfies $\sqrt{\delta} \leq \lambda_j \leq 1$, $j = 1, \ldots, n$.

**Remark.** The above assumption implies that $M$ has the sectional curvature satisfying $\delta \leq K_x \leq 1$ for all tangent two planes $\sigma$. And from the continuity of the eigen-values of the linear map $A_\zeta: T_x(M) \to T_x(M)$, called the principal curvatures of $M$, the above assumption also implies that $M$ is orientable. Therefore we can choose a global field $\zeta$ of unit normals on $M$ which satisfies the above condition, and then we can write $A_\zeta = A$.

3. To estimate the left hand side of (s) we begin with the space of functions $\mathcal{F} = \{ \psi | M; \psi: \mathbb{R}_{n+1} \to \mathbb{R} \text{ is linear} \}$, and define

$$
\mathcal{V} = \{ \nabla \psi; \psi \in \mathcal{F} \}.
$$

Then there is a natural isomorphism

(4)

$$
\mathcal{V} \cong \mathbb{R}^{n+1}
$$

which associates to $\psi \in \mathbb{R}_{n+1}$ the gradient of the function $\psi_x(x) = \langle \psi, x \rangle$ on $M$. This identification introduces a natural inner product on $\mathcal{V}$.

To any simple unit $p$-vector $\zeta \in \Lambda^p T_x(M)$, at any $x \in M$, we can associate a quadratic form $Q_\zeta$ on $\mathcal{V}$ as follows. For $V \in \mathcal{V}$, let $\phi_1, \ldots, \phi_{n+1}$ be
be the flow generated by \( V \), and define
\[
Q_\varepsilon(V) = \frac{d^2}{dt^2} |\phi_{tv}\zeta|_{t=0}.
\]
Then we have the following.

**Proposition 2.** Under the assumptions as expressed above, we have
\[
\text{tr} \, Q_\varepsilon \leq p(p + 1 - n\delta - \delta).
\]

*Proof.* Suppose \( V \in T^\varepsilon \) corresponds to \( v \in R^{n+1} \) under the isomorphism (4). Then at any \( y \in M \)
\[
V(y) = v - \langle v, \zeta_y \rangle \zeta_y,
\]
and then for \( X \in T_s(M), \mathcal{P}_X V = (\mathcal{P}_X V)^r = \langle v, \zeta_{\alpha} \rangle AX \), where \( (\cdot)^r \) denotes orthogonal projection \( T_s(R^{n+1}) \rightarrow T_s(M) \). Thus,
\[
(5) \quad \mathcal{A}^\varepsilon r(X) = \mathcal{P}_X V = \langle v, \zeta_{\alpha} \rangle AX.
\]
And it follows easily that
\[
(6) \quad \mathcal{R}_r X V = -\langle V, AX \rangle AX + \langle v, \zeta_{\alpha} \rangle \mathcal{P}_r (A\mathcal{X}) - \langle v, \zeta_{\alpha} \rangle A(\mathcal{P}_r \mathcal{X})
\]
where \( \mathcal{X} \) is any extension of \( X \) to a local vector field.

We now choose an orthonormal basis \( \{x_0 = \zeta_{\alpha}, x_1 = e_{1}, \ldots, x_n = e_{n}\} \)
for \( R^{n+1} \), where \( e_j \) is an eigenvector corresponding to the eigenvalue \( \lambda_{\alpha} \) of \( A \). Via (4) this fixes an orthonormal basis \( \{V_0, V_1, \ldots, V_n\} \) of \( T^\varepsilon \). It then follows from (5) and (6) that \( \mathcal{P}_{r} V_0 = \mathcal{A}^\varepsilon r 1 = \cdots = \mathcal{A}^\varepsilon r n = 0 \) and \( \mathcal{A}^\varepsilon r 0 = A, \mathcal{P}_{r} V_j = -\lambda_{\alpha} A, j = 1, \ldots, n \), as transformations of \( T_s(M) \). For given simple unit \( p \)-vector \( \xi \in \wedge^p T_s(M) \), we can choose an orthonormal basis \( \{\bar{e}_1, \ldots, \bar{e}_p, \bar{e}_{p+1}, \ldots, \bar{e}_q\} \)
of \( T_s(M) \) with \( \xi = \bar{e}_1 \wedge \cdots \wedge \bar{e}_p \). It then follows from (2), (2)', (3) and above formulas that
\[
\text{tr} \, (Q_\varepsilon) = \sum_{l=0}^{n} Q_\varepsilon(V_l)
\]
\[
= \sum_{l=0}^{n} \left\{ \left( \sum_{j=1}^{p} \left( A_{\bar{e}_j, \bar{e}_l} \right) \right)^2 + 2 \sum_{j=1}^{p} \sum_{a=p+1}^{n} \left( A_{\bar{e}_j, \bar{e}_a} \right)^2 \right. \\
+ \left. \sum_{j=1}^{p} \left( \mathcal{P}_{r_{k_j}} V_k, \bar{e}_l \right) \right\}
\]
\[
= \left( \sum_{j=1}^{p} \left( A_{\bar{e}_j, \bar{e}_j} \right) \right)^2 + 2 \left( \sum_{j=1}^{p} \sum_{a=p+1}^{n} \left( A_{\bar{e}_j, \bar{e}_a} \right)^2 \right. - \left. \sum_{j=1}^{p} \sum_{a=p+1}^{n} \left( \lambda_{\alpha} A_{\bar{e}_j, \bar{e}_a} \right) \right)
\]
\[
= \left( \sum_{j=1}^{p} \left( A_{\bar{e}_j, \bar{e}_j} \right) \right)^2 + 2 \sum_{j=1}^{p} \left( |A_{\bar{e}_j}|^2 - \sum_{i=1}^{p} \left( A_{\bar{e}_i, \bar{e}_j} \right)^2 \right)
\]
\[
- \sum_{j=1}^{p} \sum_{a=p+1}^{n} \lambda_{\alpha} A_{\bar{e}_j, \bar{e}_a}.
\]
\[ = 2 \sum_{i=1}^{p} |Ae_j|^2 + \sum_{i,j=1}^{p} \left( \langle Ae_i, e_i \rangle \langle Ae_j, e_j \rangle - 2 \langle Ae_i, e_j \rangle \right) \]
\[ - \sum_{i=1}^{p} |Ae_j|^2 - \sum_{i,j=1}^{p} \lambda_i \langle Ae_j, e_j \rangle . \]

By the assumption, \( \sqrt{\delta} \leq \lambda_j \leq 1, j = 1, j = 1, \ldots, n, \) we get \( |Ae_j|^2 \leq 1, \) and \( \sqrt{\delta} \leq \langle Ae_j, e_j \rangle \leq 1 \) for \( 1, \ldots, n. \) Thus we have
\[ \text{tr} (Q_i) \leq 2p + p(p - 1) - p\delta - np\delta \]
\[ = p(p + 1 - n\delta - \delta) . \]

Combining Proposition 1 and Proposition 2 we get the theorem and the Corollary 1. And by virtue of the basic theorems on integral currents, we have the Corollary 2, see [2] or [5].

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Toyama University
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