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NOTES ON STABLE CURRENTS

HIROSHI MORI

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With additional assumptions we answer a conjecture proposed by Lawson and Simons.

In a work [5], H. B. Lawson, Jr. and J. Simons proved that there exist no stable rectifiable currents on an n -dimensional unit sphere S^n in the $(n + 1)$ -dimensional Euclidean space R^{n+1} . And concerning to this fact, they conjectured as follows.

Conjecture. Let M be a compact, simply-connected Riemannian manifold with the sectional curvature satisfying $1/4 < K_s \leq 1$ for all tangent two planes σ . Then there exist no stable rectifiable currents on M .

We obtain the following results with respect to this conjecture.

Let M be a compact, connected n -dimensional Riemannian manifold isometrically immersed in $(n + 1)$ -dimensional Euclidean space R^{n+1} . Let δ be a constant with $0 < \delta \leq 1$, and suppose that at each point x of M , with respect to a suitable unit normal, every principal curvature λ_j of M satisfies

$$\sqrt{\delta} \leq \lambda_j \leq 1$$

$$j = 1, \dots, n.$$

THEOREM. *Let M be a compact, connected Riemannian manifold satisfying the conditions expressed above. Associate to each $\mathcal{S} \in \mathcal{R}_p(M)$ a quadratic form $Q_{\mathcal{S}}$ on \mathcal{V} as follows. For $V \in \mathcal{V}$, let ϕ_t be the flow generated by V and set*

$$Q_{\mathcal{S}}(V) = \frac{d^2}{dt^2} M(\phi_{t*} \mathcal{S})|_{t=0}.$$

Then for all $\mathcal{S} \in \mathcal{R}_p(M)$

$$\text{tr } Q_{\mathcal{S}} \leq p(p + 1 - n\delta - \delta)M(\mathcal{S}).$$

(For definitions of \mathcal{V} and $\mathcal{R}_p(M)$, see below.)

COROLLARY 1. *Under the assumptions of the Theorem, for all p with $1 \leq p < n\delta + \delta - 1$, any rectifiable p -current $\mathcal{S} \in \mathcal{R}_p(M)$ is not stable. If δ satisfies $n/(n + 1) < \delta \leq 1$, then any rectifiable p -current $\mathcal{S} \in \mathcal{R}_p(M)$ is not stable for each p with $1 \leq p \leq n - 1$.*

COROLLARY 2. Under the assumptions of the Theorem, if δ satisfies $n/(n+1) < \delta \leq 1$, then

$$H_p(M; Z) = H_p(S^n; Z)$$

for each p with $0 \leq p \leq n$. Therefore, in particular, if $n = 2$ or $n \geq 5$, then M is homeomorphic to S^n . (This conclusion follows from weaker conditions.)

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1. In the following, we use the same notation as in [5]. Also see [5] for detailed definitions. Let M be a compact n -dimensional Riemannian manifold with Riemannian metric g and canonical connection ∇ . For a point $x \in M$, $T_x(M)$ denotes the tangent space of M at x . Let $\mathcal{R}_p(M)$ be the set of all rectifiable p -currents on M , where $0 \leq p \leq n$. For a current $\mathcal{S} \in \mathcal{R}_p(M)$, $\vec{\mathcal{S}}_x$ denotes an orientation of \mathcal{S} , that is, for \mathcal{H}^p -almost all $x \in \mathcal{S}$, $\vec{\mathcal{S}}_x$ is a simple p -vector of unit length which represents $T_x(\mathcal{S})$, where \mathcal{H}^p is the Hausdorff p -measure on M . Let V be a smooth vector field on M . We define a linear mapping $\mathcal{A}^V: T_x(M) \rightarrow T_x(M)$ by $\mathcal{A}^V(X) := \nabla_X V$ for $X \in T_x(M)$. This mapping can be extended uniquely as a derivation to $\Lambda^p T_x(M)$, that is, as a linear map $\mathcal{A}^V: \Lambda^p T_x(M) \rightarrow \Lambda^p T_x(M)$ which for simple vectors is given by

$$\mathcal{A}^V(X_1 \wedge \cdots \wedge X_p) = \sum_{i=1}^p X_1 \wedge \cdots \wedge X_{i-1} \wedge \mathcal{A}^V X_i \wedge X_{i+1} \wedge \cdots \wedge X_p.$$

At $x \in M$, we define also the linear map $\nabla_V \cdot V: T_x(M) \rightarrow T_x(M)$ by $\Delta_{V,x} V := \nabla_V \nabla_{\tilde{X}} V - \nabla_{\Delta_V \tilde{X}} V$ for $X \in T_x(M)$, where \tilde{X} is any extension of X to a local vector field. The definition is independent of any extension \tilde{X} , and the map carries over uniquely as a derivation to $\Lambda^p T_x(M)$. Consider a current $\mathcal{S} \in \mathcal{R}_p(M)$ and a vector field V on M . Let $\phi_t: M \rightarrow M$, $t \in \mathcal{R}$ be the 1-parameter group of diffeomorphisms generated by V . Then for each $t \in \mathcal{R}$ we have the rectifiable current $\phi_{t\#}(\mathcal{S})$ which, as a linear functional on $\Lambda^p(M)$, is given by

$$(\phi_{t\#} \mathcal{S})(\omega) = \mathcal{S}(\phi_t^* \omega)$$

for $\omega \in \Lambda^p(M)$, where $\Lambda^p(M)$ is the space of all smooth p -forms on M . Let $\|\cdot\|$ denote the usual norm of a linear functional on $\Lambda^p(M)$ which has the usual Fréchet topology. Then,

$$M(\phi_{t\#} \mathcal{S}) = \int_M \sqrt{(\phi_t^* g)(\vec{\mathcal{S}}, \vec{\mathcal{S}})} d\|\mathcal{S}\|$$

where $\|\mathcal{S}\|$ is a measure on M defined, by using the p -dimensional Hausdorff measure \mathcal{H}^p on M , as follows: for a Borel set $X \subset M$, $\|\mathcal{S}\|(X) = \mathcal{H}^p(X \cap \mathcal{S})$.

DEFINITION. A rectifiable p -current $\mathcal{S} \in \mathcal{R}_p(M)$ is said to be stable if, for each vector field V the following two conditions hold:

$$(s_1) \quad \frac{d}{dt} \mathbf{M}(\phi_{t\#} \mathcal{S})|_{t=0} = 0 ,$$

$$(s_2) \quad \frac{d^2}{dt^2} \mathbf{M}(\phi_{t\#} \mathcal{S})|_{t=0} \geq 0 .$$

The following is obtained by Lawson and Simons in [5].

PROPOSITION 1. Let M be a compact Riemannian manifold and V a vector field on M with associated flow ϕ_t . Then for any rectifiable p -current $\mathcal{S} \in \mathcal{R}_p(M)$,

$$(1) \quad \frac{d}{dt} \mathbf{M}(\phi_{t\#} \mathcal{S})|_{t=0} = \int_M \langle \mathcal{A} \vec{\mathcal{S}}, \vec{\mathcal{S}} \rangle d\|\mathcal{S}\| ,$$

$$(2) \quad \begin{aligned} \frac{d^2}{dt^2} \mathbf{M}(\phi_{t\#} \mathcal{S})|_{t=0} = & \int_M \{ -\langle \mathcal{A}^V \vec{\mathcal{S}}, \vec{\mathcal{S}} \rangle^2 + \langle \mathcal{A}^V \mathcal{A}^V(\vec{\mathcal{S}}), \vec{\mathcal{S}} \rangle \\ & + |\mathcal{A}^V(\vec{\mathcal{S}})|^2 + \langle \nabla_{V, \vec{\mathcal{S}}} V, \vec{\mathcal{S}} \rangle \} d\|\mathcal{S}\| . \end{aligned}$$

REMARK. In the special case that $V = \nabla f$ (= the gradient of f) for some $f \in C^3(M)$, the transformation \mathcal{A}^V is symmetric and (2) simplifies to

$$(2)' \quad \begin{aligned} \frac{d^2}{dt^2} \mathbf{M}(\phi_{t\#} \mathcal{S})|_{t=0} = & \int_M \{ -\langle \mathcal{A}^V \vec{\mathcal{S}}, \vec{\mathcal{S}} \rangle^2 + 2|\mathcal{A}^V(\mathcal{S})|^2 \\ & + \langle \nabla_{V, \vec{\mathcal{S}}} V, \mathcal{S} \rangle \} d\|\mathcal{S}\| . \end{aligned}$$

For future reference we shall write the integrand of (2)' at $x \in M$ in terms of tangent vectors at x . Let $\{\bar{e}_1, \dots, \bar{e}_p, \bar{e}_{p+1}, \dots, \bar{e}_n\}$ be an orthonormal basis of $T_x(M)$ and set $\xi = \bar{e}_1 \wedge \dots \wedge \bar{e}_p$. Then

$$(3) \quad \begin{aligned} & -\langle \mathcal{A}^V \xi, \xi \rangle^2 + 2|\mathcal{A}^V(\xi)|^2 + \langle \nabla_{V, \xi} V, \xi \rangle \\ & = \left\{ \sum_{j=1}^p \langle \mathcal{A}^V(\bar{e}_j), \bar{e}_j \rangle \right\}^2 + 2 \sum_{j=1}^p \sum_{\alpha=p+1}^n \langle \mathcal{A}^V(\bar{e}_j), \bar{e}_\alpha \rangle^2 \\ & \quad + \sum_{i=1}^p \langle \nabla_{V, \bar{e}_j} V, \bar{e}_j \rangle , \end{aligned}$$

where $|\mathcal{A}^V(\xi)|$ denotes the length of p -vector $\mathcal{A}^V(\xi)$.

2. Now we assume that M is isometrically immersed in $(n + 1)$ -

dimensional Euclidean space R^{n+1} with canonical Riemannian metric \langle, \rangle and canonical Riemannian connection $\bar{\nabla}$. For all local formulas we may consider the isometric immersion f of M into R^{n+1} as an imbedding and thus identify $x \in M$ with $f(x) \in R^{n+1}$. The tangent space $T_x(M)$ is identified with a subspace of the tangent space $T_x(R^{n+1})$. The normal space T_x^\perp is the subspace of $T_x(R^{n+1})$ consisting of all $\zeta \in T_x(R^{n+1})$ which are orthogonal to $T_x(M)$ with respect to the Riemannian metric \langle, \rangle . For each point x of M , choose a field ζ of unit normal vectors defined on a neighborhood U of x . Then we have the basic formulas

$$\begin{aligned}\bar{\nabla}_x Y &= \nabla_x Y + \langle A_\zeta X, Y \rangle \zeta \\ \bar{\nabla}_x \zeta &= -A_\zeta X\end{aligned}$$

where X and Y are smooth vector fields tangent to M , and A_ζ is a tensor field of type $(1, 1)$, called the second fundamental form associated with ζ . The Gauss equation expresses the curvature tensor R of M as follows.

$$R(X, Y)Z = \langle A_\zeta Y, Z \rangle A_\zeta X - \langle A_\zeta X, Z \rangle A_\zeta Y$$

where X, Y and Z are smooth vector fields tangent to M .

Let δ be a constant with $0 < \delta \leq 1$, and suppose that at each point x of M , with respect to a suitable field ζ of unit normals, every principal curvature λ_j of M satisfies $\sqrt{\delta} \leq \lambda_j \leq 1$, $j = 1, \dots, n$.

REMARK. The above assumption implies that M has the sectional curvature satisfying $\delta \leq K_\sigma \leq 1$ for all tangent two planes σ . And from the continuity of the eigen-values of the linear map $A_\zeta: T_x(M) \rightarrow T_x(M)$, called the principal curvatures of M , the above assumption also implies that M is orientable. Therefore we can choose a global field ζ of unit normals on M which satisfies the above condition, and then we can write $A_\zeta = A$.

3. To estimate the left hand side of (s_2) we begin with the space of functions $\mathcal{F} = \{\psi | M; \psi: R^{n+1} \rightarrow R \text{ is linear}\}$, and define

$$\mathcal{V} = \{\nabla \psi; \psi \in \mathcal{F}\}.$$

Then there is a natural isomorphism

$$(4) \quad \mathcal{V} \cong R^{n+1}$$

which associates to $v \in R^{n+1}$ the gradient of the function $\psi_v(x) = \langle v, x \rangle$ on M . This identification introduces a natural inner product on \mathcal{V} .

To any simple unit p -vector $\xi \in \bigwedge^p T_x(M)$, at any $x \in M$, we can associate a quadratic form Q_ξ on \mathcal{V} as follows. For $V \in \mathcal{V}$, let ϕ_i

be the flow generated by V , and define

$$Q_\xi(V) = \frac{d^2}{dt^2} \phi_{t\xi} \Big|_{t=0}.$$

Then we have the following.

PROPOSITION 2. *Under the assumptions as expresses above, we have*

$$\text{tr } Q_\xi \leq p(p + 1 - n\delta - \delta).$$

Proof. Suppose $V \in \mathcal{V}$ corresponds to $v \in R^{n+1}$ under the isomorphism (4). Then at any $y \in M$

$$V(y) = v - \langle v, \zeta_y \rangle \zeta_y,$$

and then for $X \in T_x(M)$, $\nabla_X V = (\bar{\nabla}_X V)^T = \langle v, \zeta_x \rangle AX$, where $()^T$ denotes orthogonal projection $T_x(R^{n+1}) \rightarrow T_x(M)$. Thus,

$$(5) \quad \mathcal{A}^V(X) = \nabla_X V = \langle v, \zeta_x \rangle AX.$$

And it follows easily that

$$(6) \quad \nabla_{V,X} V = -\langle V, AV \rangle AX + \langle v, \zeta_x \rangle \nabla_V(A\tilde{X}) - \langle v, \zeta_x \rangle A(\nabla_V \tilde{X})$$

where \tilde{X} is any extension of X to a local vector field.

We now choose an orthonormal basis $\{x_0 = \zeta_x, x_1 = e_1, \dots, x_n = e_n\}$ for R^{n+1} , where e_j is an eigenvector corresponding to the eigenvalue λ_j of A , $j = 1, \dots, n$. Via (4) this fixes an orthonormal basis $\{V_0, V_1, \dots, V_n\}$ of \mathcal{V} . It then follows from (5) and (6) that $\nabla_{V_0} \cdot V_0 = \mathcal{A}^V 1 = \dots = \mathcal{A}^V n = 0$ and $\mathcal{A}^V 0 = A$, $\nabla_{V_j} \cdot V_j = -\lambda_j A$, $j = 1, \dots, n$, as transformations of $T_x(M)$. For given simple unit p -vector $\xi \in \Lambda^p T_x(M)$, we can choose an orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_p, \bar{e}_{p+1}, \dots, \bar{e}_n\}$ of $T_x(M)$ with $\xi = \bar{e}_1 \wedge \dots \wedge \bar{e}_p$. It then follows from (2), (2)', (3) and above formulas that

$$\begin{aligned} \text{tr}(Q_\xi) &= \sum_{i=0}^n Q_\xi(V_i) \\ &= \sum_{i=0}^n \left\{ \left(\sum_{j=1}^p \langle \mathcal{A}^V l \bar{e}_j, \bar{e}_j \rangle \right)^2 + 2 \sum_{j=1}^p \sum_{\alpha=p+1}^n \langle \mathcal{A}^V l \bar{e}_j, \bar{e}_\alpha \rangle^2 \right. \\ &\quad \left. + \sum_{j=1}^p \langle \nabla_{V_l, \bar{e}_j} V_l, \bar{e}_j \rangle \right\} \\ &= \left(\sum_{j=1}^p \langle A \bar{e}_j, \bar{e}_j \rangle \right)^2 + 2 \sum_{j=1}^p \sum_{\alpha=p+1}^n \langle A \bar{e}_j, \bar{e}_\alpha \rangle^2 - \sum_{l=1}^n \sum_{j=1}^p \langle \lambda_l A \bar{e}_j, \bar{e}_j \rangle \\ &= \left(\sum_{j=1}^p \langle A \bar{e}_j, \bar{e}_j \rangle \right)^2 + 2 \sum_{j=1}^p \left(|A \bar{e}_j|^2 - \sum_{i=1}^p \langle A \bar{e}_j, \bar{e}_i \rangle^2 \right) \\ &\quad - \sum_{l=1}^n \sum_{j=1}^p \lambda_l \langle A \bar{e}_j, \bar{e}_j \rangle \end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{j=1}^p |A\bar{e}_j|^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^p (\langle A\bar{e}_i, \bar{e}_i \rangle \langle A\bar{e}_j, \bar{e}_j \rangle - 2\langle A\bar{e}_j, \bar{e}_i \rangle^2) \\
&\quad - \sum_{j=1}^p \langle A\bar{e}_j, \bar{e}_j \rangle^2 - \sum_{i=1}^n \sum_{j=1}^p \lambda_i \langle A\bar{e}_j, \bar{e}_j \rangle.
\end{aligned}$$

By the assumption, $\sqrt{\delta} \leq \lambda_j \leq 1$, $j = 1, 2, \dots, n$, we get $|A\bar{e}_j|^2 \leq 1$, and $\sqrt{\delta} \leq \langle A\bar{e}_j, \bar{e}_j \rangle \leq 1$ for $j = 1, 2, \dots, n$. Thus we have

$$\begin{aligned}
\text{tr}(Q_\varepsilon) &\leq 2p + p(p-1) - p\delta - np\delta \\
&= p(p+1 - n\delta - \delta).
\end{aligned}$$

Combining Proposition 1 and Proposition 2 we get the theorem and the Corollary 1. And by virtue of the basic theorems on integral currents, we have the Corollary 2, see [2] or [5].

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Jiří Adámek, V. Koubek and Věra Trnková, <i>Sums of Boolean spaces represent every group</i>	1
Richard Neal Ball, <i>Full convex l-subgroups and the existence of a^*-closures of lattice ordered groups</i>	7
Joseph Becker, <i>Normal hypersurfaces</i>	17
Gerald A. Beer, <i>Starshaped sets and the Hausdorff metric</i>	21
Dennis Dale Berkey and Alan Cecil Lazer, <i>Linear differential systems with measurable coefficients</i>	29
Harald Boehme, <i>Glättungen von Abbildungen 3-dimensionaler Mannigfaltigkeiten</i>	45
Stephen LaVern Campbell, <i>Linear operators for which T^*T and $T + T^*$ commute</i>	53
H. P. Dikshit and Arun Kumar, <i>Absolute summability of Fourier series with factors</i>	59
Andrew George Earnest and John Sollion Hsia, <i>Spinor norms of local integral rotations. II</i>	71
Erik Maurice Ellentuck, <i>Semigroups, Horn sentences and isolic structures</i>	87
Ingrid Fotino, <i>Generalized convolution ring of arithmetic functions</i>	103
Michael Randy Gabel, <i>Lower bounds on the stable range of polynomial rings</i>	117
Fergus John Gaines, <i>Kato-Taussky-Wielandt commutator relations and characteristic curves</i>	121
Theodore William Gamelin, <i>The polynomial hulls of certain subsets of C^2</i>	129
R. J. Gazik and Darrell Conley Kent, <i>Coarse uniform convergence spaces</i>	143
Paul R. Goodey, <i>A note on starshaped sets</i>	151
Eloise A. Hamann, <i>On power-invariance</i>	153
M. Jayachandran and M. Rajagopalan, <i>Scattered compactification for $N \cup \{P\}$</i>	161
V. Karunakaran, <i>Certain classes of regular univalent functions</i>	173
John Cronan Kieffer, <i>A ratio limit theorem for a strongly subadditive set function in a locally compact amenable group</i>	183
Siu Kwong Lo and Harald G. Niederreiter, <i>Banach-Buck measure, density, and uniform distribution in rings of algebraic integers</i>	191
Harold W. Martin, <i>Contractibility of topological spaces onto metric spaces</i>	209
Harold W. Martin, <i>Local connectedness in developable spaces</i>	219
A. Meir and John W. Moon, <i>Relations between packing and covering numbers of a tree</i>	225
Hiroshi Mori, <i>Notes on stable currents</i>	235
Donald J. Newman and I. J. Schoenberg, <i>Splines and the logarithmic function</i>	241
M. Ann Piech, <i>Locality of the number of particles operator</i>	259
Fred Richman, <i>The constructive theory of KT-modules</i>	263
Gerard Sierksma, <i>Carathéodory and Helly-numbers of convex-product-structures</i>	275
Raymond Earl Smithson, <i>Subcontinuity for multifunctions</i>	283
Gary Roy Spoor, <i>Differentiability conditions and bounds on singular points</i>	289
Rosario Strano, <i>Azumaya algebras over Hensel rings</i>	295